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CHARACTERIZING STUDENTS' THINKING: ALGEBRAIC, INEQUALITIES AND EQUATIONS

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This paper presents the findings of a study that explores the viability of using students' act of anticipating as a means to characterize the way students think while solving problems in algebra. Two types of anticipating acts were identified: predicting a result and foreseeing an action. These acts were characterized using Harel's framework, which involves the concepts of mental act, way of understanding, and way of thinking. Categories for characterizing acts of predicting and foreseeing were identified and developed based on thirteen 11th graders' responses to problems involving algebraic inequalities and equations. The quality of students' acts of predicting and foreseeing was found to be related to the quality of their interpretations of inequalities and equations.

Upon seeing a problem, students commonly rush into action without analyzing the problem situation. As teachers, we witness some students' inappropriate use of procedures, what Fischbein and Barash (1993) call improper application of algorithmic models. On the other hand, we also witness engagement in exploration and analysis of the problem situation among certain students. This research seeks to characterize the differences in the manner students solve problems involving algebraic inequalities and equations.

Theoretical Framework

This research combines multiple perspectives: Piaget's (1967/1971) notion of anticipation, von Glasersfeld's (1998) three general kinds of anticipation, Harel's (2001, in press) notions of way of understanding and way of thinking, and Cobb's (1985) hierarchical levels of anticipation. According to Piaget (1967/1971), anticipation is one of the two functions of knowing; the other function being conservation-of-information, an instrument of which is a scheme. The anticipation function deals with the application of a scheme to a new situation. It allows us to have foresights, strategize and plan, make predictions, formulate conjectures, engage in thought experiments, etc. Such foresights and predictions are possible because of our ability to assimilate situations into our existing scheme(s); "anticipation is nothing other than a transfer or application of the scheme … to a new situation before it actually happens" (p. 195). A scheme, as outlined by von Glasersfeld (1995), involves three components: *the perceived situation*, *the activity*, and *the expected result*. The expected result component provides the anticipatory feature of a scheme. This component constitutes the fundamental difference between a Piagetian scheme and a condition-action pair in information processing or a stimulus-response association in behaviorism.

Von Glasersfeld (1998) elaborated on Piaget's notion of anticipation by pointing to three general kinds of anticipation: (a) implicit expectations that are present in our actions, e.g., the preparation and control of our movements when we grope in the dark; (b) prediction of an outcome, e.g., predicting that it will soon rain upon noticing that the sky is being covered by dark clouds; and (c) foresight of a desired event and the means for attaining it, e.g., a child's anticipation of the capitulation of his parent if he were to throw a temper tantrum in public. In

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my attempt to adapt these three kinds of anticipation to the context of solving problems in mathematics, I was not able to infer students' implicit anticipation from their statements and actions. I therefore focused on students' prediction of a result and foresight of an action. *Predicting* is defined as the act of conceiving an expectation for the result of an event without actually performing the operations associated with the event. *Foreseeing* is defined as the act of conceiving an expectation that leads to an action, prior to performing the operations associated with the action.

Harel's (2006, in press) MA-WoU-WoT framework is suitable for analyzing students' *mental acts* (MA_s) of predicting and foreseeing. Predicting and foreseeing are among the many mental acts that one might carry out when one solves a mathematics problem. Other mental acts include interpreting, symbolizing, transforming, generalizing, justifying, inferring, etc. *Way of understanding* (WoU) refers to the product of a particular mental act and *way of thinking* (WoT) refers to a character of the mental act. For example, in the act of proving WoU refers to the proof a student produces and WoT refers to the *proof scheme* that underlies the student's act of proving. Harel and Sowder (1998) have developed a taxonomy of students' proof schemes, examples of which are *authoritative proof scheme* (one derives conviction mainly from the authority of the teacher or textbook), *empirical proof scheme* (one derives conviction from empirical evidence or visual perceptions), and *deductive proof scheme* (one derives conviction based on the application of rules of logic). Similarly, for the act of predicting (foreseeing), WoU refers to the result (action) a student actually predicts (foresees), whereas WoT characterizes the manner in which the student predicts (foresees).

Cobb (1985) identifies three hierarchical levels of anticipation: beliefs, problem-solving heuristics, and conceptual structures. At the most specific level, one's expressed conceptual structure (i.e., evoked scheme) dictates one's anticipation. An expressed conceptual structure can be viewed as a WoU associated with the mental act of interpreting. In the domain of algebraic inequalities and equations, the dependence of anticipations on conceptual structures suggests a relationship between students' ways of understanding (W_soU) inequalities and equations and their ways of thinking (W_soT) associated with the mental act of anticipating.

The research, part of which this paper reports, has three objectives: (a) to categorize students' W_soT associated with the mental acts of predicting and foreseeing, (b) to identify the relationship between these W_soT and their W_soU algebraic inequalities/equations, and (c) to explore the potential for advancing students' W_soT through a short-term instructional intervention. Figure 1 provides a schematic representation of the framework for analyzing students' act of problem-solving in terms of mental acts of predicting, foreseeing, and interpreting. Each dotted curve refers to the relationship between students' W_soT associated with anticipating (predicting or foreseeing) and their interpretations of inequalities/equations.

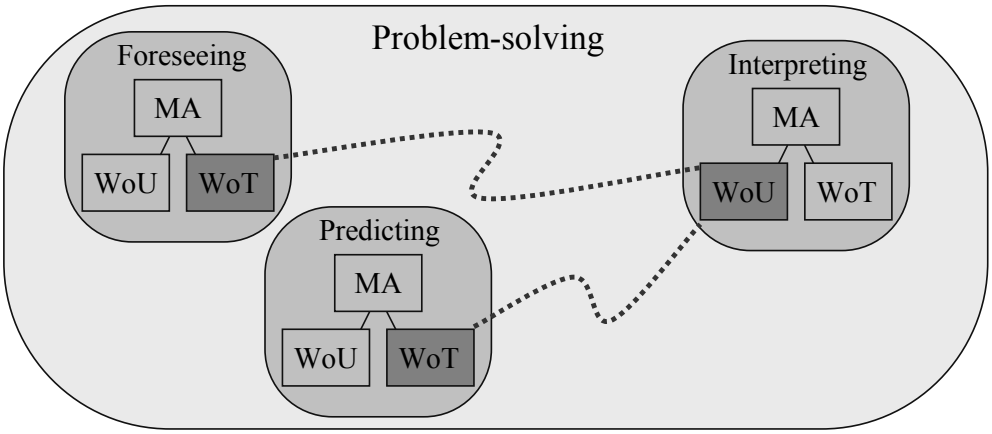

Figure 1: A schematic representation of the framework used in this research method

Fourteen 11th graders were interviewed, each for approximately 60 minutes. Four of these interviewees participated in a one-on-one teaching intervention, which was comprised of five problem-solving sessions followed by a post-interview. This research was conducted in a university-based charter school in Southern California. This school practices detracking: there is only one track for all the students; but different students at a particular grade level may be at different stages along the track. 4 interviewees were taking Algebra II, 4 were taking Pre-calculus, and 6 were taking Calculus. This distribution allowed me to observe a greater variety in students' WsoT.

The purpose of this research was to study 11th graders' thinking as they solved non-routine problems involving Algebra I concepts. Tasks used in the clinical interviews include: (a) Is there a value for x that will make $(2x – 6)(x – 3) < 0$ true? (b) Given that $5a = b + 5$, which is larger: a or b? And (c) p and q are odd integers between 20 and 50. For these values, is $5p – q > 2p + 15$ always true, sometimes true or never true?

These tasks differ from typical tasks in textbooks in that they do not direct students to perform a specific task such as "solve for x" or "simplify." This non-directive feature is found to be effective at eliciting a greater variety of anticipatory behaviors. All the tasks were phrased in the form of a question to allow students to predict the answer, if they chose to, prior to performing any actions.

All the interviews and problem-solving sessions were videotaped and transcribed. One interview was discarded because the interviewee was struggling with her arithmetic. Observation concepts (Clement, 2000) for students' WsoT associated with predicting and foreseeing and students' WsoU inequalities/equations were identified. These categories were derived from the data using a constant comparative approach (Glaser & Strauss, 1967), in which categories were constantly revised by comparing current data with previously analyzed data. The analysis involved identifying instances of the mental acts of predicting and foreseeing (inferred from student's actions and statements), generating, comparing, and refining categories for WsoU and WsoT, and consolidating and collapsing some of the categories. The consolidated categories were revised and refined in light of new information generated in subsequent phases of the analysis (e.g., analysis to account for learners' improvement).

Results and Discussion

This paper reports the research findings for the first two objectives. Excerpts of two interviewees' work are presented to illustrate the viability of using students' acts of

predicting/foreseeing as a means to characterize students' thinking. Categories for ways of thinking (WsoT) associated with predicting/foreseeing will be discussed. The same excerpts are also used to highlight the relations between WsoT associated with predicting/foreseeing and ways of understanding (WsoU) inequalities/equations.

Contrasting Two Students' Work

Consider two interviewees' response to this item: "Is there a value for x that will make the following statement true? $(6x – 8 – 15x) + 12 > (6x – 8 – 15x) + 6$". Both interviewees, Talia and Pham, were 11th graders enrolled in Calculus.

Excerpt 1: Talia's initial response

Talia: Is there a value for x that will make the following statement true? Of course there is. Let see, umm.

Lim: Why did you say "of course, there is"?

Talia: Because, well, I figure there should be an answer to this problem, and, um, let's see, I was taught to combine like terms. I was taught this (>) is actually an equal sign.

Lim: OK.

Talia: To solve it like I would solve an equation. … (She obtained $-9x + 6 = -9x$ and then wrote $6 > 0$.) Umm, that doesn't [seem] right, because x has canceled out. What did I do wrong? … OK. Is there a value for x that will make the following statement true? Maybe there isn't.

Excerpt 2: Pham's initial response

Pham: OK. Let's see. The stuffs in the parentheses are the same. Umm, OK, first I guess I would combine all like terms. … (He got $-9x + 4 > -9x – 2$). Umm, now it's asking is there a value for *x* that will make the following statement true. Umm, let me see, I think 4 and -2, so you have a common term (i.e. $-9x$). OK, so it's, you have a -9, so anything [positive] that you multiply will [make it] a negative number, and this (+4) is positive. Let's see, yes, there is a value because… this, this [left] side will be greater. I guess, if it ($-9x$) was positive then, so is this side ($-9x$). So any negative number would make the statement true. … Umm, I think all numbers would make the statement true.

One difference between these two responses is that Pham arrived at the correct answer but Talia did not. Another difference is that Talia's WoU inequalities is weaker than Pham's. Talia interpreted the inequality as a signal to isolate x and treated it as an "equation," whereas Pham treated the inequality as a comparison of two algebraic expressions. A third difference is the manner in which they approach the problem. How can we characterize the thinking that underlies the actions they took to solve this problem?

Categories for Ways of Thinking Associated with Predicting/Foreseeing

Both Talia and Pham combined like terms. From a Piagetian perspective, action presupposes anticipation. So we can assume that Talia and Pham had anticipated the expediency of combining like terms. Since a WoU associated with foreseeing refers to the action one actually anticipates, both Talia and Pham are said to have the same WoU: combining like terms. Both of them were spontaneous in their foresight of combining like terms. However, the spontaneity in Talia's anticipation was characteristically different from that in Pham's. Upon seeing the

problem, Talia immediately thought of what she could do to the inequality, rather than thinking about what the question was asking. Her act of anticipating had an element of impulsiveness, impulsive in the sense that she had routinized a particular WoU (i.e., combining like terms is a routine for her to solve certain inequalities/equations). I categorized her WoT associated with foreseeing as impulsive anticipation. This WoT is generally inferred when a student immediately applies a procedure without considering its appropriateness.

Pham, on the other hand, noticed that "the stuffs in the parentheses are the same" and combined like terms with the probable intent of obtaining a simpler form. He might have predicted in his mind that the left side was always larger than the right side and was confirming his prediction. He seemed to have interiorized the usefulness of combining like terms and was capitalizing on his understanding that it would be easier to reason with simpler expressions. Thus his WoT was coded as interiorized anticipation. By "interiorized", I mean one has not only internalized (i.e. gained the ability to autonomously and spontaneously apply one's WoU to another similar situation) a particular WoU but has also reorganized and abstracted the WoU to a higher level of understanding.

With respect to the mental act of predicting, Talia predicted "of course there is" upon seeing the problem. She seemed to have associated her having a procedure for isolating x with the inequality having a solution. Because of this, I categorized her WoT characterizing her prediction as associated-based prediction. This WoT is inferred when a student predicts by merely associating two ideas without establishing the basis for making such an association. Talia's prediction of "maybe there isn't" upon observing the disappearance of x from the inequality is also considered association-based because she associated the disappearance of x with the nonexistence of a value for x that would make the inequality true.

Pham, on the other hand, did not explicitly make a prediction. Instead, he reasoned with $9x + 4 > 9x - 2$. His WoT associated with foreseeing is considered analytic anticipation because he identified the goal of determining whether there is a value of x that will make the new inequality true, and foresaw the usefulness of reasoning with the common term -9x.

So far, I have introduced four WsoT associated with foreseeing/predicting: impulsive anticipation, interiorized anticipation, analytic anticipation, and association-based prediction. A total of five WsoT associated with foreseeing and three WsoT associated with predicting emerged from the data. Descriptions for these WsoT are presented in Table 1. These WsoT are elaborated in my doctoral dissertation (Lim, 2006). Relations between students' WsoT and the quality of their solutions are also discussed in that manuscript.

Ways of Thinking	Descriptions
Impulsive anticipation	Spontaneously proceeds with an action that comes to mind without analyzing the problem situation and without considering the relevance of the anticipated action to the problem situation
Tenacious anticipation	Maintains and does not re-evaluate one's way of understanding (prediction, problem-solving approach, claim, or conclusion) of the problem situation in light of new information
Explorative anticipation	Explores an idea to gain a better understanding of the problem situation
Analytic Anticipation	Analyzes the problem situation and establishes a goal or a criterion to guide one's actions

Interiorized anticipation	Spontaneously proceeds with an action without having to analyze the problem situation because one has interiorized the relevance of the anticipated action to the situation at hand
Association-based prediction	Predicts by associating two ideas without establishing the basis for making such an association
Comparison-based prediction	Predicts by comparing two elements or situations in a static manner
Coordination-based prediction	Predicts by coordinating quantities or attending to relationships among quantities

Table 1. Categories for WsoT associated with foreseeing and predicting

Relations between WsoT associated with Anticipating and WsoU Inequalities/Equations

In Excerpt 1, Talia seemed to interpret the inequality as a signal to isolate x and treated it as an equation within which she could manipulate symbols. Accordingly, her WoU was coded as inequality/equation-as-a-signal-for-a-procedure interpretation. This interpretation is inferred when a student treats the inequality/equation (I/E) as an object to be worked on and does not appear to have other WsoU. Her impulsive anticipation and association-based prediction appeared to be a consequence of her interpreting the inequality as a signal to do something.

In Excerpt 2, Pham's reasoning with -9x suggests that he was interpreting it as a function whose output depends on the input variable x. Thus his WoU was coded as I/E-as-a-comparison-of-functions interpretation. His analytic anticipation of reasoning with the common term -9x was supported by this WoU.

Three additional WsoU inequalities/equations (I/E) emerged from the data: I/E-as-a-static-comparison interpretation, I/E-as-a-proposition interpretation, and I/E-as-a-constraint interpretation. In general, less sophisticated WsoU were found to be related to less desirable WsoT associated with predicting/foreseeing. For example, the I/E-as-a-signal-for-a-procedure interpretation tends to lead to impulsive anticipation. Conversely, more sophisticated WsoU are related to more desirable WsoT associated with predicting/foreseeing. For example, the I/E-as-a-constraint interpretation facilitates goal-oriented reasoning, which is an attribute of analytic anticipation. As depicted in Table 2 (the entries are based on interviewees' responses to two interview items), 9 of the 13 interviewees exhibited analytic anticipation while interpreting an inequality as a constraint. This table demonstrates that significantly more students exhibited a more desirable WoT when they showed a more sophisticated WoU, and most of those who exhibited a less desirable WoT showed a less sophisticated WoU.

		More Sophisticated WoU		**Less Sophisticated WoU**	
		I/E-as-a-Comparison-of-Functions	I/E-as-a-Constraint	I/E-as-a-Static-Comparison	I/E-as-a-Signal-for-a-Procedure
More Desirable WoT	Interiorized Anticipation	4	3		
	Analytic Anticipation	2	9	1	1
	Coordination Prediction	2	3	2	
Less Desirable WoT	Association Prediction			3	1
	Tenacious Anticipation		1	1	
	Impulsive Anticipation				2

Table 2. Number of interviewees exhibiting a particular WoT and a particular WoU

Conclusion

One objective of this research was to develop categories for ways of thinking associated with the mental acts of foreseeing and predicting. Ways of thinking associated with foreseeing provide mathematics educators with the vocabulary to communicate the way students approach a problem: whether they (a) hastily apply a procedure, (b) are tenacious in their way of understanding, (c) explore different ideas, (d) analyze the problem situation and identify a goal, and (e) spontaneously apply their ways of understanding that are pertinent to the problem situation. These descriptions correspond to impulsive anticipation, tenacious anticipation, explorative anticipation, analytic anticipation, and interiorized anticipation. An awareness of these categories can help mathematics teachers to be more explicit about their goal of advancing students from being impulsive and tenacious to being analytic and explorative.

Instruction that leads students to predict can help counteract students' tendency of rushing to apply procedures when they are assigned a problem. Having explicit terms to characterize the ways students predict allows teachers to differentiate desirable ways of thinking associated with predicting from less desirable ones. For example, coordination-based prediction is desirable because it promotes reasoning that involves change and coordination whereas association-based prediction is undesirable because it tends to foster the non-referential symbolic way of thinking. Having made these distinctions explicit, mathematics educators can design and implement instructional activities that aim to help students progress from association-based prediction to coordination-based prediction.

The relationship between the desirability of students' W_soT associated with predicting/foreseeing and the sophistication in their W_soU inequalities/equations suggests that we, as teachers, should attend to students' W_soT associated with predicting/foreseeing while helping students to develop sophisticated W_soU inequalities/equations, and vice versa. This recommendation is in keeping with Harel's (2006) call to incorporate desirable W_soT and sophisticated W_soU as cognitive objectives for instruction: "In designing, developing, and implementing mathematics curricula, ways of thinking and ways of understanding must be the ultimate cognitive objectives, and they must be addressed simultaneously, for each affects the other."

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