Solving Coupled Systems of Differential Equations Using the Length Factor Artificial Neural Network Method

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ANN training by gradient descent

\[ E(w) = \sum_x (N_{\text{desired}} - N)^2 \]

- **Weight vector** \( w \)
- **Spatial vector** \( x \)
- **Output** \( N(x, w) \)
- **Error**
- **Sensitivity of** \( E \) **to changes in** \( w \)

**Diagram:**
- **Global minimum**
- **Saddle point**
- **Local minimum**
- **Plateau**

**Equation:**

\[ \frac{dE}{dw} = -2 \sum_x (N_{\text{desired}} - N) \frac{\partial N}{\partial w} \]
Solving BVPs with ANNs

Heat conduction in board

\[
\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0
\]

\[
T = T_L + \frac{2(T_H - T_L)}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} + 1}{n} \sin \left( \frac{n\pi x}{L} \right) \frac{\sinh \left( \frac{n\pi y}{L} \right)}{ \sinh \left( \frac{n\pi W}{L} \right)}
\]

Ann output

\[ T_t = N(x, y, w) \]

approximate solution

ANN weights

no guarantee BCs are satisfied

\[ E(w) = \sum_{x,y} \left( \frac{\partial^2 T_t}{\partial x^2} + \frac{\partial^2 T_t}{\partial y^2} \right)^2 \]

\[ \frac{dE}{dw} \] used to drive gradient descent
**Automatic BC satisfaction**

\[ \psi_t = f(x, y, N) = A(x, y) + L(x, y)N(x, y, \mathbf{w}) \]

- **Boundary function** $A$
  - Satisfy BCs on domain boundary
  - Value of $A$ inside domain unimportant

- **Length factor** $L$
  - Zero for locations on boundary
  - Non-zero everywhere inside domain

- **Approximate solution** $\psi_t$
  - Satisfies BCs regardless of ANN output $N$
  - Train ANN to satisfy DE inside domain
    - Values of $\psi_t$ inside domain should be accurate
**Example of training**

\[ \psi_t = A + LN \quad \text{Approximate solution} \]

\[ G = \frac{\partial^2 \psi_t}{\partial x^2} + \frac{\partial^2 \psi_t}{\partial y^2} + e^\nu - 1 - x^2 - y^2 - \frac{4}{(1 + x^2 + y^2)^2} \quad \text{Error in DE} \]

\[ D = \psi - \psi_t \quad \text{Error in solution} \]

\[ \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + e^\nu = 1 + x^2 + y^2 + \frac{4}{(1 + x^2 + y^2)^2} \]

\[ \psi(x, y) = \ln(1 + x^2 + y^2) \]
Solving coupled systems of DEs

\[ \text{Nu}_x = 0.332 \text{Re}^{\frac{1}{2}} \text{Pr}^{\frac{1}{3}} \]

**Energy**

\[ G_3 = u_t^* \frac{\partial T_t^*}{\partial x^*} + v_t^* \frac{\partial T_t^*}{\partial y^*} - \frac{1}{\text{Re Pr}} \frac{\partial^2 T_t^*}{\partial y^*} \]

**Momentum**

\[ G_2 = u_t^* \frac{\partial u_t^*}{\partial x^*} + v_t^* \frac{\partial u_t^*}{\partial y^*} - \frac{1}{\text{Re}} \frac{\partial^2 u_t^*}{\partial y^*} \]

**Continuity**

\[ G_1 = 0 \]

Self-similar method

\[ \eta = y^* \sqrt{\frac{\text{L} u_x}{\nu x^*} u_x, T_x} \]

\[ f''' + ff'' = 0 \]

\[ T^* = 1 - \int_0^\eta \frac{(f^n)^{\text{Pr}}}{d\eta} \]

\[ u_t^* = A_1 + L_1 N_1 \]

\[ v_t^* = A_2 + L_2 N_2 \]

\[ T_t^* = A_3 + L_3 N_3 \]

\[ \text{Nu}_x = x^* \frac{\partial T_t^*}{\partial y^*} \bigg|_{y^*=0} \]

omitted BCs at \( y = \infty \)

% error

\[ E(w_1, w_2, \ldots, w_n) = \sum_{i=1}^n \left( \sum_{x,y} G_i^2 \right) \]

ASME Early Career Technical Conference

October 1-2, 2010

Dr. Kevin McFall
Entrance length problem

- Parallel flat plates

\[
G_1 = \frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y}
\]

\[
G_2 = \frac{\partial p_t^*}{\partial x^*} + \left( u^* \frac{\partial u_t^*}{\partial x^*} + v_t^* \frac{\partial u_t^*}{\partial y^*} \right) - \frac{1}{\text{Re}} \left( \frac{\partial^2 u_t^*}{\partial x^* \partial y^*} + \frac{\partial^2 u_t^*}{\partial y^* \partial y^*} \right)
\]

\[
G_3 = \frac{\partial p_t^*}{\partial y^*} + \left( u_t^* \frac{\partial v_t^*}{\partial x^*} + v_t^* \frac{\partial v_t^*}{\partial y^*} \right) - \frac{1}{\text{Re}} \left( \frac{\partial^2 v_t^*}{\partial x^* \partial y^*} + \frac{\partial^2 v_t^*}{\partial y^* \partial y^*} \right)
\]

### Volumetric flow rate

\[
\int u^* \, dA^* = \frac{1}{2} w^*
\]

### Pressure gradient

\[
\left. \frac{\partial p^*}{\partial x^*} \right|_{x^*>l_e^*} = \frac{48}{\text{Re}} = 3.2
\]

\[
\left. \frac{\partial p_t^*}{\partial x^*} \right|_{x^*>l_e^*} = 3.19
\]
Benefits of ANN method

- Traditional solution method
  - Finite element method (FEM)

- Discretization comparison
  - FEM: polygon shaped meshing elements
  - ANN: rectangular grid

- Applicability comparison
  - FEM: complexity of method and DE linked
  - ANN: same method independent of DE

- Comparison with other ANN methods
  - Simpler to train
    - Unconstrained optimization
    - Boundary function $A$ similar to true solution $\psi$