Kennesaw State University

From the SelectedWorks of Kevin McFall

October, 2010

Solving Coupled Systems of Differential Equations Using the Length Factor Artificial NeuralNetwork Method

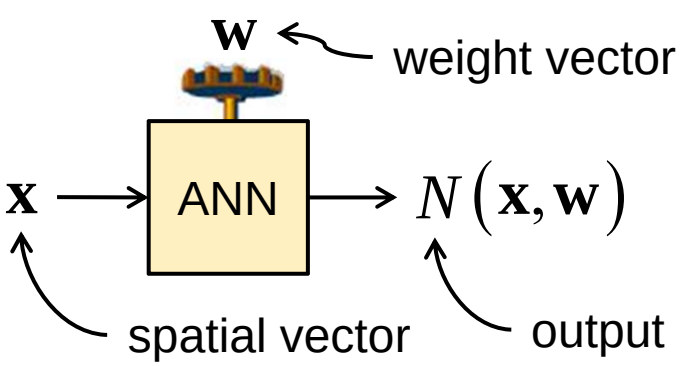
Kevin McFall, *Kennesaw State University*

Available at: https://works.bepress.com/kevin-mcfall/22/

Solving Coupled Systems of Differential Equations Using the Length Factor Artificial Neural Network Method

Dr. Kevin McFall
Assistant Professor of Engineering
The Pennsylvania State University
Lehigh Valley Campus

ANN training by gradient descent



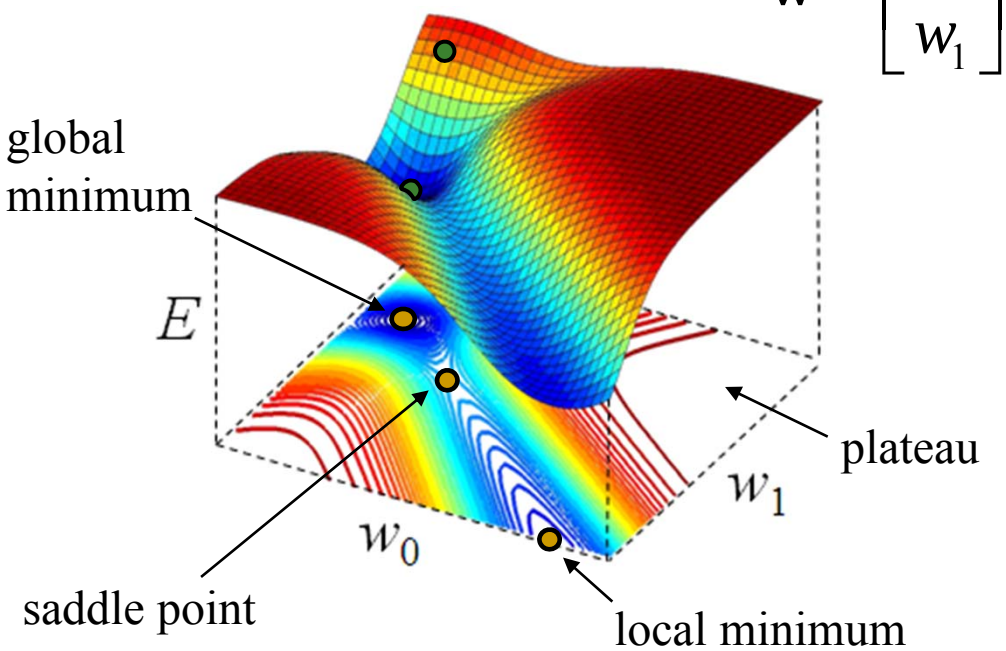
error

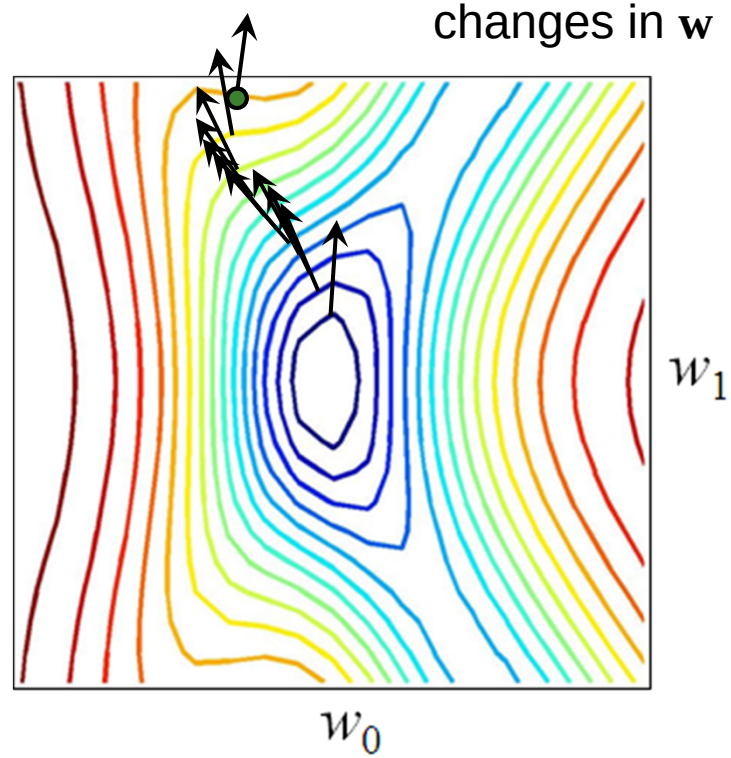
$$E(\mathbf{w}) = \sum_{\mathbf{x}} \left(N_{\text{desired}} - N \right)^2$$

$$\frac{dE}{d\mathbf{w}}$$

sensitivity of E to changes in $\mathbf{w}$

$$\mathbf{w} = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}$$





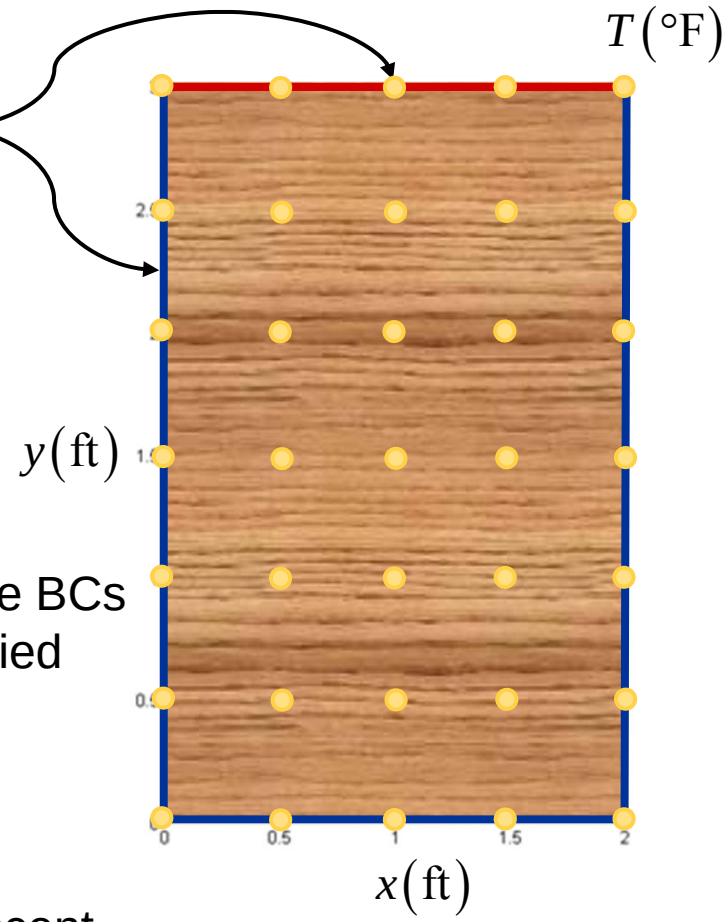
Solving BVPs with ANNs

Heat conduction in board

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

boundary conditions (BCs)

$$T = T_L + \frac{2\left(T_H - T_L\right)}{\pi} \sum_{n=1}^{\infty} \frac{\left(-1\right)^{n+1} + 1}{n} \sin\left(\frac{n\pi x}{L}\right) \frac{\sinh\left(\frac{n\pi y}{L}\right)}{\sinh\left(\frac{n\pi W}{L}\right)}$$



ANN output

ANN weights

$$T_t = N\left(x, y, \mathbf{w}\right)$$

approximate solution

no guarantee BCs are satisfied

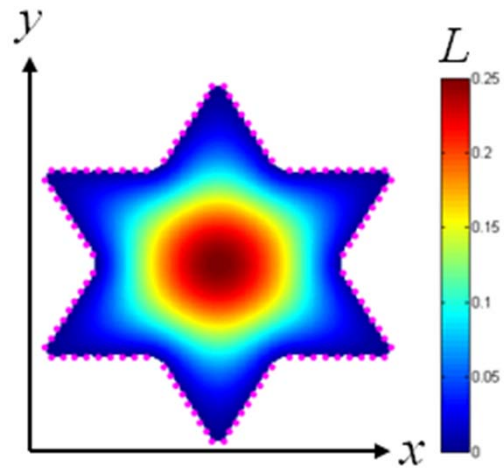
$$E\left(\mathbf{w}\right) = \sum_{x,y} \left(\frac{\partial^2 T_t}{\partial x^2} + \frac{\partial^2 T_t}{\partial y^2}\right)^2$$

$\frac{dE}{d\mathbf{w}}$ used to drive gradient descent

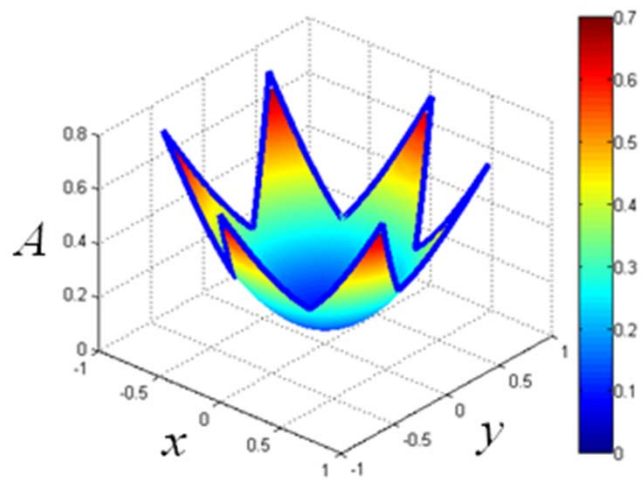
Automatic BC satisfaction

$$\psi_t = f(x, y, N) = A(x, y) + L(x, y)\, N(x, y, \mathbf{w})$$

approximate solution — boundary function — length factor — ANN output



- Boundary function A
  - Satisfy BCs on domain boundary
  - Value of A inside domain unimportant
- Length factor L
  - Zero for locations on boundary
  - Non-zero everywhere inside domain
- Approximate solution ψ_t
  - Satisfies BCs regardless of ANN output N
  - Train ANN to satisfy DE inside domain
    - Values of ψ_t inside domain should be accurate



Example of training

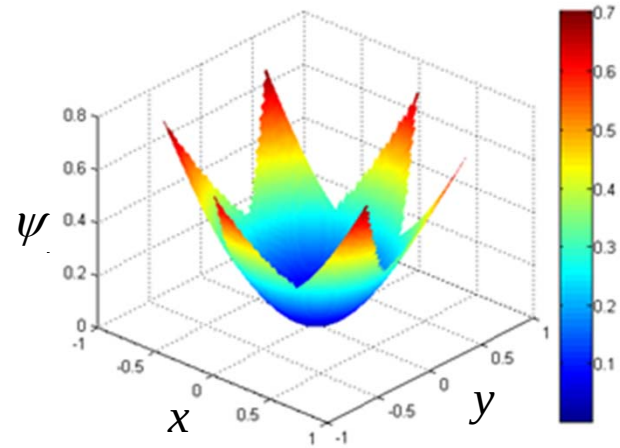
$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+e^{\psi}=1+x^2+y^2+\frac{4}{\left(1+x^2+y^2\right)^2}$$

$$\psi\left(x,y\right)=\ln\left(1+x^2+y^2\right)$$

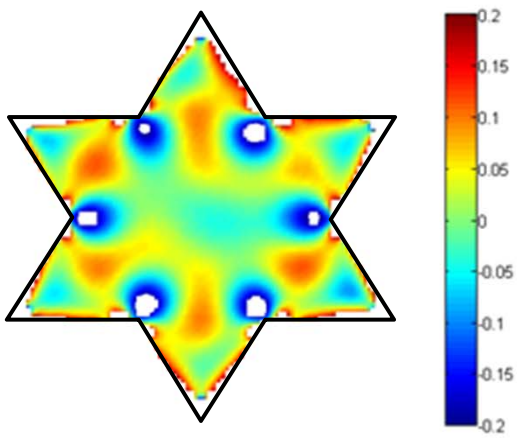
$\psi_t = A + LN$ Approximate solution

$$G=\frac{\partial^2\psi_t}{\partial x^2}+\frac{\partial^2\psi_t}{\partial y^2}+e^{\psi_t}-1-x^2-y^2-\frac{4}{\left(1+x^2+y^2\right)^2}$$ Error in DE

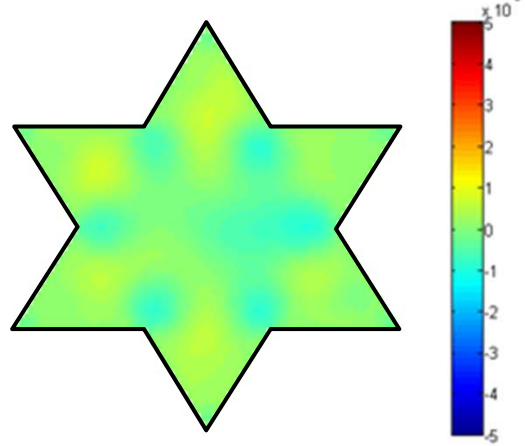
$D=\psi-\psi_t$ Error in solution

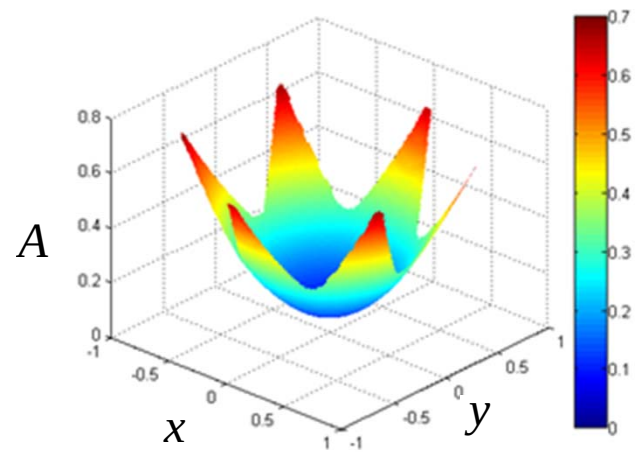


Error in DE



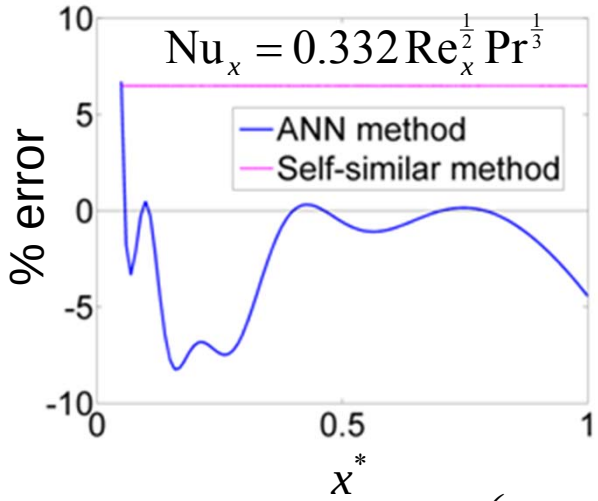
Error in solution





Solving coupled systems of DEs

omitted BCs at $y = \infty$



$$E\left(\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_n\right) = \sum_{i=1}^{n}\left(\sum_{x,y} G_i^2\right)$$

continuity $\quad G_1 = \dfrac{\partial u_t^*}{\partial x} + \dfrac{\partial v_t^*}{\partial y}$

momentum $\quad G_2 = u_t^* \dfrac{\partial u_t^*}{\partial x^*} + v_t^* \dfrac{\partial u_t^*}{\partial y^*} - \dfrac{1}{\mathrm{Re}} \dfrac{\partial^2 u_t^*}{\partial y^{*2}}$

energy $\quad G_3 = u_t^* \dfrac{\partial T_t^*}{\partial x^*} + v^* \dfrac{\partial T_t^*}{\partial y^*} - \dfrac{1}{\mathrm{Re}\,\mathrm{Pr}} \dfrac{\partial^2 T_t^*}{\partial y^{*2}}$

Self-similar method

$$\eta = y^* \sqrt{\frac{L u_\infty}{\nu x^*}}\, u_\infty, T_\infty$$

$$f''' + f f'' = 0$$

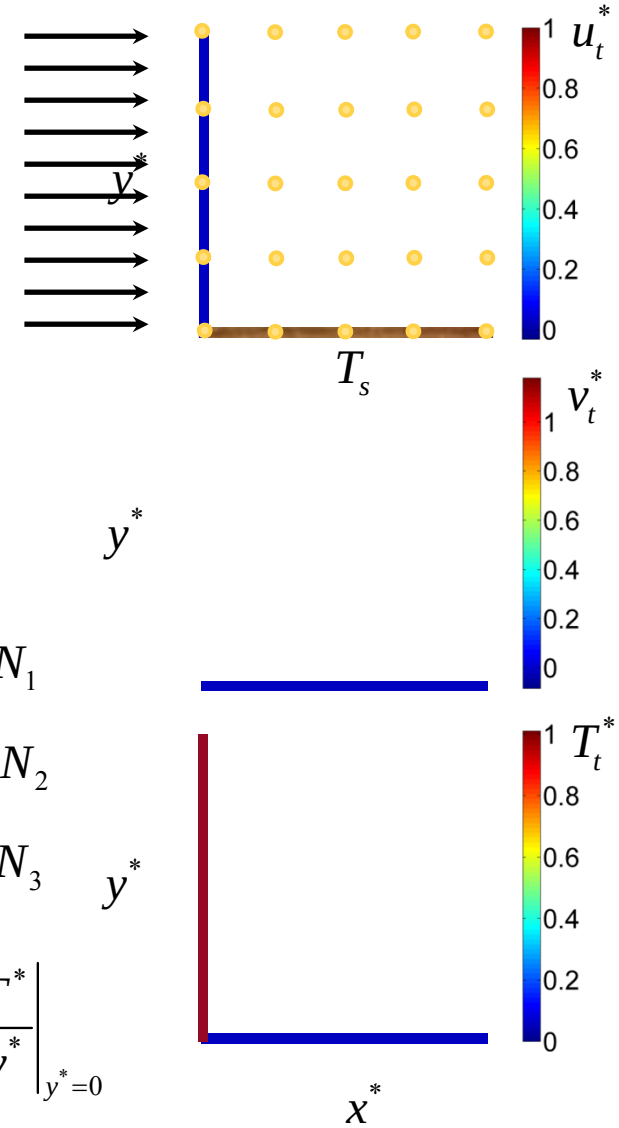
$$T^* = 1 - \frac{\int_0^\eta (f'')^{\mathrm{Pr}}\, d\eta}{\int_0^\infty (f'')^{\mathrm{Pr}}\, d\eta}$$

$$u_t^* = A_1 + L_1 N_1$$

$$v_t^* = A_2 + L_2 N_2$$

$$T_t^* = A_3 + L_3 N_3$$

$$\mathrm{Nu}_x = x^* \left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=0}$$



Entrance length problem

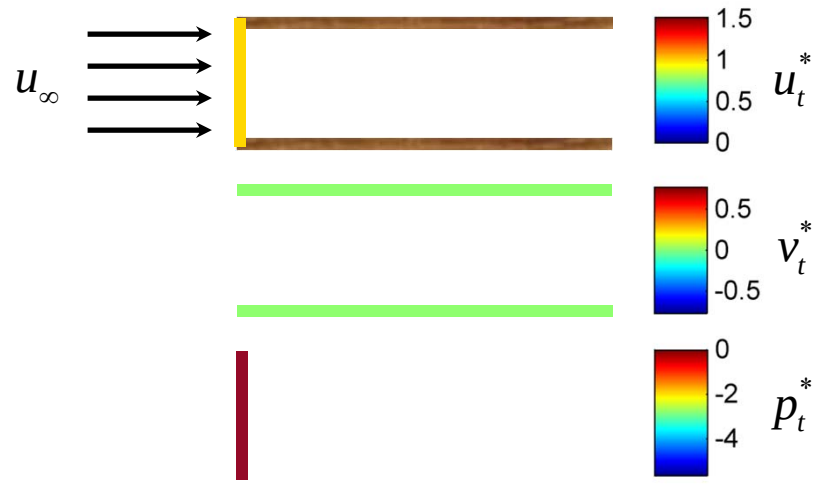
- Parallel flat plates

$$G_1 = \frac{\partial u_t^*}{\partial x} + \frac{\partial v_t^*}{\partial y}$$

$$G_2 = \frac{\partial p_t^*}{\partial x^*} + \left(u_t^* \frac{\partial u_t^*}{\partial x^*} + v_t^* \frac{\partial u_t^*}{\partial y^*} \right) - \frac{1}{\mathrm{Re}} \left(\frac{\partial^2 u_t^*}{\partial x^{*2}} + \frac{\partial^2 u_t^*}{\partial y^{*2}} \right)$$

$$G_3 = \frac{\partial p_t^*}{\partial y^*} + \left(u_t^* \frac{\partial v_t^*}{\partial x^*} + v^* \frac{\partial v_t^*}{\partial y^*} \right) - \frac{1}{\mathrm{Re}} \left(\frac{\partial^2 v_t^*}{\partial x^{*2}} + \frac{\partial^2 v_t^*}{\partial y^{*2}} \right)$$

$\mathrm{Re} = 15$



<u>Volumetric flow rate</u>

$$\int u^* dA^* = \tfrac{1}{2} w^*$$

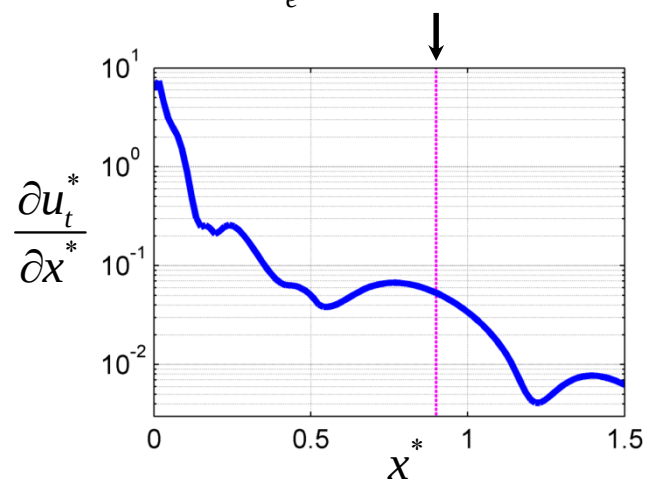
$$\int u_t^* dA^* = 0.500 w^*$$

<u>Pressure gradient</u>

$$\left. \frac{\partial p^*}{\partial x^*} \right|_{x^* > l_e^*} = \frac{48}{\mathrm{Re}} = 3.2$$

$$\left. \frac{\partial p_t^*}{\partial x^*} \right|_{x^* > l_e^*} = 3.19$$

$l_e^* = 0.06\,\mathrm{Re} = 0.9$



Benefits of ANN method

- Traditional solution method
  - Finite element method (FEM)
- Discretization comparison
  - FEM: polygon shaped meshing elements
  - ANN: rectangular grid
- Applicability comparison
  - FEM: complexity of method and DE linked
  - ANN: same method independent of DE
- Comparison with other ANN methods
  - Simpler to train
    - Unconstrained optimization
    - Boundary function A similar to true solution ψ