1998

# A Multimodal Traffic Network Equilibrium Model with Emission Pollution Permits

Anna Nagurney, *University of Massachusetts - Amherst*
Kathy K Dhanda
Padma Ramanujam, *University of Massachusetts - Amherst*

Pergamon

*Transpn Res.-D*, Vol. 3, No. 5, pp. 349–374, 1998

Printed in Great Britain
1361-9209/98 $19.00+0.00

PII: S1361-9209(98)00016-9

# A MULTIMODAL TRAFFIC NETWORK EQUILIBRIUM MODEL WITH EMISSION POLLUTION PERMITS: COMPLIANCE VS NONCOMPLIANCE

ANNA NAGURNEY,* PADMA RAMANUJAM
Department of Finance and Operations Management, School of Management, University of Massachusetts, Amherst, MA 01003, U.S.A.

and

KANWALROOP KATHY DHANDA
Department of Business, College of Business Administration, North Dakota State University, Fargo, N D 58105-5137, U.S.A.

*(Received 1 December 1997; accepted 1 April 1998)*

Abstract—In this paper, we develop a multimodal traffic network equilibrium model with vehicular emission pollution permits using the theory of variational inequalities. We consider both the case of compliance in which travelers emit pollutants no more than that mandated by their license holdings and that of non-compliance. In the former case, we prove that environmental standards imposed by the authorities are met provided that the initial license allocation meets the environmental target. For the latter model, we establish a penalty scheme that guarantees that there will be no nonconcompliant behavior. Qualitative analysis of the model is conducted and existence and uniqueness results of the solution obtained. An algorithm is proposed, with convergence results, to compute the multimodal equilibrium link load, travel demand, marginal cost of emission abatement, emission underflow, overflow, license, and license price pattern. The algorithm is then applied to several numerical examples. © 1998 Elsevier Science Ltd. All rights reserved

*Keywords*: environmental policy, mobile emission pollution permits, multimodal traffic network equilibrium, variational inequalities, noncompliance, penalty scheme.

## 1. INTRODUCTION

The protection of the environment has become a major challenge facing public policy makers today with the deterioration of air quality in urban areas creating an urgent need for environmentally friendly and effective policy instruments and measures. The role that traffic congestion, in particular, plays in the deterioration of the environment through the generation of air pollution in the form of vehicular emissions is now well-documented. Indeed, cars and other motor vehicles are responsible for at least 50% of the air pollution in urban areas (cf. *The Economist*, 1996, 1997). Vehicles are responsible for 90% of the carbon monoxide generated, 50% of the nitrogen oxide, and 15% of the carbon dioxide, the principal global warming gas (see Button, 1990). It is also well-recognized that vehicles are distinct in terms of the amount and type of the emissions that they generate with factors that affect the emission generation including: age of vehicle, type of vehicle, size, etc. (cf. Cadle, Gorse, and Lawson, 1993).

Transportation policy interventions date to Pigou, 1920 and Knight, 1924 and include such instruments as tolls (see also Dafermos, 1973), congestion pricing (see e.g. Jones and Hervik, 1992), and area licenses (Toh, 1992), but these have focused, principally, on traffic management, and not on pollution reduction, per se. Governments in recent years have introduced a variety of legislation that has spurred the further development of policy instruments aimed at pollution reduction. In the United States, for example, the 1990 Clean Air Act Amendments (cf. U.S. DOT, 1992*a*) and the 1991 Intermodal Surface Transportation Efficiency Act (U.S. DOT, 1992*b*) have stimulated a growing interest in the development, analysis, and application of environmental transportation policies. Various transportation control measures have also been applied around

*Author for correspondence. Fax: 0014135453858.

the globe in such cities as Athens, Amsterdam, Barcelona, and Munich, as well as, urban road pricing in Singapore (see e.g. Goddard, 1997; *The Economist*, 1997). It is clear that transportation policies to reduce pollution must be coupled with congestion reduction. Indeed, this point has been reiterated by many studies (cf. Lawson, et. al., 1990).

The model that we develop in this paper explicitly considers distinct modes of transportation, each of which is characterized by its own user link travel cost functions and travel disutility functions associated with traveling between the origin/destination pairs. Here we consider a mode of transportation in the most general sense so that it can also include a distinct class of motor vehicle. Moreover, we assume that each mode of transportation emits in its own particular way and we focus on modal permits or licenses along roads or links of the network. The key in the calculation of the mobile source emission estimates is the relationship that the volume of emissions is equal to the product of a composite emission factor times the vehicular activity at the link levels (cf. DeCorla-Souza, *et al.*, 1995).

The model that we develop builds upon the idea of marketable pollution permits proposed by Montgomery, 1972, within the framework of the general multimodal traffic network equilibrium model proposed by Dafermos, 1982. Note that our model, in contrast to the model proposed by Montgomery, 1972, has polluters which are not fixed in location (as were the firms) but, rather, are mobile. Moreover, we explicitly model noncompliant behavior and associate penalties with such behavior. One must recognize that, under any system of pollution control, whether it be marketable pollution permits, emission standards, or Pigouvian taxes, there is bound to be noncompliant behavior on the part of the users of the system. In the case of marketable pollution permits for firms that are fixed in location, Malik, 1990 noted that problems encountered in enforcing market-based pollution control measures are related to the difficulties encountered in the continuous monitoring of pollutants and the absence of well-developed mechanisms to assess penalties for noncompliance. Van Egteren and Weber, 1996, on the other hand, emphasized that limited budgets and prohibitive monitoring costs may make complete enforcement difficult. Moreover, they stated that firms may cheat if the marginal cost of compliance, that is, the cost of purchasing a pollution permit to cover an additional unit of emissions is higher than the marginal penalty assessed for cheating.

We utilize the theory of variational inequalities to study this new model both qualitatively and numerically. We note that, in the context of congested multimodal urban transportation systems, a modeling framework that incorporates a marketable pollution permit system has yet to be developed although discussions along such lines have taken place (cf. Krugman, 1996, Goddard, 1997). Moreover, there has not, heretofore, been a modeling framework that can handle mobile emission permits and noncompliance. Recently, however, Nagurney, Ramanujam, and Dhanda, 1997 introduced a single modal traffic network quilibrium model with emission pollution permits but did not investigate the issue of noncompliance.

The methodology of variational inequalities has been utilized by Nagurney, Thore, and Pan, 1996 in their analysis of spatial market policies with targets on the supply points, demand points, and transportation links. Nagurney and Ramanujam, 1995, 1996, in turn, used variational inequality theory to develop a pricing and command and control system model for congested multimodal urban transportation systems with targeted flows on links, along with penalty functions for failure to comply. In the latter models, however, there was no mapping of vehicle flows to emissions nor was there an emission pollution permit or license system superimposed on the traffic network system. The appeal of the emission permit system is that it provides an option for its users whereby the users have the choice to either purchase and consume their share of emission licenses or, alternatively, to not travel or to use another mode of transportation.

The paper is organized as follows. In Section 2, the multimodal traffic network equilibrium model for mobile emission permits on links is developed, the equilibrium conditions derived for both the compliant and the noncompliant cases, and then formulated as variational inequality problems. Moreover, here we establish that the multimodal equilibrium pattern is independent of the initial allocation of licenses in the compliant case, and that the environmental quality standards are met. This may not hold, as expected, in the noncompliant case. In Section 3, we study the noncompliant model qualitatively using the theory of variational inequalities. We also provide a penalty scheme that guarantees that there will be no overemissions. In Section 4, we propose an algorithm for the computation of the multimodal equilibrium link load, travel demand, marginal cost of emission abatement, emission underflow, emission overflow, license, and license price pattern.

Conditions for convergence are also provided. The algorithm is then applied to compute solutions to several numerical examples in Section 5. Finally, we summarize the results and present our conclusions in Section 6.

## 2. THE MULTIMODAL TRAFFIC NETWORK EQUILIBRIUM MODEL WITH EMISSION POLLUTION PERMITS

In this Section, we develop a multimodal transportation network policy model in which vehicular travel and, thereby, pollution, due to emissions, is restrained by the incorporation of emission pollution licenses or permits. The market-based fee, or equilibrium license price, for these emission permits stimulates the reduction in travel and pollution. We first present the compliant model in Section 2.1 and, subsequently, in Section 2.2, the noncompliant model.

In the model we assume that the government does not discriminate between the modes in that the modes can purchase permits at the same price and that licenses from one mode are transferrable into licenses from another mode. Also, we assume that the government informs the users of the transportation system of the initial license allocations on the links and the total license availability. The licenses are road or link-based since it is congestion at this level that generates the pollution emissions.

We begin with the notation that is common to both models. We consider a transportation network $G = [N, L]$ consisting of the set of nodes $N$ and a set of directed links $L$. We let $a, b$, etc., denote the links and we let $p, q$, etc., denote the paths, which are assumed to be acyclic. We let $i, j$ denote the modes of transportation and assume that there are $m$ modes of transportation contained in the set $M$. Examples of modes include cars, trucks, and public transportation, etc. Assume that there are $J$ origin/destination (O/D) pairs in the network, with a typical O/D pair denoted by $w$ and the set of O/D pairs denoted by $W$.

The flow of mode $i$ on a link $a$ is denoted by $f_a^i$, and the user cost associated with traveling by mode $i$ on link $a$ is denoted by $c_a^i$. We group the link loads into a column vector $f \in R^{mn}$, and the user travel costs into a row vector $c \in R^{mn}$, where $n$ is the number of links in the network.

The user travel cost on a link as experiences by mode $i$ will, in general, depend upon the entire link load pattern, that is

$$c = c(f) \tag{1}$$

where $c$ is a known smooth function.

A user traveling by mode $i$ on path $p$ incurs a user travel cost $C_p^i$, where

$$C_p^i(f) = \sum_{a \in L} c_a^i(f)\delta_{ap} \tag{2}$$

where $\delta_{ap} = 1$, if link $a$ is contained in path $p$, and 0, otherwise. Here we consider the case of elastic travel demands where travelers using a particular mode can forego travel entirely if the cost associated with traveling by that mode is too high or they can switch modes. The travel disutility function associated with traveling by mode $i$ between O/D pair $w$ is denoted by $\lambda_w^i$, whereas the demand for O/D pair and mode $i$ is denoted by $d_w^i$. We assume that, in general, the travel disutilities can depend on the entire vector of travel demands, that is

$$\lambda = \lambda(d) \tag{3}$$

where $\lambda$ and $d$ are, respectively, the $mJ$-dimensional row and column vectors of travel disutilities and travel demands.

The flow on path $p$ by mode $i$ is denoted by $x_p^i$, with the path flows grouped into a column vector $x \in R^{mQ}$, where $Q$ denotes the number of paths in the network.

The following conservation of flow equations must be satisfied by the flows in the network:

$$d_w^i = \sum_{p \in P_w} x_p^i, \forall i, \forall w \tag{4}$$

where $P_w$ denotes the set of paths connecting the O/D pair $w$, and

$$f_a^i = \sum_{p \in P} x_p^i \delta_{ap}, \forall i, \forall a \tag{5}$$

where $P$ denotes the set of paths in the network.

The conservation of flow eqn (4) states that the sum of the path flows of a mode on paths connecting an O/D pair must be equal to the travel demand for that mode and O/D pair. Equation (5), in turn, states that the flow of a mode on a link equals the sum of the path flows of that mode on paths that use that link. We let $K$ denote the set:

$$K \equiv \{(f, d) \mid \text{such that there exists a vector } x \geq 0 \text{ satisfying (4) and (5)}\}.$$

Before we introduce the notation necessary for the modeling of the permit market system we recall the well-known multimodal traffic network equilibrium conditions in the case of elastic travel demands (cf. Beckmann, McGuire, and Winsten, 1956, Dafermos, 1982): For each mode $i \in M$, every O/D pair $w \in W$, and each path $p \in P_w$ :

$$C_p^i(f^*) \begin{cases} = \lambda_w^i(d^*), & \text{if } x_p^{i*} > 0 \\ \geq \lambda_w^i(d^*), & \text{if } x_p^{i*} = 0 \end{cases} \tag{6}$$

that is, only those paths that have minimal and equal travel costs for each mode and O/D pair are utilized and the used path costs must be equal to the travel disutility of that mode. Recall that (cf. Beckmann, McGuire, and Winsten, 1956), in the special case where the Jacobian of the user link travel cost functions $[\frac{\partial c_a^i(f)}{\partial f_b^j} = \frac{\partial c_b^j(f)}{\partial f_a^i}]$, for all modes $i, j$, and all links $a, b$, is symmetric and positive semidefinite, whereas the Jacobian of the travel disutility functions is also symmetric, that is, $[\frac{\partial \lambda_w^i(d)}{\partial d_\omega^j(d)} = \frac{\partial \lambda_\omega^j(d)}{\partial d_w^i(d)}]$, for all modes $i, j$, and all O/D pairs $w, \omega$, and minus this Jacobian is positive semidefinite, then the above multimodal traffic network equilibrium conditions can be reformulated as the solution to the following convex optimization problem:

$$Minimize_{(f,d) \in K} \int_0^f c(x)dx - \int_0^d \lambda(y)dy \tag{7}$$

where the integrals in (7) are line integrals.

We now introduce the notation for the permit system. We let $a^i$ denote the emission factor on link $a$ due to mode $i$ and we let $l_a^i$ denote the number of licenses or permits on link $a$ that allow the travelers by mode $i$ to emit pollutants at a certain rate. We let $l_a^{i0}$ denote the initial allocation of licenses on link $a$ for mode $i$, which is assumed to be nonnegative. We group the licenses into the column vector $l \in R_+^{mn}$. Later, we discuss how the government should allocate the initial licenses in order to comply with the imposed environmental standards. Note that here we consider, for simplicity sake, a single receptor point for the emissions. If one were to model multiple receptor points, then one would need a matrix $H$ akin to the pollution dispersion matrices used in Montgomery, 1972 and Nagurney and Dhanda, 1996. Moreover, we consider that each mode of transportation emits a single pollutant and this pollutant is the same for all modes.

Also, we let $\rho$ denote the price of a license and we let $t_a^i$ denote the marginal cost of emission abatement on link $a$ by mode $i$ and return to this cost later. We group the marginal costs of abatement into the column vector $t \in R^{mn}$. As shall be shown shortly, in the case of compliance, the marginal costs must be nonnegative, whereas in the case of noncompliance they may be negative.

### *2.1. The model with compliance*

In the case of compliance, we must guarantee that each link in the network can not exceed the average rate of emission that it is licensed to produce, where the emission factor times the vehicle activity is equal to the rate of emission (cf. DeCorla-Souza and Kane, 1996, Anderson, *et al.*, 1996, Allen, 1996). Hence, in the case of compliance, we must have that

$$h_a^i f_a^i \leq l_a^i, \forall i, \forall a. \tag{8}$$

Note that here we want to guarantee compliance by modes in the strictest sense to ensure that each mode of transportation does not emit over and above the amount that its license holdings on the links allow it to. One could, in contrast, postulate a constraint of the following form:

$$\sum_{i \in M} h_a^i f_a^i \leq l_a \tag{8a}$$

where $l_a$ would be the licenses available for link $a$, but such an inequality would still hold if, for example, a particular mode exceeds its allowable emissions provided that some other modes emit accordingly less than what they are entitled to.

Furthermore, the total number of required licenses in the transportation system cannot exceed the initial allocation of the licenses on all links of the transportation network system, that is

$$\sum_{i \in M} \sum_{a \in L} l_a^i \leq \sum_{i \in M} \sum_{a \in L} l_a^{i0}. \tag{9}$$

Note that in our model we allow transfers of permits between different modes of transportation and, hence, there will be a single price for a permit or license, regardless of the mode that holds a license. Therefore, we do not allow the government to discriminate between the modes. Indeed, if one did not allow transfers of licenses between the modes then inequality (9) would take the form:

$$\sum_{a \in L} l_a^i \leq \sum_{a \in L} l_a^{i0}, \text{ for each } i \tag{9a}$$

in which case, as we shall shortly see, there would be a distinct price $\rho^{i*}$ in equilibrium for the license of mode $i$.

We now turn to deriving the equilibrium conditions for the model with compliance.

We associate with the constraint (8), for each mode $i$ and for each link $a$, the marginal cost of emission abatement $t_a^i$, which, in equilibrium, must satisfy the following system: For each mode $i \in M$ and link $a \in L$:

$$h_a^i f_a^{i*} \begin{cases} = l_a^{i*}, & \text{if } t_a^{i*} > 0 \\ \leq l_a^{i*}, & \text{if } t_a^{i*} = 0 \end{cases} \tag{10}$$

In other words, if the marginal cost of emission abatement, $t_a^{i*}$, is positive in equilibrium for a mode $i$ and link $a$, then the emissions by that mode on that link are precisely equal to the pollution license holdings of that mode for that link; if the number of licenses held by a mode on a link exceeds the emissions on a link, then, in equilibrium, the marginal cost of abatement is zero. Analogous marginal costs of abatement apply also in the case of permit systems for firms (cf. Montgomery, 1972 Nagurney and Dhanda, 1996). In this model, however, there is no explicit objective function to be optimized (except under certain restrictive symmetry assumptions discussed in Corollary 1) since the traffic network equilibrium problem is a competitive equilibrium problem (see e. g. Nagurney, 1993). The following condition must also be met at equilibrium: For each mode $i \in M$ and link $a \in L$:

$$t_a^{i*} \begin{cases} = \rho^*, & \text{if } l_a^{i*} > 0 \\ \leq \rho^*, & \text{if } l_a^{i*} = 0 \end{cases} \tag{11}$$

Hence, in equilibrium, a positive holding of licenses by a mode for a link implies that the marginal cost of abatement must be equal to the price of the license on a link. However, if the price of the license exceeds the marginal cost of abatement, then the number of licenses for that mode on the link will be zero.

The traveler on a path $p$ by mode $i$ is now subject not only to the user travel cost, given by (2), but also is subject to the payment of the price or cost of his emissions. In particular, we now have that the cost on a link $a$ incurred by mode $i$ is given by: $c^i_a(f) + h^i_a t^i_a$, and, thus, the *generalized* cost on a path $p$ incurred by mode $i$, denoted by $\hat{C}^i_p(f, t)$, is given by: $\hat{C}^i_p(f, t) = C^i_p(f) + \sum_{a\in L} h^i_a t^i_a \delta_{ap}$. Consequently, the traffic network equilibrium conditions (6), in the presence of the emission pollution permit system, take on the form: For each mode $i \in M$, each O/D pair $w \in W$, and each path $p \in P_w$:

$$\hat{C}^i_p(f^*, t^*) = C^i_p(f^*) + \sum_{a\in L} h^i_a t^{i*}_a \delta_{ap} \begin{cases} = \lambda^i_w(d^*), & \text{if} x^{i*}_p > 0 \\ \geq \lambda^i_w(d^*), & \text{if} x^{i*}_p = 0 \end{cases} \tag{12}$$

Indeed, equilibrium conditions (12) state that a traveler by a mode on a path of the network is now subjected to payment of the true cost of his emissions while traveling by mode $i$ on the path $p$. The emission payment for traveling on path $p$ by that mode is equal to the sum over all links that comprise the path $p$ of the marginal cost of emission abatement for that mode times the emission factor of that mode on the links. Thus, the total payment borne by a treveler on path $p$ is $\sum_{a\in L} h^i_a t^{i*}_a$. Later, we will establish that if a mode is used on a path in equilibrium then the marginal costs on all links on that path of mode $i$ are precisely equal to the equilibrium license price.

Note that, in this framework, the transportation authority is responsible for informing the travelers of the license price and the corresponding payments required, as well as the availability of the licenses or permits on the links.

We now state the well-known economic quilibrium conditions that express that, in equilibrium, if a price of a good (which here is the license) is positive, then the market clears for that good; if there is an excess supply of the good, then the price is zero in equilibrium. Mathematically, hence, the equilibrium price, $\rho^*$, of a license must satisfy the following equilibrium condition:

$$\sum_{i\in M} \sum_{a\in L} (l^{i0}_a - l^{i*}_a) \begin{cases} = 0, & \text{if} \rho^* > 0 \\ \geq 0, & \text{if} \rho^* = 0 \end{cases} \tag{13}$$

In (13), one can see that $\sum_{i\in M} \sum_{a\in L} l^{i0}_a$ is the supply of the licenses available in the network, whereas $\sum_{i\in M}\sum_{a\in L} l^{i*}_a$ is the demand for licenses in equilibrium.

Note that, if the government did not allow the transfer of permits between modes, that is, the supply of licenses for each mode was fixed and given by the initial allocation, in which case, instead, the inequalities in (9a) would hold, then (13) would take the form: for each mode $i \in M$:

$$\sum_{a\in L} (l^{i0}_a - l^{i*}_a) \begin{cases} = 0, & \text{if} \rho^{i*} > 0 \\ \geq 0, & \text{if} \rho^{i*} = 0 \end{cases} \tag{13a}$$

In this initial allocation scheme one could still guarantee (as we shall discuss shortly) that the environmental standards are met.

We let $\mathcal{K}$ denote the feasible set such that $\mathcal{K} \equiv K \times R^{2mn+1}_+$, since both the vectors $l$ and $t$ are $mn$-dimensional vectors and $\rho$ is single dimensional.

We are now ready to define the equilibrium state:

*2.1.1. Definition 1 (equilibrium in the case of compliance).* A vector$(f^*, d^*, t^*, l^*, \rho^*) \in \mathcal{K}$ is an equilibrium of the multimodal traffic network equilibrium emission permits market model in the case of compliance if and only if it satisfies the systems of equalities and inequalities (10), (11), (12), and (13).. We now derive the variational inequality formulation of the equilibrium conditions for the model. We then consider a special case.

*2.1.2. Theorem 1 (variational inequality formulation—compliant case).* A vector of link loads, travel demands, marginal costs of emission abatement, licenses, and license price, $(f^*, d^*, t^*, l^*, \rho^*) \in \mathcal{K}$, is an equilibrium of the traffic network equilibrium problem with emission pollution permits if and only if it is a solution to the variational inequality problem:

$$\sum_{i\in M}\sum_{a\in L}(c_a^i(f^*)+h_a^i t_a^{i*})\times(f_a^i-f_a^{i*})-$$
$$\sum_{i\in M}\sum_{w\in W}\lambda_w^i(d^*)\times(d_w^i-d_w^{i*})+\sum_{i\in M}\sum_{a\in L}(l_a^{i*}-h_a^i f_a^{i*})\times(t_a^i-t_a^{i*})+ \tag{14}$$
$$\sum_{i\in M}\sum_{a\in nL}(\rho^*-t_a^{i*})\times(l_a^i-l_a^{i*})+\sum_{i\in M}\sum_{a\in L}(l_a^{i0}-l_a^{i*})\times(\rho-\rho^*)\geq 0,\quad \forall(f,d,t,l,\rho)\in\mathcal{K}.$$

*2.1.3. Proof.* We first establish that a solution to the equilibrium conditions (10), (11), (12), and (13) satisfies the variational inequality problem (14)

Equilibrium conditions (12) imply that, for a fixed mode $i\in M$, a fixed O/D pair $w\in W$, and a fixed path $p\in P_w$

$$((C_p^i(f^*)+\sum_{a\in L}h_a^i t_a^{i*}\delta_{ap})-\lambda_w^i(d^*))\times(x_p^i-x_p^{i*})\geq 0,\forall x_p^i\in R_+. \tag{15}$$

Summing (15) over all paths $p$ and O/D pairs $w$, and all modes $i$, and using (2), (4), and (5) yields:

$$\sum_{i\in M}\sum_{a\in L}(c_a^i(f^*)+h_a^i t_a^{i*})\times(f_a^i-f_a^{i*})-\sum_{i\in M}\sum_{w\in W}\lambda_w^i(d^*)\times(d_w^i-d_w^{i*})\geq 0,\forall(f,d)\in K. \tag{16}$$

From (10) and (11) we have, in turn, that, for a fixed mode $i$, and link $a$:

$$(l_a^{i*}-h_a^i f_a^{i*})\times(t_a^i-t_a^{i*})\geq 0,\forall t_a^i\in R_+ \tag{17}$$

and

$$(\rho^*-t_a^{i*})\times(l_a^i-l_a^{i*})\geq 0,\forall l_a^i\in R_+ \tag{18}$$

Summing now (17) and (18) over all modes $i$, and all links $a$, we obtain

$$\sum_{i\in M}\sum_{a\in L}(l_a^{i*}-h_a^i f_a^{i*})\times(t_a^i-t_a^{i*})+\sum_{i\in M}\sum_{a\in L}(\rho-t_a^{i*})\times(l_a^i-l_a^{i*})\geq 0,\forall(t,l)\in R_+^{2mn}. \tag{19}$$

From equilibrium conditions (13), we can conclude that

$$\sum_{i\in M}\sum_{a\in L}(l_a^{i0}-l_a^{i*})\times(\rho-\rho^*)\geq 0,\forall\rho\in R_+ \tag{20}$$

Summing now (16), (19), and (20) yields variational inequality (14).

We now prove that a solution to the variational inequality problem (14) also satisfies the equilibrium conditions (10), (11), (12), and (13). Let $(f^*,d^*,t^*,l^*,\rho^*)\in\mathcal{K}$ be a solution to (14). Let $f_a^i=f_a^{i*}$, for all $i$ and $a$, $d_w^i=d_w^{i*}$, for all $i$ and $w$, $l_a^i=l_a^{i*}$, for all $i$ and $a$, and $\rho=\rho^*$, and substitute these values in (14). We then obtain

$$\sum_{i\in M}\sum_{a\in L}(l_a^{i*}-h_a^i f_a^{i*})\times(t_a^i-t_a^{i*})\geq 0,\forall t\in R_+^{mn} \tag{21}$$

which implies the equilibrium conditions (10).

Similarly, if we let $f_a^i=f_a^{i*}$, for all $i$ and $a$, $d_w^i=d_w^{i*}$, for all $i$ and $w$, $t_a^i=t_a^{i*}$, for all $i$ and $a$, and $\rho=\rho^*$, and substitute these values in (14), we get

$$\sum_{i\in M}\sum_{a\in L}(\rho^*-t_a^{i*})\times(l_a^i-l_a^{i*})\geq 0,\forall l\in R_+^{mn} \tag{22}$$

which implies the equilibrium conditions (11).

Also, if we let $l^i_a = l^{i*}_a$, $t^i_a = t^{i*}_a$, for all $i$ and $a$, and $\rho = \rho^*$, and substitute these values into (14), we obtain

$$\sum_{i\in M}\sum_{a\in L}(c^i_a(f^*) + h^i_a t^{i*}_a) \times (f^i_a - f^{i*}_a) - \sum_{i\in M}\sum_{w\in W}\lambda^i_w(d^*) \times (d^i_w - d^{i*}_w) \geq 0, \forall (f, d) \in K \quad (23)$$

which implies the equilibrium conditions (12).

Finally, if we let $f^i_a = f^{i*}_a$, for all $i$ and $a$, $d^i_w = d^{i*}_w$, for all $i$ and $w$, $t^i_a = t^{i*}_a$, and $l^i_a = l^{i*}_a$, for all $i$ and $a$, then upon substitution into (14), we obtain

$$\sum_{i\in M}\sum_{a\in L}(l^{i0}_a - l^{i*}_a) \times (\rho - \rho^*) \geq 0, \forall \rho \in R_+ \quad (24)$$

which implies the economic equilibrium conditions (13). This completes the proof of the theorem.

We now put the variational inequality (14) in standard form (cf. Nagurney, 1993). We define column vector $X \equiv (f, d, t, l, \rho) \in \mathcal{K}$ and the row vector $F(X)$, where

$$F(X) \equiv (C(X), \lambda(X), T(X), L(X), P(X)).$$

$C(X)$, $L(X)$, $T(X)$ are each $mn$-dimensional vectors with component $(i, a)$ given, respectively, as follows:

$$C^i_a(X) : c^i_a(f) + h^i_a t^i_a$$

$$L^i_a(X) : \rho - t^i_a$$

$$T^i_a(X) : l^i_a - h^i_a f^i_a.$$

$\lambda(X)$ is the $mJ$-dimensional vector with component $(i, w)$ given by

$$\lambda^i_w(X) : -\lambda^i_w(d)$$

whereas $P(X)$ is the one-dimensional vector with the single component

$$P(X) : \sum_{i\in M}\sum_{a\in L}(l^{i0}_a - l^i_a).$$

Thus, variational inequality (14) can now be expressed as

$$\langle F(X^*), X - X^*\rangle \geq 0, \forall X \in \mathcal{K}.$$

In the special case where there is only a single mode of transportation the above model collapses to the model with compliance developed in Nagurney, Ramanujam, and Dhanda (1997). We now establish in the following corollary that, in the case of *integrable* link user travel cost functions and disutility functions, the solution to the above transportation market policy model can be obtained using standard mathematical programming. This can be seen as the analogue to the classic optimization reformulation of the multimodal traffic network equilibrium conditions in (7).

*2.1.4. Corollary 1 (optimization reformulation in a special case).* Assume that the user link cost functions (1) and the travel disutility functions (3) are integrable, that is, that their Jacobian matrices are symmetric. Assume also that the Jacobian matrix of the user link travel cost functions and minus the Jacobian of the travel disutility functions are positive semidefinite. Then the equilibrium conditions (10)–(13) have an equivalent convex optimization formulation given by:

$$\text{Minimize}_{(f,d)\in K} \int_0^f c(x)dx - \int_0^d \lambda(y)dy \tag{25}$$

subject to:

$$h^i_a f^i_a \leqslant l^i_a, \forall i, \forall a \tag{26}$$

$$\sum_{i\in M}\sum_{a\in L} l^i_a \leqslant \sum_{i\in M}\sum_{a\in L} l^{i0}_a \tag{27}$$

$$l^i_a, f^i_a \geqslant 0, \forall i, \forall a, \tag{28}$$

where the integrals in (25) are line integrals.

*2.1.5. Proof.* Let $t^{i*}_a$ be the Lagrange multiplier associated with the $(i, a)$-th constraint in (26) and let $\rho^*$ be the Lagrange multiplier associated with (27). Then it follows that the Kuhn-Tucker conditions of the above optimization problem coincide with the equilibrium conditions (10), (11), (12), and (13). We now prove that if a mode on a path is used in equilibrium then the marginal costs on all links on that path incurred by that mode are equal to the equilibrium license price.

*2.1.6. Corollary 2.* If $x^{i*}_p > 0$, then $t^{i*}_a = \rho^*$, for all links $a \in p$.

*2.1.7. Proof.* If $x^{i*}_p > 0$, then from (5) it follows that the link load $f^{i*}_a > 0$, $\forall a \in p$, and, hence, from (10) that $l^{i*}_a > 0$, $\forall a \in p$. From (11) we then have that $t^{i*}_a = \rho^*$, for all $a \in p$. The proof is complete. Note that in (12), the general cost on a path that is used in equilibrium is, in this case:

$$C^i_p(f^*, t^*) + \rho^* \sum_{a\in L} h^i_a \delta_{ap}.$$

now turn to determining whether the multimodal equilibrium pattern is independent of the initial allocation of the licenses on the links and how to ensure that the environmental emission standards imposed by the governing body are met in equilibrium. The question as to whether the initial allocation of licenses affects the equilibrium pattern is answered in the following corollary.

*2.1.8. Corollary 3. (equilibrium independence of initial license allocation)*
$l^{i0}_a \geq 0$ for all $i$ and $a$, and $\sum_{i\in M}\sum_{a\in L} l^{i0}_a = \bar{Q}$, with $\bar{Q}$ fixed and positive, then the equilibrium pattern $(f^*, d^*, t^*, l^*, \rho^*)$ is independent of the initial allocation.

*2.1.9. Proof.* The terms in the variational inequality (14) are either independent of $l^{i0}_a$ or depend only on the sum, $\sum_{i\in M}\sum_{a\in L} l^{i0}_a$. The conclusion follows. In the following proposition, we show that the environmental standards are met by the multimodal equilibrium pattern, provided that the sum of the initial allocation of licenses is equal to the imposed environmental standard given by $\bar{Q}$.

*2.1.10. Proposition 1 (achievement of environmental standards)* If $\sum_{i\in M}\sum_{a\in L} l^{i0}_a = \bar{Q}$, the equilibrium vector achieves the environmental quality standard, $\bar{Q}$.

*2.1.11. Proof.* We have from equilibrium conditions (10) and (13) that

$$\sum_{i\in M}\sum_{a\in L} h^i_a f^{i*}_a \leq \sum_{i\in M}\sum_{a\in L} l^{i*}_a \leq \sum_{i\in M}\sum_{a\in L} l^{i0}_a = \bar{Q}$$

Hence, the environmental standards are met by the equilibrium pattern. One can also infer from the above arguments that if the government does not allow transfers of the initial license allocations between the modes of transportation in which case (13a) would replace the equilibrium conditions governing the license prices then the equilibrium pattern can still guarantee that the environmental standards are met under the same conditions as above.

*2.2. The model with noncompliance*

We now present the noncompliant version of the multimodal model and compare it with and contrast it to the compliant model just described. Recently, Nagurney and Dhanda (1997) modeled noncompliant firms in the presence of a system of marketable pollution permits, but the firms were fixed in location, in contrast to users of a transportation system

We now discuss the variables used in the modeling of noncompliance. We let $\delta_a^{i+}$ denote the possible overemission on link $a$ by mode $i$, over and above the license holding. We let $\delta_a^{i-}$ denote the possible underemission on link $a$ by mode $i$. In other words, mode $i$ may not pollute on link $a$ up to what is permitted by the license holdings for that link and mode. We group the overemissions into the column vector $\delta^+ \in R_+^{mn}$ and the underemissions into the column vector $\delta^- \in R_+^{mn}$.

We assume, as given, the total penalty function $\mu_a^i$ for each link $a$ and mode $i$, which is associated with overemitting on link $a$ by mode $i$ and we group the penalties into the vector $\mu \in R^{mn}$. We assume that the penalty for each mode and link is a function of the overemission by that mode on that link, that is

$$\mu_a^i = \mu_a^i(\delta_a^{i+}), \tag{29}$$

where the marginal penalty cost functions, $\frac{\partial \mu_a^i}{\partial \delta_a^{i+}}$, is assumed to be positive for each mode and link.

Note that now, instead of inequality (8), which had to hold in the compliant case, we have the following constraints, which state that the total amount of emissions on each link $a$ by mode $i$ is equal to the amount emitted by that mode on that link plus the possible underflow minus the possible overflow of emissions on that link by that mode, that is

$$h_a^i f_a^i + \delta_a^{i-} - \delta_a^{i+} = l_a^i, \forall i, \forall a. \tag{30}$$

Inequality (9), however, is still relevant in the noncompliant case since the government does not issue new permits following the initial allocation. Hence, the total number of required licenses in the transportation system cannot exceed the initial allocation of the licenses for all modes on all links of the transportation network system, that is

$$\sum_{i\in M}\sum_{a\in L} l_a^i \le \sum_{i\in M}\sum_{a\in L} l_a^{i0}. \tag{31}$$

We are now ready to derive the equilibrium conditions governing the multimodal traffic network equilibrium model wih pollution permits in the case of noncompliance.

We associate with the constraint (30) (as we did with constraint (8) in the compliant case), for each mode $i \in M$ and for each link $a \in L$, the marginal cost of emission abatement $t_a^i$, which, in equilibrium, can now be positive, zero, or negative, and is associated with the following system: For each mode $i \in M$ and link $a \in L$:

$$l_a^{i*} - h_a^i f_a^{i*} - \delta_a^{i-*} + \delta_a^{i+*} = 0. \tag{32}$$

We now state the equilibrium conditions for the overflow and underflow emission patterns: For the underemission pattern, we must have that, for each mode $i \in M$ and for each link $a \in L$:

$$t_a^{i*}\begin{cases} = 0, & \text{if } \delta_a^{i-*} > 0 \\ \ge 0, & \text{if } \delta_a^{i-*} = 0 \end{cases} \tag{33}$$

that is, if there is an underflow of emissions on a link by a mode, then the marginal cost of abatement is zero; if there is no underflow, then the marginal cost of abatement of that mode on that link may be positive or zero. For the overemission pattern, in turn, we must have that, for each mode $i \in M$ and each link $a \in L$:

$$\frac{\partial\mu_a^i(\delta_a^{i+*})}{\partial\delta_a^{i+}} - t_a^{i*}\begin{cases} = 0, & \text{if } \delta_a^{i+*} > 0 \\ \geq 0, & \text{if } \delta_a^{i+*} = 0 \end{cases} \tag{34}$$

that is, the marginal penalty cost of overemission is equal to the marginal cost of emission abatement. The system of equalities and inequalities in (11), is still relevant in this case, that is, the following condition must also be met at equilibrium: For each mode $i \in M$ and link $a \in L$:

$$t_a^{i*}\begin{cases} = \rho^*, & \text{if } l_a^{i*} > 0 \\ \leq \rho^*, & \text{if } l_a^{i*} = 0 \end{cases} \tag{35}$$

Consequently, in equilibrium, a positive holding of licenses by a mode for a link implies that the marginal cost of abatement must be equal to the price of the license on a link. However, if the price of the license exceeds the marginal cost of abatement, then the number of licenses held by that mode for that link will be zero.

The system of equalities and inequalities (12) governing the incurred travel costs and disutilities in equilibrium still holds, that is, we must have that for each mode $i \in M$, every O/D pair $w \in W$, and path $p \in P_w$:

$$\hat{C}_p^i(f^*, t^*) = C_p^i(f^*) + \sum_{a\in L} h_a^i t_a^{i*}\delta_{ap}\begin{cases} = \lambda_w^i(d^*), & \text{if } x_p^{i*} > 0 \\ \geq \lambda_w^i(d^*), & \text{if } x_p^{i*} = 0 \end{cases} \tag{36}$$

Finally, (13) also holds in this case, that is, the equilibrium price, $\rho^*$, of a license must satisfy the following equilibrium condition:

$$\sum_{i\in M}\sum_{a\in L}(l_a^{i0} - l_a^{i*})\begin{cases} = 0, & \text{if } \rho^* > 0 \\ \geq 0, & \text{if } \rho^* = 0 \end{cases} \tag{37}$$

We let $\mathcal{K}^1$ denote the feasible set such that $\mathcal{K}^1 \equiv K \times R_+^{3mn} \times R^{mn} \times R_+$, since the vectors $l$, $\delta^+$, and $\delta^-$ are $mn$-dimensional nonnegative vectors, $t$ also an $mn$-dimensional, and $\rho$ is single dimensional.

We are now ready to define the equilibrium state:

*2.3. Definition 2 (equilibrium–noncompliant case)*

A vector $(f^*, d^*, t^*, \delta^{-*}, \delta^{+*}, l^*, \rho^*) \in \mathcal{K}^1$ is an equilibrium of the multimodal traffic network equilibrium emission permits market model in the case of noncompliance if and only if it satisfies the systems of equalities and inequalities (32)–(37).

We now derive the variational inequality formulation of the equilibrium conditions for the model with noncompliance. As we did in the case of compliance, we then give a special case.

*2.4. Theorem 2 (variational inequality formulation–noncompliant case)*

A vector of link loads, travel demands, marginal costs of emission abatement, emission underflows, and overflows, licenses, and license price, $(f^*, d^*, t^*, \delta^{-*}, \delta^{-*}, l^*, \rho^*) \in \mathcal{K}^1$, is an equilibrium of the multimodal traffic network equilibrium problem with emission pollution permits in the case of noncompliance if and only if it is a solution to the variational inequality problem:

$$\sum_{i\in M}\sum_{a\in L}(c_a^i(f^*) + h_a^i t_a^{i*}) \times (f_a^i - f_a^{i*}) - \sum_{i\in M}\sum_{w\in W}\lambda_w^i(d^*) \times (d_w^i - d_w^{i*}) \tag{38}$$

$$+ \sum_{i\in M}\sum_{a\in L}(l_a^{i*} - h_a^i f_a^{i*} - \delta_a^{i-*} + \delta_a^{i+*}) \times (t_a^i - t_a^{i*})$$

$$+ \sum_{i\in M}\sum_{a\in L} t_a^{i*} \times (\delta_a^{i-} - \delta_a^{i-*}) + \sum_{i\in M}\sum_{a\in L}(\partial\mu_a^i\frac{(\delta_a^{i+*})}{\partial\delta_a^{i+}} - t_a^{i*}) \times (\delta_a^{i+} - \delta_a^{i+*})$$

$$+ \sum_{i\in M}\sum_{a\in L}(\rho^* - t_a^{i*}) \times (l_a^i - l_a^{i*}) + \sum_{i\in M}\sum_{a\in L}(l_a^{i0} - l_a^{i*}) \times (\rho - \rho^*) \geq 0, \forall (f, d, t, \delta^-, \delta^+, l, \rho) \in \mathcal{K}^1.$$

*2.5. Proof.*

We first establish that a solution to the equilibrium conditions (32)–(37) satisfies the variational inequality problem (38).

Immediately from (36) we may write (as in the proof of Theorem 1):

$$\sum_{i\in M}\sum_{a\in L}(c_a^i(f^*)+h_a^i t_a^{i*})\times(f_a^i-f_a^{i*})-\sum_{i\in M}\sum_{w\in W}\lambda_w^i(d^*)\times(d_w^i-d_w^{i*})\geq 0,\forall(f,d)\in K. \tag{39}$$

From (32) we have that, for a fixed mode $i$ and a fixed link $a$:

$$(l_a^{i*}-h_a^i f_a^{i*}-\delta_a^{i-*}+\delta_a^{i+*})\times(t_a^i-t_a^{i*})\geq 0,\forall t_a^i\in R. \tag{40}$$

Summing now (40) over all modes $i$ and all links $a$ yields:

$$\sum_{i\in M}\sum_{a\in L}(l_a^{i*}-h_a^i f_a^{i*}-\delta_a^{i*}+\delta_a^{i+*})\times(t_a^i-t_a^{i*})\geq 0,\forall t\in R^{mn}. \tag{41}$$

From (33) and (34) we have, in turn, that, for a fixed mode $i$ and for a fixed link $a$:

$$(t_a^{i*})\times(\delta_a^{i-}-\delta_a^{i-*})\geq 0,\forall\delta_a^{i-}\in R_+, \tag{42}$$

and

$$(\frac{\partial\mu_a^i(\delta_a^{i+*})}{\partial\delta_a^{i+}}-t_a^{i*})\times(\delta_a^{i+}-\delta_a^{i+*})\geq 0,\forall\delta_a^{i+}\in R_+. \tag{43}$$

Summming now (42) and (43) over all modes $i$ and all links $a$, and adding the resultants yields:

$$\sum_{i\in M}\sum_{a\in L}(t_a^{i*})\times(\delta_a^{i-}-\delta_a^{i-*})+\sum_{i\in M}\sum_{a\in L}(\frac{\partial\mu_a^i(\delta_a^{i+*})}{\partial\delta_a^{i+}}-t_a^{i*})\times(\delta_a^{i+}-\delta_a^{i+*})\geq 0,\forall(\delta^-,\delta^+)\in R_+^{2mn}. \tag{44}$$

Also, we have, from the proof of Theorem 1, that equilibrium conditions (35) imply that:

$$\sum_{i\in M}\sum_{a\in L}(\rho-t_a^{i*})\times(l_a^i-l_a^{i*})\geq 0,\forall l\in R_+^{mn}. \tag{45}$$

whereas equilibrium conditions (37) imply that

$$\sum_{i\in M}\sum_{a\in L}(l_a^{i0}-l_a^{i*})\times(\rho-\rho^*)\geq 0,\forall\rho\in R_+. \tag{46}$$

Summing (39), (41), (44), (45), and (46) yields variational inequality (38). We now prove that a solution to the variational inequality problem (38) also satisfies the equilibrium conditions (32)–(37).

Let $(f^*,d^*,t^*,\delta^{-*},\delta^{+*},l^*,\rho^*)\in\mathcal{K}^1$ be a solution to (38). Let $f_a^i=f_a^{i*}$, for all $i$ and $a$, $d_w^i=d_w^{i*}$, for all $i$ and $w$, $\delta_a^{+i}=\delta_a^{+i*}$, $\delta_a^{-i}=\delta_a^{-i*}$, and $l_a^i=l_a^{i*}$, for all $i$ and $a$, and $\rho=\rho^*$, and substitute these values in (38). We then obtain

$$\sum_{i\in M}\sum_{a\in L}(l_a^{i*}-h_a^i f_a^{i*}-\delta_a^{i-*}+\delta_a^{i+*})\times(t_a^i-t_a^{i*})\geqslant 0,\forall t\in R^{mn}, \tag{47}$$

which implies the equilibrium conditions (32).

Also, if we let $l_a^i=l_a^{i*},t_a^i=t_a^{i*},\delta_a^{i-}=\delta_a^{i-*},\delta_a^{i+}=\delta_a^{i+*}$, for all $i$ and $a$, and $\rho=\rho^*$, and substitute these values into (38), we obtain

$$\sum_{i\in M}\sum_{a\in L}(c_a^i(f^*)+h_a^i t_a^{i*})\times(f_a^i-f_a^{i*})-\sum_{i\in M}\sum_{w\in W}\lambda_w^i(d^*)\times(d_w^i-d_w^{i*})\geq 0,\forall(f,d)\in K, \tag{48}$$

which implies the equilibrium conditions (13).

The equilibrium conditions (33) and (34) can be obtained in a similar manner using the appropriate substitutions as done above into variational inequality (38).

Equilibrium conditions (35) and (37) can also be deduced using similar arguments as in the proof of Theorem 1. The proof is complete.

We now put the variational inequality (38) in standard form We define the column vector $\hat{X} \equiv (f, d, t, \delta^-, \delta^+, l, \rho) \in \mathcal{K}^1$ and the row vector $\hat{F}(\hat{X})$, where

$$\hat{F}(\hat{X}) \equiv (\hat{C}(\hat{X}), \hat{\lambda}(\hat{X}), \hat{T}(\hat{X}), \hat{\delta}^-(\hat{X}), \hat{\delta}^+(\hat{X}), \hat{L}(\hat{X}), \hat{P}(\hat{X}))$$

and $\hat{C}(\hat{X})$, $\hat{L}(\hat{X})$, $\hat{T}(\hat{X})$, $\hat{\delta}^-$, $\hat{\delta}^+$, are $mn$-dimensional vectors with component $(i, a)$ given, respectively, as follows:

$$\hat{C}^i_a(\hat{X}) : c^i_a(f) + h^i_a t^i_a$$

$$\hat{L}^i_a(\hat{X}) : \rho - t^i_a$$

$$\hat{T}^i_a(\hat{X}) : l^i_a - h^i_a f^i_a - \delta^{i-}_a + \delta^{i+}_a$$

$$\hat{\delta}^{i+}_a(\hat{X}) : \frac{\partial \mu^i_a(\delta^{i+}_a)}{\partial \delta^{i+}_a} - t^i_a$$

$$\hat{\delta}^{i-}_a(\hat{X}) : t^i_a$$

$\hat{\lambda}(\hat{X})$ is the $mJ$-dimensional vector with component $(i, w)$ given by

$$\hat{\lambda}^i_w(\hat{X}) : -\lambda^i_w(d)$$

whereas $\hat{P}(\hat{X})$ is the one-dimensional vector with the single component

$$\hat{P}(\hat{X}) : \sum_{a \in L} (l^{i0}_a - l^i_a).$$

Thus, variational inequality (38) can now be expressed as

$$\langle \hat{F}(\hat{X}^*), X - \hat{X}^* \rangle \geq 0, \forall X \in \mathcal{K}^1.$$

We now establish in the following corollary that, in the case of *integrable* link user travel cost functions and disutility functions, the solution to the above transportation market policy model can be obtained using standard mathematical programming.

*2.6. Corollary 4 (optimization reformulation in a special case)*

Assume that the user link cost functions (1), the travel disutility functions (3), and the penalty functions are integrable, that is, that their Jacobian matrices are symmetric. Assume also that the Jacobian matrices of the link travel cost functions and the penalty cost functions, and minus the travel disutility functions are positive semidefinite. Then the equilibrium conditions (32)–(37) have an equivalent convex optimization formulation, given by:

$$\text{Minimize}_{(f,d) \in K} \int_0^f c(x)dx - \int_0^d \lambda(y)dy + \sum_{i \in M} \sum_{a \in L} \mu^i_a(\delta^{i+}_a) \tag{49}$$

subject to:

$$l^i_a - h^i_a f^i_a - \delta^{i-}_a + \delta^{i+}_a = 0, \forall i, \forall a \tag{50}$$

$$\sum_{i \in M} \sum_{a \in L} l^i_a \leqslant \sum_{i \in M} \sum_{a \in L} l^{i0}_a \tag{51}$$

$$l^i_a \geqslant 0, \delta^{i-}_a \geqslant 0, \delta^{i+}_a \geqslant 0, \forall i, \forall a, \tag{52}$$

where the integrals in (49) are line integrals.

*2.7. Proof.*

Let $t^{i*}_a$ be the Lagrange multiplier associated with the $(i, a)$-th constraint in (50) and let $\rho^*$ be the Lagrange multiplier associated with (51). Then it follows that the Kuhn-Tucker conditions of the above optimization problem coincide with the equilibrium conditions (32)–(37).

In the subsequent corollary we establish that, in equilibrium, one cannot have both a positive overemission and a positive underemission by a mode on a link, that is, $\delta^{i+*}_a \times \delta^{i-*}_a = 0$.

*2.8. Corollary 5.*

In equilibrium, $\delta^{i+*}_a \times \delta^{i-*}_a = 0$, for all modes $i$ and all links $a$.

*2.9. Proof.*

Assume, on the contrary, that for a particular mode $i$ and a particular link $a$, $\delta^{i+*}_a > 0$ and $\delta^{i-*}_a > 0$.

Consider now variational inequality (38) and make the following substitutions: $f^i_a = f^{i*}_a$, $t^i_a = t^{i*}_a$, $l^i_a = l^{i*}_a$, for all modes $i$ and all links $a$, $d^i_w = d^{i*}_w$, for all modes $i$ and O/D pairs $w$, and $\rho = \rho^*$. Also, since is it assumed that both $\delta^{i+*}_a$ and $\delta^{i-*}_a$ are positive, we can set: $\delta^{i-}_a = \delta^{i-*}_a - \epsilon$ and $\delta^{i+}_a = \delta^{i+*}_a - \epsilon$ for some $\epsilon > 0$, and sufficiently small, and $\delta^{j-}_b = \delta^{j-*}_b$ and $\delta^{j+}_b = \delta^{j+*}_b$, for all other modes and links $j, b \neq i, a$. Note that with such values, (32) still holds. Making now the above substitutions into (38) yields:

$$t^{i*}_a \times (-\epsilon) + \left(\frac{\partial \mu^i_a(\delta^{i+*}_a)}{\partial \delta^{i+}_a} - t^{i*}_a\right) \times (-\epsilon) \geq 0,$$

which implies that

$$\frac{\partial \mu^i_a(\delta^{i+*}_a)}{\partial \delta^{i*}_a} \times (-\epsilon) \geq 0, \tag{53}$$

but, under the assumption that the marginal penalty cost is positive for all links and modes, (53) implies that $\epsilon \leq 0$, which is a contradiction. Hence, it follows that $\delta^{i-*}_a \times \delta^{i+*}_a = 0$. The proof is complete.

Before providing a mechanism for the assignment of the penalty costs for overemission that guarantees that there will be no overemissions by the modes on the links, we provide an alternative formulation of the variational inequality (14) governing the model with compliance. We then utilize this *alternative* variational inequality formulation in the proof of the subsequent theorem. We first note that we can rewrite inequality (8) in the model with compliance as the following equality:

$$h^i_a f^i_a + \delta^{i-}_a = l^i_a, \forall i, \forall a, \tag{54}$$

where $\delta^{i'}_a \geqslant 0$, in which case equilibrium condition (10) becomes:

$$l^{i'}_a - h^i_a f^{i'}_a - \delta^{i-'}_a = 0, \forall i, \forall a, \tag{55}$$

with the Lagrange multiplier $t^{i'}_a$ associated with the $(i, a)$-th such constraint, and where we use ce9′cf9 now to denote the equilibrium pattern. In addition, we must now have that the underflow pattern must satisfy, for all modes $i \in M$ and all links $a \in L$:

$$t_a^{i'} \begin{cases} = 0, & \text{if } \delta_a^{i-'} > 0 \\ \geq 0, & \text{if } \delta_a^{i-'} = 0 \end{cases} \tag{56}$$

that is, if there is an underflow of emissions on a link by a mode, then the marginal cost of abatement is zero; if there is no underflow, then the marginal cost of abatement of that mode on that link may be positive or zero. Note that this condition is identical to condition (33). Hence, instead of (10), we now have (55) and (56) as a replacement.

Conditions (11), (12), and (13) remain the same, however, and, hence, equivalently, an equilibrium pattern $(f', d', t', \delta^-, l', \rho') \in K \times R^{mn} \times R_+^{2mn} \times R_+$ for the model with compliance must satisfy conditions (11)–(13), (55), and (56).

*2.10. Lemma 1.*

The equilibrium conditions governing the model with compliance (with equality license constraints), that is, (11), (12), and (13), with (10) substituted by (55) and (56), can be formulated as the alternative variational inequality: Determine $(f', d', t', \delta^-, l', \rho') \in K \times R^{mn} \times R_+^{2mn} \times R_+$ satisfying:

$$\sum_{i\in M}\sum_{a\in L} c_a^i(f') + h_a^i t_a^{i'}) \times (f_a^i - f_a^{i'}) - \sum_{i\in M}\sum_{w\in W} \lambda_w^i(d') \times (d_w^i - d_w^{i'})$$

$$+ \sum_{i\in M}\sum_{a\in L} (l_a^{i'} - h_a^i f_a^{i'} - \delta_a^{i-'}) \times (t_a^i - t_a^{i'})$$

$$+ \sum_{i\in M}\sum_{a\in L} t_a^{i'} \times (\delta_a^{i-} - \delta_a^{i-'}) + \sum_{i\in M}\sum_{a\in L} (\rho' - t_a^{i'}) \times (l_a^i - l_a^{i'}) + \sum_{i\in M}\sum_{a\in L} (l_a^{i0} - l_a^{i'}) \times (\rho - \rho') \geq 0,$$

$$\forall (f, d, t, \delta^-, l, \rho) \in K \times R^{mn} \times R_+^{2mn} \times R_+. \tag{57}$$

*2.11. Proof.*

The variational inequality follows using similar arguments as in the proofs of Theorem 1 and 2.

In the next theorem we establish that if the marginal penalty costs are set precisely equal to the equilibrium price in the model with compliance then there can be no overemissions, that is, there is no noncompliance.

*2.12. Theorem 3.*

Assume that the marginal penalty costs are set as follows: $\frac{\partial \mu_a^i}{\partial \delta_a^{i+}} = \rho'$, for all modes $i \in M$ and all links $a \in L$, where $\rho'$ is the equilibrium price in the model with compliance and satisfies variational inequality (57). Then the solution to (57) coincides with the solution to variational inequality (38) governing the model with noncompliance and, moreover, $\delta_a^{i+*} = 0$, for all modes $i$ and all links $a$.

*2.13. Proof.*

Assume that the solution $(f', d', t', \delta^-, l', \rho') \in K \times R^{mn} \times R_+^{2mn} \times R_+$ with $\delta^+ = 0$, does not satisfy variational inequality (38). Then, for some $(f, d, t, \delta^-, \delta^+, l, \rho) \in \mathcal{K}^1$ we must have that

$$\sum_{i\in M}\sum_{a\in L} c_a^i(f') + h_a^i t_a^{i'}) \times (f_a^i - f_a^{i'}) - \sum_{i\in M}\sum_{w\in W} \lambda_w^i(d') \times (d_w^i - d_w^{i'})$$

$$+ \sum_{i\in M}\sum_{a\in L} (l_a^{i'} - h_a^i f_a^{i'} - \delta_a^{i-'}) \times (t_a^i - t_a^{i'})$$

$$+ \sum_{i\in M}\sum_{a\in L} t_a^{i'} \times (\delta_a^{i-} - \delta_a^{i-'}) + \sum_{i\in M}\sum_{a\in L} (\rho' - t_a^{i'}) \times (\delta_a^{i+} - \delta_a^{i+'}) \tag{58}$$

$$+\sum_{i\in M}\sum_{a\in L}(\rho' - t_a^{i'}) \times (l_a^i - l_a^{i'}) + \sum_{i\in M}\sum_{a\in L}(l_a^{i0} - l_a^{i'}) \times (\rho - \rho') < 0, \tag{58}$$

which implies that

$$\sum_{i\in M}\sum_{a\in L}(c_a^i(f') + h_a^i t_a^{i'}) \times (f_a^i - f_a^{i'}) - \sum_{i\in M}\sum_{a\in L}\lambda_w^i(d') \times (d_w^i - d_w^{i'})$$

$$+\sum_{i\in M}\sum_{a\in L}(l_a^{i'} - h_a^i f_a^{i'} - \delta_a^{i-}) \times (t_a^i - t_a^{i'})$$

$$+\sum_{i\in M}\sum_{a\in L} t_a^{i'}(\delta_a^{i-} - \delta_a^{i-'}) + \sum_{i\in M}\sum_{a\in L}(\rho' - t_a^{i'}) \times (l_a^{i} - l_a^{i'}) + \sum_{i\in M}\sum_{a\in L}(l_a^{i0} - l_a^{i'}) \times (\rho - \rho')$$

$$< -\sum_{i\in M}\sum_{a\in L}(\rho' - t_a^{i'})\delta_a^{i+}, \tag{59}$$

since $\delta_a^{i+} = 0$ for all $a$ and $i$, and (11) must hold.

But, under the assumption that $(f', d', t', \delta^{+'}, l', \rho')$ is a solution to (57), the left-hand side of (59) must be greater than or equal to zero and, hence, we have obtained a contradiction. The conclusion in the theorem must, therefore, hold true.

## 3. QUALITATIVE PROPERTIES

In this Section, we present qualitative properties of the equilibrium pattern for the model without compliance. In particular, we provide a uniqueness result and also establish properties of the function $\hat{F}(\hat{X})$ under which convergence of the algorithm in Section 4 is guaranteed.

### *3.1. Lemma 2 (monotonicity)*

Assume that the user link cost functions are monotone increasing in the link loads, the marginal penalty functions are mononotone increasing in the overflows, and that the travel disutility functions are monotone decreasing in the travel demands, that is, for every $(f^1, d^1, \delta^{+1}), (f^2, d^2, \delta^{+2}) \in K \times R_+^{mn}$, we have that

$$\langle c(f^1) - c(f^2), f^1 - f^2\rangle - \langle\lambda(d^1) - \lambda(d^2), d^1 - d^2\rangle \geqslant 0$$

$$+\langle\frac{\partial\mu(\delta^{+1})}{\partial\delta^{+}} - \frac{\partial\mu(\delta^{+2})}{\partial\delta^{+}}, \delta^{+1} - \delta^{+2}\rangle \geq 0, \forall(f^1, d^1, \delta^{+1}), (f^2, d^2, \delta^{+2}) \in K \times R_+^{mn}. \tag{60}$$

Then $\hat{F}(\hat{X})$ is monotone, that is

$$\langle\hat{F}(\hat{X}^1) - \hat{F}(\hat{X}^2), \hat{X}^1 - \hat{X}^2\rangle \geq 0, \forall\hat{X}^1, \hat{X}^2 \in \mathcal{K}^1. \tag{61}$$

*3.1.1. Proof.* From the definition of $\hat{F}(\hat{X})$ for the model, the left-hand side term of inequality (61) is given by

$$\sum_{i\in M}\sum_{a\in L}[c_a^i(f^1) - c_a^i(f^2)] \times [f_a^{i1} - f_a^{i2}] - \sum_{i\in M}\sum_{w\in W}[\lambda_w^i(d^1) - \lambda_w^1(d^2)] \times [d_w^{i1} - d_w^{i2}]$$

$$+\sum_{i\in M}\sum_{a\in L}[t_a^{i1}h_a^i - t_a^{i2}h_a^i] \times [f_a^{i1} - f_a^{i2}] + \sum_{i\in M}\sum_{a\in L}[(l_a^{i1} - h_a^i f^{i1} + \delta_a^{i+1} - \delta_a^{i-1})$$

$$-(l_a^{i2} - h_a^i f_a^{i2} + \delta_a^{i+1} - \delta_a^{i-1})] \times [t_a^{i1} - t_a^{i2}]$$

$$+\sum_{a\in L}[(l_a^{i0}-l_a^{i1})-(l_a^{i0}-l_a^{i2})]\times[\rho^1-\rho^2]+\sum_{i\in M}\sum_{a\in L}[(\rho^1-t_a^{i1})-(\rho^2-t_a^{i2})]\times[l_a^{i1}-l_a^{i2}]+$$

$$\sum_{i\in M}\sum_{a\in L}[(\frac{\partial\mu_a^i(\delta^{+1})}{\partial\delta_a^{i+}}-t_a^{i1})-(\frac{\partial\mu_a^i(\delta^{+2})}{\partial\delta_a^{i+}}-t_a^{i2})]\times[\delta_a^{i+1}-\delta_a^{i+2}]+\sum_{i\in M}\sum_{a\in L}[t_a^{i1}-t_a^{i2}]\times[\delta_a^{i-1}-\delta_a^{i-2}]. \quad (62)$$

After rearranging the terms and simplifying, (62) reduces to

$$\sum_{i\in M}\sum_{a\in L}[c_a^i(f^1)-c_a^i(f^2)]\times[f_a^{i1}-f_a^{i2}]-\sum_{i\in M}\sum_{w\in W}[\lambda_w^i(d^1)-\lambda_w^i(d^2)]\times[d_w^{i1}-d_w^{i2}]$$

$$\sum_{i\in M}\sum_{a\in L}[\frac{\partial\mu_a^i(\delta_a^{i+1})}{\partial\delta_a^{i+}}-\frac{\partial\mu_a^i(\delta_a^{i+2})}{\partial\delta_a^{i+}}]\times[\delta_a^{i+1}-\delta_a^{i+2}], \quad (63)$$

which, by the assumption of monotonicity, is greater than or equal to zero. This proves that $\hat{F}(\hat{X})$ is monotone.

A uniqueness result is presented in the subsequent theorem.

*3.1.2. Theorem 4 (uniqueness).* Assume that the user link cost functions, the marginal penalty cost functions, and minus the travel disutility functions are strictly monotone, in $f$, $\delta^+$, and $d$, respectively, that is

$$\langle c(f^1)-c(f^2),f^1-f^2\rangle\langle\lambda(d^1)-\lambda(d^2),d^1-d^2\rangle+\langle\frac{\partial\mu(\delta^{+1})}{\partial\delta^+}-\frac{\partial\mu(\delta^{+2})}{\partial\delta^+},\delta^{+1}-\delta^{+2}\rangle>0$$

$$\forall(f^1,d^1,\delta^{+1}),(f^2,d^2,\delta^{+2})\in K\times R_+^{mn},\forall(f^1,d^1,\delta^{+1})\neq(f^2,d^2,\delta^{+2}).$$

Then the equilibrium link load, the noncompliant overflows, underflows, and travel demand pattern $(f^*,d^*,\delta^{+*})$ is unique.

*3.1.3. Proof.* Assume, on the contrary, that there are two distinct equilibrium patterns, denoted by $\hat{X}^1$ and $\hat{X}^2$. Then, both must satisfy the variational inequality (38). Hence, we have that

$$\langle\hat{F}(\hat{X}^1),X-\hat{X}^1\rangle\geq 0,\forall X\in\mathcal{K}^1 \quad (64)$$

and

$$\langle\hat{F}(\hat{X}^2),X-\hat{X}^2\rangle\geq 0,\forall X\in\mathcal{K}^1. \quad (65)$$

Substituting $\hat{X}^2$ for $X$ in (64) and $\hat{X}^1$ for $X$ in (65) and adding the two resultant inequalities, yields:

$$\sum_{i\in M}\sum_{a\in L}(c_a^i(f^1)-c_a^i(f^2))\times(f_a^{i2}-f_a^{i1})-\sum_{i\in M}\sum_{w\in W}(\lambda_w^i(d^1)-\lambda_w^i(d^2))\times(d_w^{i2}-d_w^{i1})$$

$$+\sum_{i\in M}\sum_{a\in L}(\frac{\partial\mu_a^i(\delta^{+1})}{\partial\delta_a^{i+}}-\frac{\partial\mu_a^i(\delta^{+2})}{\partial\delta_a^{i+}})\times(\delta_a^{i+2}-\delta_a^{+1})\geq 0 \quad (66)$$

but due to the assumption of strict monotonicity this implies that this inequality must hold as an equality and, hence, we must have that $f^1=f^2$ and $d^1=d^2$ and $\delta^{+1}=\delta^{+2}$. Furthermore, it follows from equality (30) that, therefore, $\delta^{-*}$ must also be unique.

*3.1.4. Theorem 5 (Lipschitz continuity).* If the travel cost functions, the travel disutility functions, and the marginal penalty cost functions have bounded first order derivatives, then the function $\hat{F}(\hat{X})$ is Lipschitz continuous, that is, there exists a positive constant $\bar{L}$, such that

$$\| \hat{F}(\hat{X}^1) - \hat{F}(\hat{X}^2) \| \leq \bar{L} \| \hat{X}^1 - \hat{X}^2 \|, \forall X^1, X^2 \in \mathcal{K}^1. \tag{67}$$

*3.1.5. Proof.* Follows along the same arguments as the proof of Lemma 3 in Nagurney (1994).

In regards to the model with compliance, one can establish uniqueness results for the link load and travel demand pattern in a straightforward manner by using the results in Nagurney, Ramanujam, and Dhanda (1997) for the single modal model with compliance.

## 4. ALGORITHM

In this Section, the modified projection method of Korpelevich (1977) is proposed for the solution of variational inequality (38). This algorithm can also be applied to compute the equilibrium for the compliant model as well.

### 4.1. Modified projection method

*4.1.1. Step 0: Initialization.* Set $X^0 \in \mathcal{K}^1$. Let $k = 1$ and let $\alpha$ be a scalar such that $0 < \alpha < \frac{1}{\bar{L}}$, where $\bar{L}$ is the Lipschitz continuity constant.

*4.1.2. Step 1: Computation.* Compute $\bar{X}^k$ by solving the variational inequality subproblem:

$$\langle [\bar{X}^k + \alpha \hat{F}(X^{k-1})^T - X^{k-1}]^T, X - \bar{X}^k \rangle \geq 0, \forall X \in \mathcal{K}^1. \tag{68}$$

*4.1.3. Step 2: Adaptation.* Compute $X^k$ by solving the variational inequality subproblem:

$$\langle [X^k + \alpha \hat{F}(\bar{X}^k)^T - X^{k-1}]^T, X - X^k \rangle \geq 0, \forall X \in \mathcal{K}^1. \tag{69}$$

*4.1.4. Step 3: Convergence Verification.* If $\| X^k - X^{k-1} \| \leq \epsilon$, for $\epsilon > 0$, a prespecified tolerance, then, stop; otherwise, set $k = k + 1$ and go to Step 1.

For completeness, we now present the above algorithm in $\hat{F}(X)$ is in expanded form for the model.

*4.1.5. Step 0: Initialization.* Set $(f^0, d^0, t^0, \delta^{-0}, \delta^{+0}, l^0, \rho^0) \in \mathcal{K}^1$. Let $k = 1$ and set $\alpha$ such that $0 < \alpha < \frac{1}{\bar{L}}$.

*4.1.6. Step 1: Computation.* Compute $(\bar{F}^k, \bar{d}^k, \bar{t}^k, \bar{\delta}^{-k}, \bar{\delta}^{+k}, \bar{l}^k, \bar{\rho}^k) \in \mathcal{K}^1$ by solving the variational inequality subproblem:

$$\sum_{i \in M} \sum_{a \in L} (\bar{f}_a^{i(k)} + \alpha(c_a^i(f^{i(k-1)}) + h_a^i t_a^{i(k-1)}) - f_a^{i(k-1)}) \times (f_a^i - \bar{f}_a^{i(k)})$$

$$- \sum_{i \in M} \sum_{w \in W} (\bar{d}_w^{i(k)} + \alpha \lambda_w(d^{i(k-1)}) - d_w^{i(k-1)}) \times (d_w^i - \bar{d}_w^{i(k)})$$

$$+ \sum_{i \in M} \sum_{a \in L} (\bar{t}_a^{i(k)} + \alpha(l_a^{i(k-1)} - h_a^i f_a^{i(k-1)} + \delta_a^{i+(k-1)} - \delta_a^{i-(k-1)}) - t_a^{i(k-1)}) \times (t_a^i - \bar{t}_a^{i(k)})$$

$$+ \sum_{i \in M} \sum_{a \in L} (\bar{\delta}_a^{i-(k)} + \alpha(t_a^{i(k-1)}) - \delta_a^{i-(k-1)}) \times (\delta_a^{i-} - \bar{\delta}_a^{i-(k)}) \tag{70}$$

$$+\sum_{i\in M}\sum_{a\in L}(\bar{\delta}_a^{i+(k)}+\alpha(\frac{\partial\mu_a^i(\delta_a^{i+(k-1)})}{\partial\delta_a^{i+}}-t_a^{i(k-1)})-\delta_a^{i+(k-1)})\times(\delta_a^{i+}-\bar{\delta}_a^{i+(k)})$$

$$+\sum_{i\in M}\sum_{a\in L}(\bar{l}_a^{i(k)}+\alpha(\rho^{(k-1)}-t_a^{i(k-1)})-l_a^{i(k-1)})\times(l_a^i-\bar{l}_a^{i(k)})$$

$$+(\bar{\rho}+\alpha(\sum_{i\in M}\sum_{a\in L}l_a^{i0}-\sum_{i\in M}\sum_{a\in L}l_a^{i(k-1)})-\rho^{(k-1)})\times(\rho-\bar{\rho}^{(k)})\geq 0,\forall(f,d,t,\delta^-,\delta^+,l,\rho)\in\mathcal{K}^1. \quad (70)$$

*4.1.7. Step 2: Adaptation.* Compute $(f^k,d^k,t^k,\delta^{-k},\delta^{+k},l^k,\rho^k)\in\mathcal{K}^1$ by solving the variational inequality subproblem:

$$\sum_{i\in M}\sum_{a\in L}(f_a^{i(k)}+\alpha(c_a^i(\bar{f}^{i(k)})+h_a^i\bar{t}_a^{i(k)})-f_a^{i(k-1)})\times(f_a^i-f_a^{i(k)})$$

$$-\sum_{i\in M}\sum_{w\in W}(d_w^{i(k)}+\alpha\lambda_w(\bar{d}^{i(k)})-d_w^{i(k-1)})\times(d_w^i-d_w^{i(k)})$$

$$+\sum_{i\in M}\sum_{a\in L}(t_a^{i(k)}+\alpha(\bar{l}_a^{i(k)}-h_a^i\bar{f}_a^{i(k)}+\delta_a^{i+(k-1)}-\delta_a^{i-(k-1)})-t_a^{i(k-1)})\times(t_a^i-t_a^{i(k)})$$

$$+\sum_{i\in M}\sum_{a\in L}(\delta_a^{i-(k)}+\alpha(\bar{t}_a^{i(k-1)})-\delta_a^{i-(k-1)})\times(\delta_a^{i-}-\delta_a^{i-(k)})$$

$$+\sum_{i\in M}\sum_{a\in L}(\delta_a^{i+(k)}+\alpha(\frac{\partial\mu_a^i(\bar{\delta}_a^{i+(k-1)})}{\partial\delta_a^{i+}}-\bar{t}_a^{i(k-1)})-\delta_a^{i+(k-1)})\times(\delta_a^{i+}-\delta_a^{i+(k)})$$

$$+\sum_{i\in M}\sum_{a\in L}(l_a^{i(k)}+\alpha(\bar{\rho}^k-\bar{t}_a^{i(k)})-l_a^{i(k-1)})\times(l_a^i-l_a^{i(k)})$$

$$+(\rho+\alpha(\sum_{i\in M}\sum_{a\in L}l_a^{i0}-\sum_{i\in M}\sum_{a\in L}\bar{l}_a^{i(k)})-\rho^{(k-1)})\times(\rho-\rho^{(k)})\geq 0,\forall(f,d,t,\delta^-,\delta^+,l,\rho)\in\mathcal{K}^1. \quad (71)$$

*4.1.8. Step 3: Convergence Verification.* If $\| f^{(k)}-f^{(k-1)}\|\leq\epsilon$, $\| d^{(k)}-d^{(k-1)}\|\leq\epsilon$, $\| t^{(k)}-t^{(k-1)}\|\leq\epsilon$, $\| \delta^{-(k)}-\delta^{-(k-1)}\|\leq\epsilon$, $\| \delta^{+(k)}-\delta^{+(k-1)}\|\leq\epsilon$, $\| l^{(k)}-l^{(k-1)}\|\leq\epsilon$, $\| \rho^{(k)}-\rho^{(k-1)}\|\leqslant\epsilon$, with $\epsilon>0$, a pre-specified tolerance, then stop; otherwise, set $k:=k+1$, and go to Step 1.

We now discuss the solution of the variational inequality subproblems (70) and (71). First recall the definition of the projection of $x$ on the closed convex subset $\mathcal{K}^1$, with respect to the Euclidean norm, and denoted as $P_{\mathcal{K}^1}x$, as

$$y=P_{\mathcal{K}^1}x=\arg\min_{x\in\mathcal{K}^1}\|\mathrm{x}-\mathrm{z}\|. \quad (72)$$

In particular, observe that (cf. Theorem 1.2 in Nagurney (1993)), $\bar{X}^T$ generated by the modified projection method as the solution to the variational inequality subproblem (70) is, in fact, the projection of $(X^{k-1}-\alpha\hat{F}(X^{k-1})^T)$ on the closed convex set $\mathcal{K}^1$, that is

$$\bar{X}^k=P_{\mathcal{K}^1}[X^{k-1}-\alpha\hat{F}(X^{k-1})^T]. \quad (73)$$

Similarly

$$\bar{X}^k=P_{\mathcal{K}^1}[X^{k-1}-\alpha F(\bar{X}^k)^T]. \quad (74)$$

We now discuss the simple form of the decomposed subproblems which can be computed very efficiently. In particular, the feasible set is a Cartesian product, where $K$ has the network structure of the traffic network equilibrium problem with elastic demands, whereas the other sets yield subproblems of box-type constraints, each of which can be solved explicitly and in closed form using a simple formula. Convergence for the algorithm is given in the following Theorem.

*4.2. Theorem 6 (convergence)*

If the user link travel cost functions $c$, the noncompliant marginal penalty functions $\frac{\partial \mu}{\partial \delta^+}$, and the travel disutility functions $-\lambda$ are assumed to be monotone and have bounded first order derivatives, then the modified projection method described above converges to the solution of the variational inequality (38), provided that a solution exists.

*4.3. Proof*

From Lemma 2, it follows that $\hat{F}(\hat{X})$ is monotone whereas from Theorem 5, $\hat{F}(\hat{X})$ is Lipschitz continuous. Hence, according to Korpelevich (1977), the modified projection method is guaranteed to converge.

## 5. NUMERICAL EXAMPLES

In this Section, we present numerical examples to illustrate both the multimodal model and the algorithm. We implemented the modified projection method in FORTRAN and utilized the IBM SP2 located at the Cornell Theory Center for the numerical work.

For the solution of the separable and standard traffic network equilibrium problem encountered in both the computation and adaptation steps [cf. (70) and (71)] we utilized the Euler method (cf. Nagurney and Zhang 1996, and the references therein). The Euler method also induces subproblems (and these are in path flow variables in this case), which can also be solved exactly and in closed form. The convergence tolerance was set to $\epsilon = 0.001$ for all the examples.

We set $\alpha = 0.05$ for all the examples. We initialized the modified projection method by setting the initial demand for each O/D pair equal to 100 and setting the flow on a path equal to the travel demand for the O/D pair that the path belongs to divided by the number of paths. All other variables were initialized to zero. We report the CPU time, exclusive of input/output and setup time, as well as, the number of iterations for each example.

For all the examples we considered the network topology given by the Braess (1968) network and depicted in Fig. 1. There were two modes of transportation in each example.

*5.1. Example 1*

The first example, network with user link travel cost functions given by:

*5.2. Mode 1*

$$c_1^1(f) = 0.0005 f_1^{1^4} + 10 f_1^1 + 2 f_2^2, \; c_2^1(f) = 0.00001 f_2^{1^4} + f_2^1 + .5 f_1^1 + 50,$$

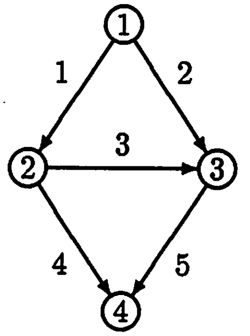


Fig. 1. Set 1 network topology.

$$c_3^1(f) = 0.0004f_3^{1^4} + f_3^1 + .2f_2^2 + 10,\ c_4^1(f) = 0.0001f_4^{1^4} + f_4^1 + .1f_3^2 + 50,$$

$$c_5^1(f) = 0.0001f_5^{1^4} + 10f_5^1 + 4f_1^1.$$

### 5.3. *Mode 2*

$$c_1^2(f) = 0.0005f_1^{2^4} + 11f_1^2 + 2f_2^2 + 10,\ c_2^2(f) = 0.00001f_2^{2^4} + 2f_2^2 + 7f_1^1 + 50,$$

$$c_3^2(f) = 0.0004f_3^{1^4} + 2f_3^2 + .5f_2^1 + 15,\ c_4^2(f) = 0.0005f_4^{2^4} + 4f_4^2 + .3f_3^1 + 50,$$

$$c_5^2(f) = 0.0004f_5^{2^4} + 12f_5^2 + 7f_1^1 + 15.$$

The O/D pair was $w = (1, 4)$ with travel disutilities for the modes given by:

$$\lambda_w^1(d_w^1) = -d_w^1 + 98\ \lambda_w^2(d_w^2) = -d_w^2 + 100.$$

The paths were: $p_1 = (1, 3, 5)$, $p_2 = (1, 4)$, and $p_3 = (2, 5)$.

In the first example the emission parameters and the initial license allocation were set as follows: $h_a^i = i \times a$, $l_a^{i0} = a$, for all $i \in M$ and $a \in L$. We set the marginal penalty cost functions to: $\frac{\partial \mu_a^i}{\partial \delta_a^{i+}} = .1 \times a \times \delta_a^{i+}$ for all modes $i$ and links $a$.

The modified projection method converged in 1.01 CPU seconds and 1100 iterations yielding the following equilibrium pattern:

$$f_1^{1*} = 3.65,\ f_2^{1*} = 0.45,\ f_3^{1*} = 1.72,\ f_4^{1*} = 1.92,\ f_5^{1*} = 2.17$$

$$f_1^{2*} = 1.98,\ f_2^{2*} = .21,\ f_3^{2*} = .33,\ f_4^{2*} = 1.65,\ f_5^{2*} = .54.$$

This equilibrium link load pattern was induced by the equilibrium path flow pattern:

$$x_{p_1}^{1*} = 1.72,\ x_{p_2}^{1*} = 1.92,\ x_{p_3}^{1*} = 0.45,$$

$$x_{p_1}^{2*} = 0.33,\ x_{p_2}^{2*} = 1.65,\ x_{p_3}^{2*} = 0.21$$

and the equilibrium travel demand:

$$d_w^{1*} = 4.09,\ d_w^{2*} = 2.19.$$

The generalized user travel costs on the paths were:

$$\hat{C}_{p_1}^1 = \hat{C}_{p_2}^1 = \hat{C}_{p_3}^1 = 93.91,$$

$$\hat{C}_{p_1}^2 = \hat{C}_{p_2}^2 = \hat{C}_{p_3}^2 = 97.81.$$

The travel disutilities were:

$$\lambda_w^1 = 93.91,\ \lambda_w^2 = 97.81.$$

The equilibrium licenses were:

$$l_1^{1*} = 0.00,\ l_2^{1*} = 0.00,\ l_3^{1*} = 1.84,\ l_4^{1*} = 5.20,\ l_5^{1*} = 8.85,$$

$$l_1^{2*} = 0.00,\ l_2^{2*} = 0.00,\ l_3^{2*} = 0.00,\ l_4^{2*} = 10.67,\ l_5^{2*} = 3.43.$$

The equilibrium marginal costs were:

$$t_1^1* = 0.38,\ t_2^{1*} = 0.18,\ t_3^{1*} = 1.00,\ t_4^{1*} = 1.00,\ t_5^{1*} = 1.00,$$

$$t_1^{2*} = 0.40,\ t_2^{2*} = 0.16,\ t_3^{2*} = 0.60,\ t_4^{2*} = 1.00,\ t_5^{2*} = 1.00.$$

The price of a license in equilibrium was:

$$\rho^* = 1.00.$$

In this example, the market for licenses cleared, that is, the excess supply of licenses was zero.

All the underemissions were equal to zero, whereas the overemissions were equal to:

$$\delta_1^{1+*} = 3.63,\ \delta_2^{1+*} = 0.89,\ \delta_3^{1+*} = 3.33,\ \delta_4^{1+*} = 2.50,\ \delta_5^{1+*} = 2.00,$$

$$\delta_1^{2+*} = 3.96,\ \delta_2^{2+*} = 0.83,\ \delta_3^{2+*} = 2.01,\ \delta_4^{2+*} = 2.51,\ \delta_5^{2+*} = 1.99.$$

### 5.4. *Example 2*

We then made the following changes to the above example. We kept all the data as described above except that we increased the marginal penalty cost terms by a factor of 10. The computed equilibrium link load pattern was:

$$f_1^{1*} = 3.24, f_2^{1*} = 0.12, f_3^{1*} = 1.73, f_4^{1*} = 1.51, f_5^{1*} = 1.86$$

$$f_1^{2*} = 1.21, f_2^{2*} = 0.06, f_3^{2*} = 0.09, f_4^{2*} = 1.12, f_5^{2*} = 0.15.$$

This equilibrium link load pattern was induced by the equilibrium path flow pattern:

$$x_{p_1}^1* = 1.73, x_{p_2}^{1*} = 1.51, x_{p_3}^{1*} = 0.12,$$

$$x_{p_1}^{2*} = 0.09, x_{p_2}^{2*} = 1.12, x_{p_3}^{2*} = 0.06$$

and the equilibrium travel demand:

$$d_w^{1*} = 3.37,\ d_w^{2*} = 1.27.$$

The generalized user travel costs on the paths were:

$$\hat{C}_{p_1}^1 = \hat{C}_{p_2}^1 = \hat{C}_{p_3}^1 = 94.62,$$

$$\hat{C}_{p_1}^2 = \hat{C}_{p_2}^2 = \hat{C}_{p_3}^2 = 98.73.$$

The travel disutilities were:

$$\lambda_w^1 = 94.62, \ \lambda_w^2 = 98.73.$$

The equilibrium licenses were:

$$l_1^{1*} = 1.19, \ l_2^{1*} = 0.00, \ l_3^{1*} = 4.51, \ l_4^{1*} = 5.52, \ l_5^{1*} = 8.88,$$

$$l_1^{2*} = 0.36, \ l_2^{2*} = 0.00, \ l_3^{2*} = 0.00, \ l_4^{2*} = 8.43, \ l_5^{2*} = 1.11.$$

The equilibrium marginal costs were:

$$t_1^{1*} = 2.06, \ t_2^{1*} = 0.50, \ t_3^{1*} = 2.06, \ t_4^{1*} = 2.06, \ t_5^{1*} = 2.06,$$

$$t_1^{2*} = 2.06, \ t_2^{2*} = 0.51, \ t_3^{2*} = 1.70, \ t_4^{2*} = 2.08, \ t_5^{2*} = 2.04.$$

The price of a license in equilibrium was:

$$\rho^* = 2.06.$$

In this example, the market for licenses also cleared, that is, the excess supply of licenses was zero. All the underemissions were equal to zero, whereas the overemissions were equal to:

$$\delta_1^{1+*} = 2.05, \ \delta_2^{1+*} = 0.25, \ \delta_3^{1+*} = 0.69, \ \delta_4^{1+*} = 0.51, \ \delta_5^{1+*} = 0.41,$$

$$\delta_1^{2+*} = 2.06, \ \delta_2^{2+*} = 0.26, \ \delta_3^{2+*} = 0.57, \ \delta_4^{2+*} = 0.52, \ \delta_5^{2+*} = 0.41.$$

As one would expect, higher penalty costs for overemissions resulted on lower overemissions.

*5.5. Example 3*

We then constructed the following example. We kept all the data as in Example 2 but we solved for the model with compliance. Note that now there will be no penalty cost functions nor any overemission or underemission variables. Note also, based on our construction of Example 2 from Example 1, that this model with compliance also holds for Example 1.

The modified projection method converged in 3850 iterations and 1.09 seconds of CPU time yielding the following equilibrium link load pattern:

$$f_1^{1*} = 3.00, \ f_2^{1*} = 0.00, \ f_3^{1*} = 1.70, \ f_4^{1*} = 1.30, \ f_5^{1*} = 1.70$$

$$f_1^{2*} = 0.82, \ f_2^{2*} = 0.00, \ f_3^{2*} = 0.00, \ f_4^{2*} = 0.82, \ f_5^{2*} = 0.00.$$

This equilibrium link load pattern was induced by the equilibrium path flow pattern:

$$x_{p_1}^{1*} = 1.70, \ x_{p_2}^{1*} = 1.30, \ x_{p_3}^{1*} = 0.00,$$

$$x_{p_1}^{2*} = 0.00, \ x_{p_2}^{2*} = 0.82, \ x_{p_3}^{2*} = 0.00$$

and the equilibrium travel demand:

$$d_w^{1*} = 3.00, \ d_w^{2*} = 0.82.$$

The generalized user travel costs on the paths were:

$$\hat{C}^1_{p_1} = \hat{C}^1_{p_2} = 95.00,$$

$$\hat{C}^1_{p_3} = 97.41,$$

$$\hat{C}^2_{p_2} = 99.18,$$

$$\hat{C}^2_{p_1} = 100.45, \hat{C}^2_{p_3} = 103.76.$$

The travel disutilities were:

$$\lambda^1_w = 95, \ \lambda^2_w = 99.18.$$

The equilibrium licenses were:

$$l_1^{1*} = 3.02, \ l_2^{1*} = 0.00, \ l_3^{1*} = 5.08, \ l_4^{1*} = 5.21, \ l_5^{1*} = 8.50,$$

$$l_1^{2*} = 1.63, \ l_2^{2*} = 0.00, \ l_3^{2*} = 0.00, \ l_4^{2*} = 6.55, \ l_5^{2*} = 0.00.$$

The equilibrium marginal costs were:

$$t_1^{1*} = 2.71, \ t_2^{1*} = 1.72, \ t_3^{1*} = 2.68, \ t_4^{1*} = 2.69, \ t_5^{1*} = 2.69,$$

$$t_1^{2*} = 2.68, \ t_2^{2*} = 1.62, \ t_3^{2*} = 2.40, \ t_4^{2*} = 2.69, \ t_5^{2*} = 2.60.$$

The price of a license in equilibrium was:

$$\rho^* = 2.69.$$

In this example, the market for licenses also cleared, that is, the excess supply of licenses was zero.

### 5.6. *Example 4*

In the final example, in order to provide an illustration of *Theorem 3*, which provides us with a mechanism to guarantee that there will be no overemissions and the solution to such a model will guarantee compliance, we proceeded as follows: We set the marginal penalty cost functions equal to the equilibrium price computed in *Example 3* for all modes and links and we solved for equilibrium for the model with noncompliance. As *Theorem 3* established, the equilibrium solution coincided with the equilibrium solution for the model with compliance solved in *Example 3*.

## 6. SUMMARY AND CONCLUSIONS

In this paper, we developed a multimodal traffic network equilibrium model with emission pollution permits. We considered two versions of the model with the first being the model with compliance and the second–the model with noncompliance in which we incorporated penalty costs associated with overemissions on the links over and above those allowed by the permit or license holdings. We provided the equilibrium conditions for both models as well as their variational inequality formulations. We addressed certain qualitative issues and also provided results that could be of benefit to policy-makers. In particular, we showed that if one sets all marginal penalty costs associated with overemissions precisely equal to the equilibrium license price in the model with compliance, then there will be no overemissions. We also proposed a numerical scheme, along with convergence results and several illustrative numerical examples to highlight the results obtained in the paper.

In practice, the operationalism of the model could be summarized as follows. Each user traveling on a certain path on a specific mode would incur a *generalized cost* equal to the sum of the travel cost plus the total emission cost (which is equal to the sum over all links that comprise the path of the emission factor on each link times the marginal emission abatement cost). It should be noted that while the emission factor is based on the geographical position of the roads and the mode of transport, the marginal abatement cost would equal the price of the license at equilibrium, in the case of compliance, or the marginal penalty cost associated with overemission in the case of noncompliance. It should be noted that an initial allocation of licenses would ensure the maintenance of the preset environmental quality standard, but only in the case of compliance. The emission cost is thus market derived and replaces the tolls and costs of externalities proposed by the current fiscal and pricing models in literature.

Regarding implementation, the transportation authority would be responsible for broadcasting the price of licenses, the individual payments, and the availability of licenses. Sensory stickers or tokens (that might be specific to license plates to avoid transfers) could be made available at gas stations, motor vehicle registries, etc., at the market price based on the chosen path between a certain origin and destination pair. Purchase of licenses prior to travel and the availability of licenses, however, could pose a problem to the traveler. Hence, an imposition of a time frame, for example, a week, for the validity of licenses would help to solve this problem.

Ultimately, our objective is to develop a dynamic, electronic, market-oriented emission permit system differentiating pollutants and pollutant receptors for traffic network equilibrium problems. Indeed, here, for simplicity, we considered only a single receptor point and a single pollutant, although we explicitly modeled multiple modes of transportation. This work serves as one of the first steps towards achieving this goal. There are definite hurdles regarding implementation especially in relation to the payment and the collection of the associated travelers' emission costs. However, these difficulties are faced by other fiscal and pricing policies as well.

This work represents the first multimodal traffic network equilibrium model with elastic travel demands in the presence of a permit or license market system to reduce pollution emissions (as well as congestion). Moreover, it is the first time that both compliant and noncompliant behavior is modeled in such a setting.

*Acknowledgements*—The first author's work was supported by NSF Grant No. DMS-9024071. This support is gratefully acknowledged. The authors are grateful to the anonymous referees for helpful suggestions and comments.

## REFERENCES

Allen, Jr and W. G., (1996) Model improvements for evaluating pricing strategies. *Transportation Research Record* **1498**, 75–81.

Anderson, W. P., Kanaroglou, P. S., Miller, E. J. and Buliung, R. N. (1996) Simulating automobile emissions in an integrated urban model. *Transportation Research Record* **1520**, 71–80.

Beckmann, M. J., Mcguire, C. B. and Winsten, C. B. (1956) *Studies in the Economics of Transportation.* Yale University Press, New Haven, Connecticut.

Braess, D. (1968) Uber ein parodoxon der verkehrsplanung. *Unternehmenforschung* **12**, 258–268.

Button, K. (1990) Environmental externalities and transport policy. *Oxford Review of Economic Policy* **6**, 61–75.

Cadle, S. H., Gorse, R. A. and Lawson D. R., (1993) Real-world vehicle emissions: A summary of the third annual CRC-APRAC on-road vehicle emissions workshop. *Air & Waste* **43**, 1084–090.

Dafermos, S. (1973) Toll patterns for multiclass-user transportation networks. *Transportation Science* **7**, 211–223.

Dafermos, S. (1982) The general multimodal network equilibrium problem with elastic demand. *Networks* **12**, 57–72.

Decorla-Souza, P., Everett, J., Cosby, J. and Lim, P. (1995) Trip based approach to estimate emissions with Environmental Protection Agency's MOBILE model. *Transportation Research Record* **1444**, 118–125.

Decorla-Souza, P. and Kane, A. (1996) Peak period tolls: precepts and prospects. *Transportation* **19**, 293–311.

*The Economist* (1996) Living with the car. June, 3–18.

*The Economist* (1997) Living with the car. December, 21–23.

Goddard, H. C. (1997) Optimal restrictions on vehicle use for urban sustainability for Mexico city. *International Journal of Environment and Pollution* **7**, 357–374.

Jones, P. and Hervik, A. (1992) Restraining car traffic in European cities: an emerging role for road pricing. *Transportation Research* **26A**, 133–145.

Knight, F. H. (1924) Some fallacies in the interpretation of social costs. *Quarterly Journal of Economics* **38**, 582–606.

Korpelevich, G. M. (1977) The extragradient method for finding saddle points and other problems. *Matekon 13*, 35–49.

Krugman, P. R. (1996) The tax-reform obsession. *The New York Times Magazine* April, 36–37.

Lawson, D. R., Groblicki, P. J., Stedman, D. H., Bishop, G. A. and Guenther, P.L. (1990) Emissions from in-use motor vehicles in Los Angeles: a pilot study of remote sensing and the inspection and maintenance program. *Journal of Air Waste Management Association* **40**, 1096–1105.

Malik, A. S. (1990) Markets for pollution control when firms are noncompliant. *Journal of Environmental Economics and Management* **18**, 97–106.

Montgomery, W. D. (1972) Markets in licenses and efficient pollution control programs. *Journal of Economic Theory* **5**, 395–418.

Nagurney, A. (1993) *Network Economics: A Variational Inequality Approach*. Kluwer Academic Publishers, Boston, Massachusetts.

Nagurney, A. (1994) Variational inequalities in the analysis and computation of multisector, multi-instrument financial equilibria. *Journal of Economic Dynamics and Control* **18**, 161–184.

Nagurney, A. and Dhanda, K. (1996) Variational inequality approach for marketable pollution permits. *Computational Economics* **9**, 360–384.

Nagurney, A. and Dhanda, K. K. (1997) Noncompliant oligopolistic firms and marketable pollution permits: statics and dynamics. School of Management, University of Massachusetts, Amherst, MA.

Nagurney, A. and Ramanujam, P. (1995) Command and control for congestion pricing of general multimodal transportation networks. *Scientia Iranica* **2**, 185–195.

Nagurney, A. and Ramanujam, P. (1996) Transportation network policy modeling with goal targets and generalized penalty functions. *Transportation Science* **30**, 3–13.

Nagurney, A., Ramanujam, P. and Dhanda, K. (1997) An integrated traffic network equilibrium model with emission pollution permits: qualitative analysis and computation. School of Management, University of Massachusetts, Amherst, MA.

Nagurney, A., Thore, S. and Pan, J. (1996) Spatial market policy modeling with goal targets. *Operations Research* **44**, 393–406.

Nagurney, A. and Zhang, D. (1996) *Projected Dynamical Systems and Variational Inequalities with Applications*. Kluwer Academic Publishers, Boston, Massachusetts.

Pigou, A.C. (1920) *The Economics of Welfare*. MacMillan, London.

Toh, R. S. (1992) Experimental measures to curb road congestion in Singapore: pricing and quotas. *The Logistics and Transportation Review* **28**, 289–317.

United States Department of Transportation (1992*a*) *A summary: Transportation Programs and Provisions of the Clean Air Act Aamendments of 1990*. Publication No. FHWA-PD-92-023.

United States Department of Transportation (1992*b*) *A summary: Air Quality Programs and Provisions of the Intermodal Surface Transportation Efficiency Act of 1991, 1992*. Publication No. FHWA-PD-92-022.

Van Egteren, H. and Weber, M. (1996) Marketable permits, market power and cheating. *Journal of Environmental Economics and Management* **30**, 161–173.