2000

Noncompliant Oligopolistics Firms and Marketable Pollution Permits: Statics and Dynamics

Anna Nagurney, *University of Massachusetts - Amherst*
Kathy K Dhanda

Noncompliant oligopolistic firms and marketable pollution permits: Statics and dynamics

Anna Nagurney [a] and Kanwalroop Kathy Dhanda [b]

[a] *Department of Finance and Operations Management, Isenberg School of Management, University of Massachusetts, Amherst, MA 01003, USA*
E-mail: nagurney@gbfin.umass.edu

[b] *School of Business Administration, The University of Portland, 5000 North Willamette Boulevard, Portland, OR 97203-5798, USA*

In this paper, we consider the modeling, analysis, and computation of solutions to both static and dynamic models of multiproduct, multipollutant noncompliant oligopolistic firms who engage in a market for pollution permits. In the case of the static model, we utilize variational inequality theory for the formulation of the governing equilibrium conditions as well as the qualitative analysis of the equilibrium pattern, including sensitivity analysis. We then propose a dynamic model, using the theory of projected dynamical systems, whose set of stationary points coincides with the set of solutions to the variational inequality problem. We propose an algorithm, which is a discretization in time of the dynamic adjustment process, and provide convergence results using the stability analysis results that are also provided herein. Finally, we apply the algorithm to several numerical examples to compute the profit-maximized quantities of the oligopolistic firms' products and the quantities of emissions, along with the equilibrium allocation of licenses and their prices, as well as the possible noncompliant overflows and underflows. This is the first time that these methodologies have been utilized in conjunction to study a problem drawn from environmental policy modeling and analysis.

Keywords: environmental policy, pollution permits, oligopolistic markets, noncompliance, variational inequalities, projected dynamical systems

1. Introduction

Environmental policy modeling and analysis is an area of research and application which is not only timely but also one to which tools from both operations research and economics can be brought to bear. In this paper, we consider the modeling, analysis, and computation of a particular environmental policy problem – that of noncompliance in the case of multiproduct, multipollutant oligopolistic firms who can trade in marketable pollution permits. For the study of the new models developed here we utilize variational inequality theory for the statics surrounding the equilibrium state and projected dynamical systems theory for the dynamics and disequilibrium behavior.

Finite-dimensional variational inequality theory (cf. Nagurney [9] and the references therein) has been utilized for environmental policy modeling in the framework of goal targets with associated penalties for noncompliance in the context of spatial economic markets by Thore et al. [17] and Nagurney et al. [14] and by Nagurney and Ramanujam [13] in the context of congested urban transportation systems. It has also been used in the case of marketable pollution permits by Nagurney and Dhanda [10], albeit for single product, single pollutant oligopolistic and perfectly competitive firms. Recently Nagurney et al. [11] extended the latter model to the case of multiple products and multiple pollutants.

In this paper, we develop not only a more general oligopolistic market model than those cited previously, but deal explicitly with the issue of noncompliance. Moreover, we apply, for the first time, the theory of projected dynamical systems (cf. Dupuis and Nagurney [2], Zhang and Nagurney [20], and Nagurney and Zhang [15]) for the development of a dynamic multiproduct, multipollutant oligopolistic market model with marketable pollution permits and noncompliance. Projected dynamical systems have been utilized in the dynamic modeling and analysis of oligopolistic markets but not in the presence of environmental policies (see, e.g., Nagurney et al. [12]).

The system of marketable pollution permits was initially hailed as an economic-incentive approach to address the problem of pollution. Indeed, the theoretical basis of this system is very sound (cf. Montgomery [8]). The marginal cost of pollution abatement is equalized across the polluting firms by having the firms purchase a permit which is a license to pollute a pre-specified unit of emissions. The initial allocations of licenses are made in such a manner so that the regional environmental standards are achieved.

However, under any system of pollution control, whether it is marketable permits, emission standards, or Pigouvian taxes, there is bound to be noncompliant behavior on the part of the firms. Malik [6] stated that problems encountered in enforcing market based pollution control policies pertain to the technical difficulty encountered in the continuous monitoring of pollutants and the absence of well-developed mechanisms to assess penalties for noncompliance.

Van Egteren and Weber [19] reiterated that limited budgets and prohibitive monitoring costs make complete enforcement impossible. Furthermore, they stated that firms will cheat if the marginal cost of compliance, that is, the cost of purchasing a pollution permit to cover an additional unit of emissions, is higher than the marginal penalty for cheating.

Malik [6] showed that, in the case of noncompliance, a participant in the marketable pollution permit system does not minimize abatement costs since a change in the permit price also affects the number of cheating firms and the allocation of abatement costs across the firms. Additionally, if the number of participants in the permit market is small, then noncompliance on the part of a smaller subset of firms, or even one firm, can have a significant impact on equilibrium outcome through the equilibrium permit price.

There has been some work done to compare various systems of pollution control in the presence of noncompliant behavior. Harford [4] presented simple theoretical models of firm behavior under effluent standards and taxes. Keeler [5] focused on the shape of the penalty functions to make a specific comparison between systems of marketable pollution permits and uniform emission standards when full compliance is not achieved by regulatory authorities. Malik [7] examined whether enforcement costs vary across policy instruments and compared marketable permits with uniform emission standards to determine if an economic-incentive approach is less costly to enforce than a direct control approach.

In this paper, we model multiproduct, multipollutant oligopolistic firms engaged in markets in ambient-based pollution permits. Montgomery [8] presented two systems of pollution licenses: a system that defines allowable pollution concentrations at a set of receptor points, referred to as the ambient-based permit system (APS), and a system of emission licenses that confers the right to emit pollutants at a certain rate, referred to as the emission-based permit system (EPS). In the ambient-based system, a target level of environmental quality is established by the governmental authority with the level of pollution being defined in terms of total allowable emissions.

In our models, the oligopolistic firms face production costs, emission costs, and the cost of purchasing licenses in excess of their endowment of initial allocations. In addition, a noncompliant firm faces a penalty or a fine in the case that a firm emits more pollutant than it is licensed to emit.

We focus on oligopolistic firms, rather than on perfectly-competitive firms, since perfect competition is not applicable to major polluting generating firms which are oligopolistic in their output markets. We do, however, assume that the firms are perfectly-competitive in the permit markets. More specifically, each source of pollution takes the price of the license to pollute at a particular pollutant at a certain point as given. We note that the license trading system, as an economic-incentive approach, designs the license markets in order to achieve environmental goals in a cost-effective way. Indeed, unlike the firms' production outputs, the supply of the initial licenses is fixed and determined so that the environmental goals are achieved. The "market" power, hence, as regards the licenses, is completely dependent on the initial license allocation and is controlled by the regulatory agency. The effectiveness of the license trading system, consequently, depends upon perfect competition in the permit market.

An application of the permit system is the sulfur allowance program Administered by the Environmental Protection Agency (EPA) whereby allowances to emit sulfur oxides have been granted to existing electricity generating plants. The idea is to reduce the amount of sulfur oxides emissions by 10 million tons from the year 1980 to the year 2010. The unique aspect of this program is that the allowances are traded in an auction market conducted by the Chicago Board of Trade (CBOT). In addition any interested party can purchase these allowances, hence, the environmental groups are buyers as well as polluting electricity plants. For a detailed discussion on the acid rain program, we refer the reader to Tietenberg [18]. The clearing prices of these allowances in the spot market were lower than anticipated. These market-clearing prices dropped from \$150 per allowance in August, 1984 to \$70 in March, 1996 and then jumped back to \$189 in August, 1998. For information on the monthly average price of sulfur dioxide allowances under the acid rain program, the reader may access EPA's web site at *http://www.epa.gov/acidrain/ats/prices.html.*

The models also deal explicitly with spatial differentiation through the use of a diffusion matrix that maps emissions from sources to receptor points that are dispersed in space. In an APS approach, a pollution dispersion matrix is required. An APS approach, as opposed to an EPS approach, is preferable in the case of pollutants classified as nonuniformly mixed assimilative pollutants. Assimilative pollutants are called so since the capacity to absorb them is rather large; by uniformly mixed is meant that the ambient concentration depends on the total amount of emissions but not on the distribution of these emissions among the sources (cf. Tietenberg [18]). In the case of a large region, as is the focus in this paper, nonuniformity is a reasonable assumption.

In this paper, we develop both static and dynamic models for oligopolistic markets with marketable pollution permits and noncompliant behavior. Importantly, the set of solutions to the variational inequality governing the equilibrium conditions coincides with the set of stationary points of the projected dynamical system. In addition, we generalize the oligopolistic model presented in Nagurney and Dhanda [10] not only to include noncompliance but also introduce the dynamics.

The paper is organized as follows. In section 2 we develop the optimization problem faced by the individual oligopolistic firm that might exhibit noncompliant behavior. Subsequently, we present the economic conditions governing the market model and then derive the variational inequality formulation of the equilibrium conditions. Moreover, for completeness and easy reference, we provide the multiproduct, multipollutant oligopolistic model where there is no noncompliance. We then provide certain qualitative properties of the equilibrium pattern for the model with noncompliance and include some sensitivity analysis results.

In section 3 we propose the dynamic model and present some qualitative properties. In section 4 we propose an algorithm for the discretization of the dynamic adjustment process and for the computation of the profit-maximized quantities of the multiple products, the optimal quantities of the various emissions, the equilibrium noncompliant overflows and underflows, the equilibrium allocation of the licenses, and the shadow prices as well as the license prices. We also provide convergence results. The algorithm, the Euler method, yields subproblems of very simple structure, each of which can be solved explicitly and in closed form. This algorithm is then applied to compute solutions to several numerical examples in section 5. Lastly, we summarize our results and present conclusions in section 6.

2. The static noncompliant oligopolistic market equilibrium model

In this section we develop the multiproduct, multipollutant market equilibrium model with ambient-based pollution permits in which the firms may engage in noncompliant behavior but with associated penalties for noncompliance imposed by the regulatory agencies. Here we present the variational inequality formulation of the governing equilibrium conditions and focus on questions surrounding the equilibrium state. In section 3 we then turn our attention to the dynamics and disequilibrium behavior.

We consider m firms that are also sources of industrial pollution in the region, with a typical source denoted by i. There are n receptor points, with a typical receptor point denoted by j. Also, let there be r different pollutants emitted by the sources, with a typical pollutant denoted by t. Let e_i^t denote the amount of pollutant t emitted by firm i and group the firm's emissions into a vector $e_i \in R_+^r$. We group then all the firms' emissions into the vector $e \in R_+^{mr}$.

We assume (cf. Montgomery [8]), as given, an $r \times m \times n$ diffusion matrix H, where the component h_{ij}^t denotes the contribution that one unit of emission by source i makes to average pollutant concentration of type t at receptor point j.

Let a permit denote a license, the possession of which will allow a source to pollute a specific pollutant at some specific receptor point. Each polluter, hence, will have to hold a portfolio of licenses to cover all of the relevant monitored receptor points. Let l_{ij}^t denote the number of licenses for pollutant t at point j held by source i and group the licenses for each firm i into a vector $l_i \in R_+^{rn}$. We assume throughout that some initial allocation of licenses l_{ij}^{t0}, $i = 1, \ldots, m$, $j = 1, \ldots, n$, $t = 1, \ldots, r$, has been made by the regulatory agency and we group this initial allocation of licenses into a vector $l_i^0 \in R_+^{rn}$. We then further group all the licenses of the firms into the vector $l \in R_+^{mrn}$.

We now discuss the variables used in the modeling of noncompliance. We let δ_{ij}^{t+} denote the possible overflow of pollution by firm i of pollutant t at point j as mandated by its license holdings for that point and that pollutant. We let δ_{ij}^{t-} denote the possible underflow of pollution by firm i of pollutant t at point j. In other words, firm i may not pollute to the level allowed by its license holdings if $\delta_{ij}^{t-} > 0$. We group the overflows for each firm i into the vector $\delta_i^+ \in R_+^{rn}$ and the underflows into the vector $\delta_i^- \in R_+^{rn}$. We then group the overflows and the underflows for all the firms into the vectors $\delta^+ \in R_+^{mrn}$ and $\delta^- \in R_+^{mrn}$.

We let p_j^t denote the price of the license for pollutant t that affects receptor point j and group the prices of the licenses into the vector $p \in R_+^{rn}$. Also, as discussed in the introduction, we assume that the market in pollution licenses is perfectly competitive, that is, each source of pollution takes the price of the license to pollute at a specific point as given and cannot affect the price itself.

Let there be s distinct products that are produced by the firms in a noncooperative manner, with a typical output denoted by d and the quantity of product d produced by firm i denoted by q_{id}. These quantities are first grouped into the vector $q_i \in R_+^s$ and then further into the vector $q \in R_+^{ms}$. Each of the products is assumed to be homogeneous, that is, the consumers are indifferent as to which firm was the producer.

The fundamental idea behind the market in pollution permits is that firms or sources of pollution must be encouraged to trade permits. A typical firm participating in a permit market has to take into account various costs, such as the cost of production, the cost of emission-abatement, the cost of purchasing pollution licenses, and, finally, the costs involved in noncompliance, in the presence of such behavior.

Each firm i in the oligopoly is faced with a cost f_i of producing the vector of quantities q_i where

$$f_i = f_i(q_i), \tag{1}$$

that is, this cost is dependent upon the vector of quantities produced by the firm.

Each firm i in the region is faced with a cost G_i of emitting the vector of emissions e_i where

$$G_i = G_i(e_i, q_i). \tag{2}$$

This cost is dependent upon the vector of emissions and the vector of production quantities.

Each firm i, if it engages in noncompliant behavior, is penalized by a unit amount M_{ij}^t, for each unit overflow of pollutant t at point j, with M_{ij}^t assumed to be positive. The total cost associated with noncompliance is, hence, given by

$$\sum_{t=1}^{r}\sum_{j=1}^{n} M_{ij}^t \delta_{ij}^{t+}. \tag{3}$$

We note that perfect monitoring of the emitters is implicitly assumed by expression (3) in order to be able to assess the total penalty to be paid by a particular firm due to possible overemissions.

Since we assume that the firms are oligopolistic in their output markets, they affect the prices of their outputs. Hence, if we denote the price of product d by ρ_d, we can write

$$\rho_d = \rho_d\left(\sum_{i=1}^{m} q_{id}\right). \tag{4}$$

Consequently, each firm i acquires a revenue of

$$\sum_{d=1}^{s} \rho_d\left(\sum_{i=1}^{m} q_{id}\right) q_{id}. \tag{5}$$

Since the regulatory body bestows upon the firm i a portfolio of an initial allocation of licenses, l_i^0, the value of a firm's initial endowment of licenses is given by $\sum_{j=1}^{n}\sum_{t=1}^{r} p_j^{t*} l_{ij}^{t0}$, where p_j^{t*} denotes the given price of a license to pollute for the specific pollutant t at receptor point j, which, under the assumption of perfect competition in the license markets, is assumed given.

Also, the cost of purchasing licenses for a specific pollutant t that affects receptor point j for source i is given by

$$\sum_{j=1}^{n} p_j^{t*}\left(l_{ij}^t - l_{ij}^{t0}\right). \tag{6}$$

We assume that each firm in the oligopoly is profit-maximizing and, thus, can be characterized by a utility function that measures its profit or net revenue. The utility function u_i faced by each such firm i, $i = 1, \dots, m$, can, hence, be expressed as the difference between total revenue acquired and the total cost incurred by the firm:

$$u_i = u_i(q, e_i, \delta_i^+, \delta_i^-, l_i) = \sum_{d=1}^{s}\left(\rho_d\left(\sum_{i=1}^{m} q_{id}\right) q_{id}\right) - f_i(q_i) - G_i(e_i, q_i)$$
$$- \sum_{t=1}^{r}\sum_{j=1}^{n} M_{ij}^t \delta_{ij}^{t+} - \sum_{t=1}^{r}\sum_{j=1}^{n} p_j^{t*}\left(l_{ij}^t - l_{ij}^{t0}\right). \tag{7}$$

An oligopolistic firm's optimization problem is then expressed as

$$\text{Maximize} \quad u_i(q, e_i, \delta_i^+, \delta_i^-, l_i) \tag{8}$$
$$\text{subject to} \quad h_{ij}^t e_i^t + \delta_{ij}^{t-} - \delta_{ij}^{t+} = l_{ij}^t, \quad t = 1, \dots, r, \ j = 1, \dots, n, \tag{9}$$

and the nonnegativity constraints

$$q_{id} \geqslant 0, \quad e_i^t \geqslant 0, \quad \delta_{ij}^{t-} \geqslant 0, \quad \delta_{ij}^{t+} \geqslant 0, \quad l_{ij}^t \geqslant 0,$$
$$d = 1, \dots, s, \ t = 1, \dots, r, \ j = 1, \dots, n. \tag{10}$$

Constraint (9) states that each firm's license holding for a particular point and a particular pollutant is equal to the average rate of emission plus a possible underflow minus a possible overflow, due to noncompliance. Once can also establish that, in equilibrium, one can only have a positive overflow or underflow of a particular pollutant at a particular point, that is, $\delta_{ij}^{t+} \times \delta_{ij}^{t-} = 0$, for all i, j, t (see, e.g., [13]).

Let λ_{ij}^t denote the Lagrange multiplier associated with the tj-th constraint of the form (9), and let λ_i denote the vector of Lagrange multipliers in R^{rn}. Finally, group the vectors λ_i into the vector $\lambda \in R^{mrn}$. Note that λ_{ij}^t may be interpreted as a shadow price and, henceforth, we, interchangeably, refer to this Lagrange multiplier as the marginal abatement cost.

We define the feasible set $K_i \equiv \{q_i \in R_+^s, e_i \in R_+^r, \delta_i^+ \in R_+^{rn}, \delta_i^- \in R_+^{rn}, l_i \in R_+^{rn}, \lambda_i \in R^{rn}\}$, such that (9) is satisfied, and the feasible set K for all firms as the Cartesian product: $K \equiv \prod_{i=1}^m K_i$.

As stated earlier, the oligopolistic firms are assumed to operate in a noncooperative manner in their product markets, where the governing equilibrium concept is that of Nash–Cournot (cf. Nash [16], Cournot [1]), and is defined as follows.

Definition 1. A Nash equilibrium is a vector of production outputs $q^* \in R_+^{ms}$, emission quantities $e^* \in R_+^{mr}$, noncompliant overflows $\delta^+ \in R_+^{mrn}$, underflows $\delta^- \in R_+^{mrn}$, and licenses $l^* \in R_+^{rmn}$, such that

$$u_i\left(q_i^*, \hat{q}_i^*, e_i^*, \delta_i^{+*}, \delta_i^{-*}, l_i^*\right) \geqslant u_i\left(q_i, \hat{q}_i^*, e_i, \delta_i^+, \delta_i^-, l_i\right),$$
$$\forall q_i \in R_+^s, \ \forall e_i \in R_+^r, \ \forall \delta_i^+ \in R_+^{rn}, \ \forall \delta_i^- \in R_+^{rn}, \ \forall l_i \in R_+^{rn},$$

satisfying (9) for all firms i, (11)

where $\hat{q}_i^* = (q_1^*, \ldots, q_{i-1}^*, q_{i+1}^*, \ldots, q_m^*)$.

In other words, the rationality postulate here is that each firm selects its production outputs so that its utility is maximized, given the production output vector decisions of other oligopolistic firms, and also its emissions, its overflows and underflows, and licenses.

Optimality condition for a firm

If we assume that the utility function, $u_i(q, e_i, \delta_i^+, \delta_i^-, l_i)$, for each firm i is concave with respect to its arguments, and is continuously differentiable, the necessary and sufficient conditions for an optimal firm-specific product, emission, noncompliant overflow and underflow, license, and marginal abatement cost pattern, $(q_i^*, e_i^*, \delta_i^{+*}, \delta_i^{-*}, l_i^*, \lambda_i^*)$, given p^*, is that this pattern lies in K_i, and satisfies the inequality:

$$\begin{aligned}
&\sum_{d=1}^{s}\left[\frac{\partial f_i(q_i^*)}{\partial q_{id}} + \frac{\partial G_i(e_i^*, q_i^*)}{\partial q_{id}} - \frac{\partial \rho_d(\sum_{i=1}^m q_{id}^*)}{\partial q_{id}} q_{id}^* - \rho_d\left(\sum_{i=1}^{m} q_{id}^*\right)\right] \times \left[q_{id} - q_{id}^*\right] \\
&\quad + \sum_{t=1}^{r}\left[\frac{\partial G_i(e_i^*, q_i^*)}{\partial e_i^t} + \sum_{j=1}^{n} \lambda_{ij}^{t*} h_{ij}^t\right] \times \left[e_i^t - e_i^{t*}\right] \\
&\quad + \sum_{t=1}^{r}\sum_{j=1}^{n}\left[M_{ij}^t - \lambda_{ij}^{t*}\right] \times \left[\delta_{ij}^{t+} - \delta_{ij}^{t+*}\right] + \sum_{t=1}^{r}\sum_{j=1}^{n}\left[\lambda_{ij}^{t*}\right] \times \left[\delta_{ij}^{t-} - \delta_{ij}^{t-*}\right] \\
&\quad + \sum_{t=1}^{r}\sum_{j=1}^{n}\left[p_j^{t*} - \lambda_{ij}^{t*}\right] \times \left[l_{ij}^t - l_{ij}^{t*}\right] \\
&\quad + \sum_{t=1}^{r}\sum_{j=1}^{n}\left[l_{ij}^{t*} - h_{ij}^t e_i^{t*} - \delta_{ij}^{t-*} + \delta_{ij}^{t+*}\right] \times \left[\lambda_{ij}^t - \lambda_{ij}^{t*}\right] \geqslant 0, \qquad (12) \\
&\quad \forall \bigl(q_i, e_i, \delta_i^+, \delta_i^-, l_i, \lambda_i\bigr) \in K_i.
\end{aligned}$$

Note that an inequality similar to (12) needs to hold for each of the other oligopolistic firms (see also, e.g., Gabay and Moulin [3]).

Market clearing conditions for licenses

We now describe the system of equalities and inequalities governing the quantities and prices of licenses in the region at equilibrium.

Mathematically, the economic system conditions governing market clearance in pollution permits are: for each pollutant t, $t = 1, \dots, r$, and for each receptor point j, $j = 1, \dots, n$:

$$\sum_{i=1}^{m} \left[l_{ij}^{t0} - l_{ij}^{t*} \right] \begin{cases} = 0, & \text{if } p_j^{t*} > 0, \\ \geqslant 0, & \text{if } p_j^{t*} = 0. \end{cases} \tag{13}$$

The system (13) states that: if the price of a license for pollutant t at a point j is positive, then the market for licenses at that point must clear; if there is an excess supply of licenses for a particular pollutant t at a receptor point, then the price of a license at that point must be zero.

We define the feasible set $\mathcal{K}$ for the entire system as: $\mathcal{K} \equiv K \times S$, where $S \equiv \{p \mid p \in R_+^{rn}\}$.

Definition 2. A vector of production outputs, emissions, noncompliance overflows and underflows, licenses, associated marginal costs of abatement, and license prices, $(q^*, e^*, \delta^{+*}, \delta^{-*}, l^*, \lambda^*, p^*) \in \mathcal{K}$, is an equilibrium of the multiproduct, multipollutant oligopoly with ambient-based pollution permits developed above if and only if it satisfies inequality (12) for all firms: $i = 1, \dots, m$, and the system of equalities and inequalities (13) for all receptor points: $j = 1, \dots, n$, and for all pollutants: $t = 1, \dots, r$.

We now derive the variational inequality formulation of the equilibrium conditions for the above market model.

Theorem 3. A vector of firm production outputs, emissions, noncompliance overflows and underflows, licenses, shadow prices, and license prices, $(q^*, e^*, \delta^{+*}, \delta^{-*}, l^*, \lambda^*, p^*) \in \mathcal{K}$, is an equilibrium of the model with noncompliance if and only if it satisfies the variational inequality problem:

$$\begin{aligned}
&\sum_{i=1}^{m} \sum_{d=1}^{s} \left[\frac{\partial f_i(q_i^*)}{\partial q_{id}} + \frac{\partial G_i(e_i^*, q_i^*)}{\partial q_{id}} - \frac{\partial \rho_d(\sum_{i=1}^{m} q_{id}^*)}{\partial q_{id}} q_{id}^* - \rho_d\left(\sum_{i=1}^{m} q_{id}^* \right) \right] \times \left[q_{id} - q_{id}^* \right] \\
&\quad + \sum_{i=1}^{m} \sum_{t=1}^{r} \left[\frac{\partial G_i(e_i^*, q_i^*)}{\partial e_i^t} + \sum_{j=1}^{n} \lambda_{ij}^{t*} h_{ij}^t \right] \times \left[e_i^t - e_i^{t*} \right] \\
&\quad + \sum_{i=1}^{m} \sum_{t=1}^{r} \sum_{j=1}^{n} \left[M_{ij}^t - \lambda_{ij}^{t*} \right] \times \left[\delta_{ij}^{t+} - \delta_{ij}^{t+*} \right] + \sum_{i=1}^{m} \sum_{t=1}^{r} \sum_{j=1}^{n} \left[\lambda_{ij}^{t*} \right] \times \left[\delta_{ij}^{t-} - \delta_{ij}^{t-*} \right] \\
&\quad + \sum_{i=1}^{m} \sum_{t=1}^{r} \sum_{j=1}^{n} \left[p_j^{t*} - \lambda_{ij}^{t*} \right] \times \left[l_{ij}^t - l_{ij}^{t*} \right]
\end{aligned}$$

$$+\sum_{i=1}^{m}\sum_{t=1}^{r}\sum_{j=1}^{n}\left[l_{ij}^{t*}-h_{ij}^{t}e_{i}^{t*}-\delta_{ij}^{t-*}+\delta_{ij}^{t+*}\right]\times\left[\lambda_{ij}^{t}-\lambda_{ij}^{t*}\right]$$
$$+\sum_{t=1}^{r}\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\left(l_{ij}^{t0}-l_{ij}^{t*}\right)\right]\times\left[p_{j}^{t}-p_{j}^{t*}\right]\geqslant 0,\quad\forall(q,e,\delta^{+},\delta^{-},l,\lambda,p)\in\mathcal{K}.\quad(14)$$

Proof. Assume that $(q^*, e^*, \delta^{+*}, \delta^{-*}, l^*, \lambda^*, p^*) \in \mathcal{K}$ is an equilibrium. Note that inequality (12) holds for all firms $i = 1, \dots, m$. Hence, summing (12) over all firms, we obtain

$$\sum_{i=1}^{m}\sum_{d=1}^{s}\left[\frac{\partial f_i(q_i^*)}{\partial q_{id}}+\frac{\partial G_i(e_i^*,q_i^*)}{\partial q_{id}}-\frac{\partial\rho_d(\sum_{i=1}^{m}q_{id}^*)}{\partial q_{id}}q_{id}^*-\rho_d\left(\sum_{i=1}^{m}q_{id}^*\right)\right]\times\left[q_{id}-q_{id}^*\right]$$
$$+\sum_{i=1}^{m}\sum_{t=1}^{r}\left[\frac{\partial G_i(e_i^*,q_i^*)}{\partial e_i^t}+\sum_{j=1}^{n}\lambda_{ij}^{t*}h_{ij}^{t}\right]\times\left[e_i^t-e_i^{t*}\right]$$
$$+\sum_{i=1}^{m}\sum_{t=1}^{r}\sum_{j=1}^{n}\left[M_{ij}^{t}-\lambda_{ij}^{t*}\right]\times\left[\delta_{ij}^{t+}-\delta_{ij}^{t+*}\right]+\sum_{i=1}^{m}\sum_{t=1}^{r}\sum_{j=1}^{n}\left[\lambda_{ij}^{t*}\right]\times\left[\delta_{ij}^{t-}-\delta_{ij}^{t-*}\right]$$
$$+\sum_{i=1}^{m}\sum_{t=1}^{r}\sum_{j=1}^{n}\left[p_j^{t*}-\lambda_{ij}^{t*}\right]\times\left[l_{ij}^{t}-l_{ij}^{t*}\right]$$
$$+\sum_{i=1}^{m}\sum_{t=1}^{r}\sum_{j=1}^{n}\left[l_{ij}^{t*}-h_{ij}^{t}e_{i}^{t*}-\delta_{ij}^{t-*}+\delta_{ij}^{t+*}\right]\times\left[\lambda_{ij}^{t}-\lambda_{ij}^{t*}\right]\geqslant 0,$$
$$\forall(q,e,\delta^{+},\delta^{-},l,\lambda)\in K.\quad(15)$$

Also, from the system (13) we can conclude that the equilibrium must satisfy

$$\sum_{t=1}^{r}\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\left(l_{ij}^{t0}-l_{ij}^{t*}\right)\right]\times\left[p_{j}^{t}-p_{j}^{t*}\right]\geqslant 0,\quad\forall p\in S.\quad(16)$$

Finally, summing (15) and (16), one obtains variational inequality (14).

We now establish the converse of the proof, that is, the solution to (14) also satisfies (12) and (13).

Let $(q^*, e^*, \delta^{+*}, \delta^{-*}, l^*, \lambda^*, p^*) \in \mathcal{K}$ be a solution of (14). If one lets $q_{id} = q_{id}^*$ for all i, d; $e_i^t = e_i^{t*}$ for all i, t; $\delta_{ij}^{t+} = \delta_{ij}^{t+*}$ for all i, j, t; $\delta_{ij}^{t-} = \delta_{ij}^{t-*}$ for all i, j, t; $l_{ij}^t = l_{ij}^{t*}$ for all i, j, t; $\lambda_{ij}^t = \lambda_{ij}^{t*}$ for all i, j, t, and substitutes these values into (14), one obtains

$$\sum_{t=1}^{r}\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\left(l_{ij}^{t0}-l_{ij}^{t*}\right)\right]\times\left[p_{j}^{t}-p_{j}^{t*}\right]\geqslant 0,\quad\forall p\in S,\quad(17)$$

which implies the system conditions (13).

Similarly, if one lets $p_j^t = p_j^{t*}$ for all j, t, and substitutes these values into (14), one obtains

$$\sum_{i=1}^{m}\sum_{d=1}^{s}\left[\frac{\partial f_i(q_i^*)}{\partial q_{id}} + \frac{\partial G_i(e_i^*, q_i^*)}{\partial q_{id}} - \frac{\partial \rho_d(\sum_{i=1}^{m} q_{id}^*)}{\partial q_{id}} q_{id}^* - \rho_d\left(\sum_{i=1}^{m} q_{id}^*\right)\right] \times \left[q_{id} - q_{id}^*\right]$$
$$+\sum_{i=1}^{m}\sum_{t=1}^{r}\left[\frac{\partial G_i(e_i^*, q_i^*)}{\partial e_i^t} + \sum_{j=1}^{n}\lambda_{ij}^{t*}h_{ij}^t\right] \times \left[e_i^t - e_i^{t*}\right]$$
$$+\sum_{i=1}^{m}\sum_{t=1}^{r}\sum_{j=1}^{n}\left[M_{ij}^t - \lambda_{ij}^{t*}\right] \times \left[\delta_{ij}^{t+} - \delta_{ij}^{t+^*}\right] + \sum_{i=1}^{m}\sum_{t=1}^{r}\sum_{j=1}^{n}\left[\lambda_{ij}^{t*}\right] \times \left[\delta_{ij}^{t-} - \delta_{ij}^{t-^*}\right]$$
$$+\sum_{i=1}^{m}\sum_{t=1}^{r}\sum_{j=1}^{n}\left[p_j^{t*} - \lambda_{ij}^{t*}\right] \times \left[l_{ij}^t - l_{ij}^{t*}\right]$$
$$+\sum_{i=1}^{m}\sum_{t=1}^{r}\sum_{j=1}^{n}\left[l_{ij}^{t*} - h_{ij}^t e_i^{t*} - \delta_{ij}^{t-^*} + \delta_{ij}^{t+^*}\right] \times \left[\lambda_{ij}^t - \lambda_{ij}^{t*}\right] \geqslant 0,$$
$$\forall (q, e, \delta^+, \delta^-, l, \lambda) \in K, \tag{18}$$

which implies that (12) must hold for all the firms. The proof is complete. □

We now discuss the above model in relationship to another model that has appeared in the literature. In particular, if there is only a single pollutant and noncompliance is not considered, then the above model (and the variational inequality formulation) collapses to the single-product, single-pollutant oligopolistic model in Nagurney and Dhanda [10]. For completeness, we now present the generalization of that model to the case of multiple pollutants and multiple products. This model is a special case of the above model but in the absence of noncompliance.

The market equilibrium model with compliance

Note that, in the case, of compliance, constraint (9) takes the form

$$h_{ij}^t e_i^t \leqslant l_{ij}^t, \quad t = 1, \ldots, r, \; j = 1, \ldots, n, \tag{19}$$

and the nonnegativity constraints (10) now become

$$q_{id} \geqslant 0, \quad e_i^t \geqslant 0, \quad l_{ij}^t \geqslant 0, \tag{20}$$
$$d = 1, \ldots, s, \; t = 1, \ldots, r, \; j = 1, \ldots, n.$$

The utility function for a firm i (cf. (7)), in the case of compliance, and denoted by $\tilde{u}_i$, now takes the form

$$\tilde{u}_i = \tilde{u}_i(q, e_i, l_i) = \sum_{d=1}^{s}\left(\rho_d\left(\sum_{i=1}^{m} q_{id}\right) q_{id}\right) - f_i(q_i) - G_i(e_i, q_i) - \sum_{t=1}^{r}\sum_{j=1}^{n} p_j^{t*}\left(l_{ij}^t - l_{ij}^{t0}\right). \tag{21}$$

The optimization problem, hence, in this case, for firm i, is given by

$$\text{Maximize} \quad \tilde{u}_i(q_i, e_i, l_i) \tag{22}$$
$$\text{subject to (19) and (20).}$$

Note that a similar optimization problem is faced by each firm in the oligopolistic market in the case of compliance.

The market equilibrium conditions (13) are still relevant.

The governing equilibrium conditions for the entire system, hence, consist of the optimality conditions holding for each firm i and the market equilibrium conditions (13) holding for each pollutant t and receptor point j.

If we let $\tilde{\lambda}_{ij}^t$ denote the Lagrange multiplier associated with the jt-th constraint in (19) and we assume that the utility functions are concave in their arguments and continuously differentiable and that the behavior of the firms in the oligopoly is noncooperative and governed by a Nash–Cournot equilibrium, then the variational inequality formulation of the governing equilibrium conditions is given by:

Corollary 4. A vector of firm production outputs, emissions, licenses, shadow prices, and license prices, $(q^*, e^*, l^*, \tilde{\lambda}^*, p^*) \in R_+^{ms+rm+2mrn+rn}$, is an equilibrium of the model with compliance if and only if it satisfies the variational inequality problem:

$$\begin{aligned}
&\sum_{i=1}^{m}\sum_{d=1}^{s}\left[\frac{\partial f_i(q_i^*)}{\partial q_{id}} + \frac{\partial G_i(e_i^*, q_i^*)}{\partial q_{id}} - \frac{\partial \rho_d(\sum_{i=1}^m q_{id}^*)}{\partial q_{id}} q_{id}^* - \rho_d\left(\sum_{i=1}^m q_{id}^*\right)\right] \times \left[q_{id} - q_{id}^*\right] \\
&\quad + \sum_{i=1}^{m}\sum_{t=1}^{r}\left[\frac{\partial G_i(e_i^*, q_i^*)}{\partial e_i^t} + \sum_{j=1}^{n} \tilde{\lambda}_{ij}^{t*} h_{ij}^t\right] \times \left[e_i^t - e_i^{t*}\right] \\
&\quad + \sum_{i=1}^{m}\sum_{t=1}^{r}\sum_{j=1}^{n}\left[p_j^{t*} - \tilde{\lambda}_{ij}^{t*}\right] \times \left[l_{ij}^t - l_{ij}^{t*}\right] + \sum_{i=1}^{m}\sum_{t=1}^{r}\sum_{j=1}^{n}\left[l_{ij}^{t*} - h_{ij}^t e_i^{t*}\right] \times \left[\tilde{\lambda}_{ij}^t - \tilde{\lambda}_{ij}^{t*}\right] \\
&\quad + \sum_{t=1}^{r}\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\left(l_{ij}^{t0} - l_{ij}^{t*}\right)\right] \times \left[p_j^t - p_j^{t*}\right] \geqslant 0, \\
&\quad \forall (q, e, l, \tilde{\lambda}, p) \in R_+^{ms+mr+2mrn+rn}.
\end{aligned} \tag{23}$$

We first put variational inequality (14) into standard form and then variational inequality (23) (cf. Nagurney [9]). Define the column vectors: $X \equiv (q, e, \delta^+, \delta^-, l, \lambda, p) \in \mathcal{K}$ and $F(X) \equiv (G(X), E(X), \Delta^+(X), \Delta^-(X), L(X), \Lambda(X), P(X))$, where $G(X)$ is the ms-dimensional vector with component id given by: $-\partial u_i/\partial q_{id}$, $E(X)$ is the mr-dimensional vector with component it given by: $-\partial u_i/\partial e_i^t + \sum_{j=1}^n \lambda_{ij}^t h_{ij}^t$, $\Delta^+(X)$ is the mrn-dimensional vector with component ijt given by: $M_{ij}^t - \lambda_{ij}^t$, $\Delta^-(X)$ is the mrn-dimensional vector with component ijt given by: λ_{ij}^t, $L(X)$ is the mnr-dimensional vector with component ijt given by: $p_j^t - \lambda_{ij}^t$, $\Lambda(X)$ is the mnr-dimensional vector with component ijt given by: $l_{ij}^t - h_{ij}^t e_i^t - \delta_{ij}^{t-} + \delta_{ij}^{t+}$, and $P(X)$ is the nr-dimensional vector with jt-th component given by: $\sum_{i=1}^m (l_{ij}^{t0} - l_{ij}^t)$.

Variational inequality (14) can now be expressed as

$$F(X^*)^{\mathrm{T}} \cdot (X - X^*) \geqslant 0, \quad \forall X \in \mathcal{K}, \tag{24}$$

where $\cdot$ denotes the inner product in R^N.

Variational inequality (24) is referred to as $\mathrm{VI}(F, \mathcal{K})$.

We now put variational inequality (23) into standard form. We redefine the column vectors: X as $X \equiv (q, e, l, \tilde{\lambda}, p) \in R_+^{ms+mr+2mrn+rn}$ and $F(X) \equiv (\tilde{G}(X), \tilde{E}(X), \tilde{L}(X), \tilde{\Lambda}(X), P(X))$, where $\tilde{G}(X)$ is the ms-dimensional vector with component id given by: $-\partial\tilde{u}_i/\partial q_{id}$, $E(X)$ is the mr-dimensional vector with component it given by: $-\partial\tilde{u}_i/\partial e_i^t + \sum_{j=1}^n \tilde{\lambda}_{ij}^t h_{ij}^t$, $\tilde{L}(X)$ is the mnr-dimensional vector with component ijt given by: $p_j^t - \tilde{\lambda}_{ij}^t$, $\tilde{\Lambda}(X)$ is the mnr-dimensional vector with component ijt given by: $l_{ij}^t - h_{ij}^t e_i^t$, and $P(X)$ is as defined previously.

In a manner analogous to the corresponding proofs in Nagurney and Dhanda [10], one can establish that, in the case of the model with compliance, the equilibrium pattern is independent of the initial license allocation, provided that the sum of licenses for each pollutant and receptor point is fixed. Moreover, in this case, environmental standards can be achieved, provided that the sum of initial licenses for each receptor point and pollutant is equal to the imposed environmental quality standard for that receptor point and that pollutant.

2.1. *Qualitative properties*

In this subsection we investigate certain qualitative properties of the equilibrium. In particular, we establish certain properties of the function $F(X)$, which are useful also in the subsequent analysis of the dynamic model presented in section 3, state an existence result, and present some sensitivity analysis results.

Lemma 5. If the utility functions u_i are concave for each firm i, then $F(X)$ is monotone.

Proof. We will now establish the monotonicity of $F(X)$, that is, that

$$\left[F(X^1) - F(X^2)\right]^{\mathrm{T}} \cdot \left[X^1 - X^2\right] \geqslant 0, \quad \forall X^1, X^2 \in \mathcal{K}. \tag{25}$$

In view of the definition of $F(X)$ in the above model, (25) takes the form:

$$\sum_{i=1}^m \sum_{d=1}^s -\left[\frac{\partial u_i(q^1, e_i^1, \delta_i^{+1}, \delta_i^{-1}, l_i^1)}{\partial q_{id}} - \frac{\partial u_i(q^2, e_i^2, \delta_i^{+2}, \delta_i^{-2}, l_i^2)}{\partial q_{id}}\right] \times \left[q_{id}^1 - q_{id}^2\right]$$

$$+\sum_{i=1}^{m}\sum_{t=1}^{r}\left[\left(-\frac{\partial u_i(q^1,e_i^1,\delta_i^{+1},\delta_i^{-1},l_i^1)}{\partial e_i^t}+\sum_{j=1}^{n}\lambda_{ij}^{t1}h_{ij}^t\right)\right]$$
$$-\left[\left(-\frac{\partial u_i(q^2,e_i^2,\delta_i^{+2},\delta_i^{-2},l_i^2)}{\partial e_i^t}+\sum_{j=1}^{n}\lambda_{ij}^{t2}h_{ij}^t\right)\right]\times\left[e_i^{t1}-e_i^{t2}\right]$$
$$+\sum_{i=1}^{m}\sum_{t=1}^{r}\sum_{j=1}^{n}\left[(M_{ij}^t-\lambda_{ij}^{t1})-(M_{ij}^t-\lambda_{ij}^{t2})\right]\times\left[\delta_{ij}^{t+1}-\delta_{ij}^{t+2}\right]$$
$$+\sum_{i=1}^{m}\sum_{t=1}^{r}\sum_{j=1}^{n}\left[\lambda_{ij}^{t1}-\lambda_{ij}^{t2}\right]\times\left[\delta_{ij}^{t-1}-\delta_{ij}^{t-2}\right]$$
$$+\sum_{i=1}^{m}\sum_{t=1}^{r}\sum_{j=1}^{n}\left[(p_j^{t1}-\lambda_{ij}^{t1})-(p_j^{t2}-\lambda_{ij}^{t2})\right]\times\left[l_{ij}^{t1}-l_{ij}^{t2}\right]$$
$$+\sum_{i=1}^{m}\sum_{t=1}^{r}\sum_{j=1}^{n}\left[(l_{ij}^{t1}-h_{ij}^te_i^{t1}-\delta_{ij}^{t-1}+\delta_{ij}^{t+1})-(l_{ij}^{t2}-h_{ij}^te_i^{t2}-\delta_{ij}^{t-2}+\delta_{ij}^{t+2})\right]$$
$$\times\left[\lambda_{ij}^{t1}-\lambda_{ij}^{t2}\right]+\sum_{t=1}^{r}\sum_{j=1}^{n}\left[\sum_{i=1}^{m}(l_{ij}^{t0}-l_{ij}^{t1})-(l_{ij}^{t0}-l_{ij}^{t2})\right]\times\left[p_j^{t1}-p_j^{t2}\right]. \quad (26)$$

After combining and simplifying terms, the expression (26) reduces to

$$\sum_{i=1}^{m}\sum_{d=1}^{s}-\left[\frac{\partial u_i(q^1,e_i^1,\delta_i^{+1},\delta_i^{+1},l_i^1)}{\partial q_{id}}-\frac{\partial u_i(q^2,e_i^2,\delta_i^{+2},\delta_i^{-2}l_i^2)}{\partial q_{id}}\right]\times\left[q_{id}^1-q_{id}^2\right]$$
$$+\sum_{i=1}^{m}\sum_{t=1}^{r}-\left[\frac{\partial u_i(q^1,e_i^1,\delta_i^{+1},\delta_i^{-1},l_i^1)}{\partial e_i^t}-\frac{\partial u_i(q^2,e_i^2,\delta_i^{+2},\delta_i^{-2},l_i^2)}{\partial e_i^t}\right]\times\left[e_i^{t1}-e_i^{t2}\right]. \quad (27)$$

But under the assumption that the utility functions are concave, we know that minus the gradient of the utility function is monotone (cf. Nagurney [9]) and, hence, the expression in (27) must be greater than or equal to zero. The proof is complete. □

Lemma 6. The function $F(X)$ is Lipschitz continuous, that is, there exists a positive constant L, such that

$$\|F(X^1)-F(X^2)\|\leqslant L\|X^1-X^2\|,\quad \forall X^1,X^2\in\mathcal{K}, \quad (28)$$

under the assumption that the utility functions have bounded second-order derivatives.

Proof. Follows from the same arguments as the proof of lemma 8.1 in Nagurney [9]. □

We now provide an existence result, but first recall the definition of coercivity.

Definition 7. A function $F(X)$, from a feasible set $\mathcal{K}$ to R^N, is said to be coercive, if

$$\frac{(F(X) - F(X^1))^{\mathrm{T}} \cdot (X - X^1)}{\|X - X^1\|} \to \infty, \tag{29}$$

as $\|X\| \to \infty$ for $X \in \mathcal{K}$, and for some $X^1 \in \mathcal{K}$.

Theorem 8. If $(q^*, e^*, \delta^{+*}, \delta^{-*}, l^*, \lambda^*, p^*) \in \mathcal{K}$ satisfies variational inequality (14) then the equilibrium production, emission, and noncompliant overflow vector is a solution to the variational inequality problem:

$$\sum_{i=1}^{m}\sum_{d=1}^{s} -\frac{\partial u_i(q^*, e_i^*, \delta_i^{+*}, \delta_i^{-*}, l_i^*)}{\partial q_{id}} \times \left[q_{id} - q_{id}^*\right]$$
$$+\sum_{i=1}^{m}\sum_{t=1}^{r} -\frac{\partial u_i(q^*, e_i^*, \delta_i^{+*}, \delta_i^{-*}, l_i^*)}{\partial e_i^t} \times \left[e_i^t - e_i^{t*}\right]$$
$$+\sum_{i=1}^{m}\sum_{t=1}^{r}\sum_{j=1}^{n} \left[M_{ij}^t\right] \times \left[\delta_{ij}^{t+} - \delta_{ij}^{t+*}\right] \geqslant 0, \quad \forall (q, e, \delta^+, \delta^-) \in K^1, \tag{30}$$

where

$$K^1 = (q, e, \delta^+, \delta^-) \in R_+^{ms+mr+2mrn}, \quad h_{ij}^t e_i^t - \delta_{ij}^{t+} + \delta_{ij}^{t-} = l_{ij}^t,$$
$$\sum_{i=1}^{m} \left(l_{ij}^{t0} - l_{ij}^t\right) \geqslant 0, \quad \forall i, d, t, j. \tag{31}$$

A solution to (30) is guaranteed to exist provided that $-\nabla u(\cdot)$ is coercive (cf. [9]), where ∇u is the gradient of u. Moreover, if $(q^*, e^*, \delta^{+*}, \delta^{-*})$ is a solution to (34), there exist $l^* \in R_+^{rmn}$, $\lambda^* \in R_+^{rmn}$, and $p^* \in R_+^{rn}$ such that $(q^*, e^*, \delta^{+*}, \delta^{-*}, l^*, \lambda^*, p^*)$ is a solution to variational inequality (14), and, hence, an equilibrium.

Proof. Follows similar arguments as the existence proof given in theorem 4 of Nagurney and Dhanda [10]. □

As mentioned in the introduction, Van Egteren and Weber [19] stated that firms will cheat, that is, not comply, if the cost of purchasing a pollution permit to cover an additional unit of emission is higher than the marginal penalty of cheating. We now establish this situation rigorously in the context of our general model.

Corollary 9. If $\delta_{ij}^{t+*} > 0$, then $p_j^{t*} \geqslant M_{ij}^t$, that is, in the presence of an equilibrium noncompliant overflow, the unit penalty associated with noncompliance cannot exceed the equilibrium price of the corresponding license.

Proof. Consider the variational inequality problem (14) and construct the following feasible pattern: $q_{id} = q_{id}^*$, $e_i^t = e_i^{t*}$, $\delta_{ij}^{t-} = \delta_{ij}^{t*}$, $\lambda_{ij}^t = \lambda_{ij}^{t*}$, $p_j^t = p_j^{t*}$, for all i, d, j, t.

Let $\delta_{ij}^{t+} = \delta_{ij}^{t*}$ and $l_{ij}^{t} = l_{ij}^{t*}$, for all $i, j, t \neq k, l, \tau$.

Select a small $\varepsilon > 0$, and let $\delta_{kl}^{\tau+} = \delta_{kl}^{\tau+^*} - \varepsilon$ and $l_{kl}^{\tau} = l_{kl}{}^{\tau*} + \varepsilon$.

Making such a substitution into variational inequality (14) yields

$$\left[M_{kl}^{\tau} - \lambda_{kl}^{\tau*}\right] \times [-\varepsilon] + \left[p_l^{\tau*} - \lambda_{kl}^{\tau*}\right] \times [\varepsilon] \geqslant 0, \tag{32}$$

which implies that

$$p_l^{\tau*} \geqslant M_{kl}^{\tau}. \tag{33}$$

Note that this result holds for any firm i, receptor point j, and pollutant t, so the proof is complete. □

Subsequently, we now establish a sensitivity analysis result.

Corollary 10. Consider a change Δ_{kl}^{τ} to the fixed unit penalty cost term M_{kl}^{τ}, that is, the new penalty for noncompliance for firm k, receptor point l and pollutant τ is given by: $M_{kl}^{\tau} + \Delta_{kl}^{\tau}$. Let $(q^*, e^*, \delta^{+*}, \delta^{-*}, l^*, \lambda^*, p^*) \in \mathcal{K}$ denote the solution to variational inequality (14) before the change. Let $(q', e', \delta^{+'}, \delta^{-'}, l', \lambda', p') \in \mathcal{K}$ denote the solution to the analogous variational inequality problem after the cost perturbation.

Then

$$\text{if } \Delta_{kl}^{\tau} > 0, \quad \delta_{kl}^{\tau+^*} \geqslant \delta_{kl}^{\tau+'}, \tag{34}$$

and

$$\text{if } \Delta_{kl}^{\tau} < 0, \quad \delta_{kl}^{\tau+^*} \leqslant \delta_{kl}^{\tau+'}, \tag{35}$$

that is, if the penalty associated with noncompliance is increased, the equilibrium noncompliant overflow cannot increase; similarly, if the penalty is decreased, the equilibrium noncompliant flow cannot decrease.

Proof. Hence, we must have that

$$\sum_{i=1}^{m}\sum_{d=1}^{s}\left[\frac{\partial f_i(q_i')}{\partial q_{id}} + \frac{\partial G_i(e_i', q_i')}{\partial q_{id}} - \frac{\partial \rho_d(\sum_{i=1}^{m} q_{id}')}{\partial q_{id}} q_{id}^* - \rho_d\left(\sum_{i=1}^{m} q_{id}'\right)\right] \times \left[q_{id} - q_{id}^*\right]$$
$$+ \sum_{i=1}^{m}\sum_{t=1}^{r}\left[\frac{\partial G_i(e_i', q_i')}{\partial e_i^t} + \sum_{j=1}^{n} \lambda_{ij}^{t'} h_{ij}^t\right] \times \left[e_i^t - e_i^{t'}\right]$$
$$+ \sum_{\substack{i=1\\ i\neq k}}^{m}\sum_{\substack{t=1\\ t\neq\tau}}^{r}\sum_{\substack{j=1\\ j\neq l}}^{n}\left[M_{ij}^t - \lambda_{ij}^{t'}\right] \times \left[\delta_{ij}^{t+} - \delta_{ij}^{t+'}\right] + \left[M_{kl}^{\tau} + \Delta_{kl}^{\tau} - \lambda_{kl}^{\tau'}\right] \times \left[\delta_{kl}^{\tau+} - \delta_{kl}^{\tau+'}\right]$$
$$+ \sum_{i=1}^{m}\sum_{t=1}^{r}\sum_{j=1}^{n}\left[\lambda_{ij}^{t'}\right] \times \left[\delta_{ij}^{t-} - \delta_{ij}^{t-'}\right] + \sum_{i=1}^{m}\sum_{t=1}^{r}\sum_{j=1}^{n}\left[p_j^{t'} - \lambda_{ij}^{t'}\right] \times \left[l_{ij}^t - l_{ij}^{t'}\right]$$

$$+\sum_{t=1}^{r}\sum_{j=1}^{n}\left[l_{ij}^{t'} - h_{ij}^{t}e_i^{t'} - \delta_{ij}^{t-'} + \delta_{ij}^{t+'}\right] \times \left[\lambda_{ij}^{t} - \lambda_{ij}^{t'}\right]$$
$$+\sum_{t=1}^{r}\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\left(l_{ij}^{t0} - l_{ij}^{t'}\right)\right] \times \left[p_j^t - p_j^{t'}\right] \geqslant 0, \quad \forall (q, e, \delta^+, \delta^-, l, \lambda, p) \in \mathcal{K}. \tag{36}$$

Let $(q, e, \delta^+, \delta^-, l, \lambda, p) = (q', e', \delta^{+'}, \delta^{-'}, l', \lambda', p')$ and substitute this vector into variational inequality (14). Also, let $(q, e, \delta^+, \delta^-, l, \lambda, p) = (q^*, e^*, \delta^{+*}, \delta^{-*}, l^*, \lambda^*, p^*)$, and substitute the vector into (36). Adding both resulting inequalities, after some algebraic simplifications yields

$$\begin{aligned}
&\Delta_{kl}^{\mathcal{T}}\left[\delta_{ij}^{\mathcal{T}*} - \delta_{kl}^{\mathcal{T}'}\right] \\
&\geqslant \sum_{i=1}^{m}\sum_{d=1}^{s} -\left[\frac{\partial u_i(q', e_i', \delta_i^{+'}, \delta_i^{+'}, l_i')}{\partial q_{id}} - \frac{\partial u_i(q^*, e_i^*, \delta_i^{+*}, \delta_i^{-*}, l_i^*)}{\partial q_{id}}\right] \times \left[q_{id}' - q_{id}^*\right] \\
&+\sum_{i=1}^{m}\sum_{t=1}^{r} -\left[\frac{\partial u_i(q', e_i', \delta_i^{+'}, \delta_i^{-'}, l_i')}{\partial e_i^t} - \frac{\partial u_i(q^*, e_i^*, \delta_i^{+*}, \delta_i^{-*}, l_i^*)}{\partial e_i^t}\right] \times \left[e_i^{t'} - e_i^{t*}\right].
\end{aligned} \tag{37}$$

But the right-hand side of (37) must be greater than or equal to zero due to monotonicity of $F(X)$ established in (25). Both (34) and (35), therefore, hold true. □

3. The dynamic noncompliant oligopolistic market model

In this section we provide the dynamic counterpart of the static noncompliant oligopolistic market model developed in section 2. We first recall some basic definitions and results from projected dynamical systems. What is notable about projected dynamical systems theory is that it provides a rigorous foundation for the investigation of a class of dynamical system characterized by a discontinuous right-hand side, a characteristic that arises in numerous applications in which constraints are essential.

We first recall the definition of a projected dynamical system (PDS). The relationship between a projected dynamical system and its associated variational inequality problem with the same constraint set is then identified. For additional discussion and applications of this methodology, as well as proofs of the theoretical results in this section, we refer the reader to Dupuis and Nagurney [2], Zhang and Nagurney [20], and Nagurney and Zhang [15].

Definition 11. Given $X \in \mathcal{K}$ and $v \in R^n$, define the projection of the vector v at X (with respect to $\mathcal{K}$) by

$$\Pi_{\mathcal{K}}(X, v) = \lim_{d \to 0} \frac{(P_{\mathcal{K}}(X + dv) - X)}{d}, \tag{38}$$

where $P_{\mathcal{K}}$ is the standard projection defined as

$$y = P_{\mathcal{K}}(X) = \operatorname{argmin}_{z\in\mathcal{K}}\|X - z\|. \tag{39}$$

The class of ordinary differential equations that are of concern in this paper take on the following form:

$$\dot{X} = \Pi_{\mathcal{K}}\big(X, -F(X)\big), \tag{40}$$

where $\mathcal{K}$ is a closed convex set, as is the case in the noncompliant oligopoly model and $F(X)$ is a vector field defined on $\mathcal{K}$.

We note that Lipschitz continuity of $F(X)$ (cf. Nagurney and Zhang [15] and (28) provides a sufficient condition for the fundamental properties of projected dynamical systems in terms of the existence of a trajectory, subject to an initial condition, the uniqueness of that trajectory, and the continuous dependence of the solution on the initial value.

We note that the classical dynamical system, in contrast to (44), is of the form

$$\dot{X} = -F(X). \tag{41}$$

The following theorem makes a basic connection between the static world of finite-dimensional variational inequality problems (a common tool for the formulation and analysis of equilibrium problems) and the dynamic world of projected dynamical systems (a new class of dynamical system). It is due to Dupuis and Nagurney [2].

Theorem 12. Assume that $\mathcal{K}$ is a convex polyhedron. Then the equilibrium or stationary points of the PDS$(F,\mathcal{K})$ coincide with the solutions of the VI$(F,\mathcal{K})$. Hence, for $X^* \in \mathcal{K}$ and satisfying

$$0 = \Pi_{\mathcal{K}}\big(X^*, -F(X^*)\big) \tag{42}$$

also satisfies

$$F(X^*)^{\mathrm{T}} \cdot (X - X^*) \geqslant 0, \quad \forall X \in \mathcal{K}. \tag{43}$$

This theorem establishes the equivalence between the set of equilibria of a projected dynamical system and the set of solutions of a variational inequality problem. Moreover, it provides a natural underlying dynamics (out of equilibrium) to such systems.

We now present the dynamic adjustment process for the noncompliant oligopolistic market problem with marketable pollution permits. In particular, we have that

$$\dot{X} = \Pi_{\mathcal{K}}\big(X, -F(X)\big), \tag{44}$$

with $F(X)$ and X defined for our model immediately preceding inequality (24). We refer to (44) as PDS$(F,\mathcal{K})$.

We now briefly discuss this dynamic adjustment or tatonnement process. Note that, in contrast to the variational inequality formulation of the governing equilibrium conditions (cf. (14) and (24)), the dynamical system (44) also provides insight into disequilibrium behavior. Nevertheless, as stated in a later result due to Nagurney and Zhang [20], the set of stationary points of this system coincides with the set of solutions to the variational inequality (24) (equivalently, (14)), and, consequently, in view of variational inequality (14), also correspond to an equilibrium pattern.

For example, in the case that X lies in the interior of $\mathcal{K}$ then its evolution proceeds according to $\dot{X} = -F(X)$. However, if it is pushed to the boundary of $\mathcal{K}$, then the projection operator $\Pi_{\mathcal{K}}$ guarantees that the constraints will not be violated.

Recalling the definition of $F(X)$ for our model with noncompliance we see that the firms' quantities, emissions, license holdings, as well as the noncompliant overflows evolve in the direction of greatest utility maximization (subject to the nonnegativity constraints). The prices of the licenses, on the other hand, evolve according to the laws of supply and demand for the licenses. The allocation of licenses, in turn, evolves in the direction of the difference between the marginal costs of abatement and the prices. The marginal costs of abatement, on the other hand, evolve according to the emission constraints. Finally, the underflows evolve according to minus the marginal costs of abatement. Hence, this tatonnement process is meaningful and, moreover, its set of stationary points coincides with the set of solutions to variational inequality (24).

Note that one may also develop a dynamic counterpart of the multiproduct, multipollutant oligopolistic market model whose equilibrium conditions are governed by variational inequality (23). Indeed, in this case one obtains a projected dynamical system akin to (44) but with $F(X)$ and X defined as immediately following (24) in the text.

3.1. *Qualitative properties*

In this section we discuss some qualitative properties of the projected dynamical system (44). In particular, we present some stability analysis results which will be utilized to prove convergence of the iterative scheme in section 4.

We require first the following notation. The notation $X \cdot t$ and $X(t)$ denote the solution path of the initial value problem, IVP$(F, \mathcal{K}, X)$, where the initial value problem is given by

$$\dot{X} = \Pi_{\mathcal{K}}\big(X, -F(X)\big), \quad X(0) = X_0,$$

that passes through X at time $t = 0$, that is, $X \cdot 0 = X(0) = X$. We will use $B(X, R)$, hereafter, to denote the open ball with radius R and center X.

For additional discussion, see Zhang and Nagurney [20].

Definition 13. An equilibrium point X^* is *stable* if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $X \in B(X^*, \delta)$ and $t \geqslant 0$

$$X \cdot t \in B(X^*, \varepsilon).$$

The equilibrium point X^* is *unstable* if it is not stable.

Definition 14. An equilibrium point X^* is a *monotone attractor* if there exists a $\delta > 0$ such that for all $X \in B(X^*, \delta)$

$$d(X,t) = \|X \cdot t - X^*\| \tag{45}$$

is a nonincreasing function of t, X^* is a *global monotone attractor* if $d(X,t)$ is nonincreasing in t for all $X \in \mathcal{K}$.

We now recall the following theorem due to Zhang and Nagurney [20]:

Theorem 15. Suppose that X^* solves VI$(F, \mathcal{K})$. If $F(X)$ is locally monotone at X^*, then X^* is a monotone attractor for the PDS$(F, \mathcal{K})$; if $F(X)$ is monotone, then X^* is a global monotone attractor.

The following result is immediate from the above stated theorem:

Corollary 16. X^* that is a solution to variational inequality (24) and a stationary point of the projected dynamical system (44) is a global monotone attractor.

Proof. Follows from the monotonicity result in the qualitative properties and the above stated theorem. □

4. The iterative scheme

Although the projected dynamical system (44) describes a continuous time adjustment process, a discrete time process is needed for actual computation purposes. In this section we present the Euler method in order to compute a stationary point of the projected dynamical system (44). Besides computing an equilibrium point it also provides a discrete time approximation to the continuous time adjustment process (48).

Recall (cf. Dupuis and Nagurney [2]) that at an iteration T of the Euler method, one has to compute:

$$X^T = P_{\mathcal{K}}\big(X^{T-1} - a_{T-1}F\big(X^{T-1}\big)\big), \tag{46}$$

where $P_{\mathcal{K}}$ was the projection operator defined in (39), and $\{a_T\}$ is a positive sequence to be discussed later.

For completeness, we now state the above algorithm in which $F(X)$ is in expanded form for our specific model:

Step 0: Initialization:
Set $(q^0, e^0, \delta^{+0}, \delta^{-0}, l^0, \lambda^0, p^0) \in \mathcal{K}$. Let $T = 1$.

Step 1: Computation:

Compute $(q^T, e^T, \delta^{+T}, \delta^{-T}, l^T, \lambda^T, p^T) \in \mathcal{K}$ by solving the variational inequality subproblem:

$$\sum_{i=1}^{m}\sum_{d=1}^{s}\left[q_{id}^T + a_{T-1}\left(-\frac{\partial u_i(q^{T-1}, e_i^{T-1}, \delta_i^{+T-1}, \delta_i^{-T-1}, l_i^{T-1})}{\partial q_{id}}\right) - q_{id}^{T-1}\right][q_{id} - q_{id}^T]$$
$$+\sum_{i=1}^{m}\sum_{t=1}^{r}\left[e_i^{tT} + a_{T-1}\left(-\frac{\partial u_i(q^{T-1}, e_i^{T-1}, \delta_i^{+T-1}, \delta_i^{-T-1}, l_i^{T-1})}{\partial e_i^t}\right.\right.$$
$$\left.\left.+\sum_{j=1}^{n}\lambda_{ij}^{T-1}h_{ij}\right) - e_i^{tT-1}\right] \times [e_i^t - e_i^{tT}]$$
$$+\sum_{i=1}^{m}\sum_{t=1}^{r}\sum_{j=1}^{n}[\delta_{ij}^{t+T} + a_{T-1}(M_{ij}^t - \lambda_{ij}^{T-1}) - \delta_{ij}^{t+T-1}] \times [\delta_{ij}^{t+} - \delta_{ij}^{t+T}]$$
$$+\sum_{i=1}^{m}\sum_{t=1}^{r}\sum_{j=1}^{n}[\delta_{ij}^{t-T} + a_{T-1}\lambda_{ij}^{t\ T-1} - \delta_{ij}^{t-T-1}] \times [\delta_{ij}^{t-} - \delta_{ij}^{t-T}]$$
$$+\sum_{i=1}^{m}\sum_{t=1}^{r}\sum_{j=1}^{n}[l_{ij}^{tT} + a_{T-1}(p_j^{tT-1} - \lambda_{ij}^{tT-1}) - l_{ij}^{tT-1}] \times [l_{ij}^t - l_{ij}^{tT}]$$
$$+\sum_{t=1}^{r}\sum_{j=1}^{n}[\lambda_{ij}^{tT} + a_{T-1}(l_{ij}^{tT-1} - h_{ij}^t e_i^{tT-1} - \delta_{ij}^{t-T-1} + \delta_{ij}^{t+T-1}) - \lambda_{ij}^{tT-1}][\lambda_{ij}^t - \lambda_{ij}^{tT}]$$
$$+\sum_{t=1}^{r}\sum_{j=1}^{n}\left[p_j^{tT} + a_{T-1}\sum_{i=1}^{m}(l_{ij}^{t0} - l_{ij}^{tT-1}) - p_j^{tT-1}\right] \times [p_j^t - p_j^{tT}] \geqslant 0,$$
$$\forall (q, e, \delta^+, \delta^-, l, \lambda, p) \in \mathcal{K}. \tag{47}$$

Step 2: Convergence verification:

If $\max_{id}|q_{id}^T - q_{id}^{T-1}| \leqslant \varepsilon$, $\max_{it}|e_i^{tT} - e_j^{tT-1}| \leqslant \varepsilon$, $\max_{ijt}|\delta_{ij}^{t+T} - \delta_{ij}^{t+T-1}| \leqslant \varepsilon$, $\max_{ijt}|\delta_{ij}^{t-tT} - \delta_{ij}^{t-T-1}| \leqslant \varepsilon$, $\max_{ijt}|l_{ij}^{tT} - l_{ijt}^{tT-1}| \leqslant \varepsilon$, $\max_{ijt}|\lambda_{ij}^{tT} - \lambda_{ijt}^{tT-1}| \leqslant \varepsilon$, $\max_{jt}|p_j^{tT} - p_j^{tT-1}| \leqslant \varepsilon$, for all i, j, d, t, with $\varepsilon > 0$, a prespecified tolerance, then stop; else, set $T = T + 1$, and go to step 1.

We now discuss the Euler method more fully. In particular, the solution of each of the variables encountered in (47) amounts to projecting separately. Consequently, we can provide closed-form expressions for the solution of problem (47). In particular, we have that (47) can be solved as:

For all firms i, $i = 1, \ldots, m$, and all products d, $d = 1, \ldots, s$, set

$$q_{id} = \max\left\{0, a_{T-1}\left(\frac{\partial u_i(q^{T-1}, e_i^{T-1}, \delta_i^{+T-1}, \delta_i^{-T-1}, l_i^{T-1})}{\partial q_{id}}\right) + q_{id}^{T-1}\right\} \tag{48}$$

and for firms i, $i = 1, \ldots, m$, and all pollutants t, $t = 1, \ldots, r$,

$$e_i^{tT} = \max\left\{0, a_{T-1}\left(\frac{\partial u_i(q^{T-1}, e_i^{T-1}, \delta_i^{+T-1}, \delta_i^{-T-1}, l_i^{T-1})}{\partial e_i^t} - \sum_{j=1}^{n} \lambda_{ij}^{tT-1} h_{ij}^t\right) + e_i^{tT-1}\right\}. \tag{49}$$

For all firms i, $i = 1, \ldots, m$, all receptor points j, $j = 1, \ldots, n$, and all pollutants t, $t = 1, \ldots, r$, set

$$\delta_{ij}^{t+T} = \max\{0, a_{T-1}(-M_{ij}^t + \lambda_{ij}^{T-1}) + \delta_{ij}^{t+T-1}\}. \tag{50}$$

For all firms i, $i = 1, \ldots, m$, all receptor points j, $j = 1, \ldots, n$ and all pollutants t, $t = 1, \ldots, r$, set

$$\delta_{ij}^{t-T} = \max\{0, a_{T-1}(-\lambda_{ij}^{T-1}) + \delta_{ij}^{t-T-1}\}. \tag{51}$$

For all firms i, $i = 1, \ldots, m$, all receptor points $j, j = 1, \ldots, n$, and all pollutants t, $t = 1, \ldots, r$, set

$$l_{ij}^{tT} = \max\{0, a_{T-1}(-p_j^{tT-1} + \lambda_{ij}^{tT-1}) + l_{ij}^{tT-1}\}, \tag{52}$$

and

$$\lambda_{ij}^{tT} = \max\{0, a_{T-1}(-l_{ij}^{tT-1} + h_{ij}^t e_i^{tT-1} + \delta_{ij}^{t-T-1} - \delta_{ij}^{t+T-1}) + \lambda_{ij}^{tT-1}\}. \tag{53}$$

Finally, for all receptor points $j, j = 1, \ldots, n$, and all pollutants t, $t = 1, \ldots, r$, set

$$p_j^{tT-1} = \max\left\{0, a_{T-1}\left(-\sum_{i=1}^{m}(l_{ij}^{t0} - l_{ij}^{tT-1})\right) + p_j^{tT-1}\right\}. \tag{54}$$

Convergence is given in the following:

Theorem 17. Suppose that $F(X)$ is as defined above and that the utility functions are concave in their arguments and that the sequence generated by the Euler method is bounded. Assume also that $F(X)$ is strictly monotone at any equilibrium pattern. Then for any initial condition $X^0 \in \mathcal{K}$, the sequence generated by the Euler method, where

$$\lim_{T\to\infty} a_T = 0 \quad \text{and} \quad \sum_{T=0}^{\infty} a_T = \infty$$

converges to X^*.

Proof. According to Dupuis and Nagurney [2] (see also Nagurney and Zhang [15]), the general iterative scheme due to Dupuis and Nagurney [2], of which the Euler method is a special case, converges to a stationary point or equilibrium point provided $F(X)$ is Lipschitz continuous and the following assumption holds. □

Assumption 18. Fix an initial condition $X^0 \in \mathcal{K}$. Define the sequence $\{X^T, T \in N\}$ as above. Assume the following conditions:

1. $\sum_{T=0}^{\infty} a_T = \infty$, $a_T > 0$, $a_T \to 0$, as $T \to \infty$.
2. $d(F_T(X), \bar{F}(X)) \to 0$ uniformly on compact subsets of $\mathcal{K}$ as $T \to \infty$, where $d(X, A) = \inf\{\|X - y\|, y \in A\}$, and the overline indicates closure.
3. Define ϕ_y to be the unique solution to $\dot{X} = \Pi_{\mathcal{K}}(X, -F(X))$ that satisfies $\phi_y(0) = y \in \mathcal{K}$. The ω-limit set

$$\bigcup_{y \in K} \bigcap_{t \geqslant 0} \overline{\bigcup_{s \geqslant t} \{\phi_y(s)\}}$$

   is contained in the set of stationary points of $\dot{X} = \Pi_{\mathcal{K}}(X, -F(X))$.
4. The sequence $\{X^T, T \in N\}$ is bounded.
5. The solutions to $\dot{X} = \Pi_{\mathcal{K}}(X, -F(X))$ are stable in the sense that given any compact set $\mathcal{K}_1$ there exists a compact set $\mathcal{K}_2$ such that $\bigcup_{y \in \mathcal{K} \cap \mathcal{K}_1} \bigcup_{t \geqslant 0} \{\phi_y(t)\} \subset \mathcal{K}_2$.

We note that assumption 18.1 always holds true by our selection of the sequence $\{a_T\}$. Also, assumption 18.2 also holds true since we assume that $F(X)$ is continuous.

By proposition 4.1 in Nagurney and Zhang [15], under the assumption that $F(X)$ is strictly monotone at any equilibrium pattern, assumption 18.3 also holds true. Since $F(X)$ is monotone, by proposition 4.2 of Nagurney and Zhang [15] we have that assumption 18.5 is also satisfied. The sequence is bounded under our assumption. The proof is complete.

5. Numerical examples

In this section we present numerical examples to which we apply the Euler method of section 4.

We present three examples of increasing complexity. All the examples have quadratic production cost and emission cost functions.

Each firm i in each example faces a production cost function of the form

$$f_i(q_i) = \sum_{d=1}^{s} \left[\phi 1_{id} q_{id}^2 + \phi 2_{id} q_{id}\right], \tag{55}$$

with the specific terms for the parameters reported along with the examples.

The demand price function for product d in each example was given by

$$\rho_d\left(\sum_{i=1}^{m} q_{id}\right) = 5000^{1/1.1}\left(\sum_{i=1}^{m} q_{id}\right)^{-1/1.1}. \tag{56}$$

Each firm i in each example also faced an emission cost function of the form

$$G_i(e_i, q_i) = \sum_{t=1}^{r} \left[\psi 1_i^t (e_i^t)^2 + \psi 2_i^t e_i^t\right] + \sum_{d=1}^{s} [\phi 3_{id} q_{id}], \tag{57}$$

with the specific terms for the parameters reported along with the examples.

We set the penalty cost terms $M_{ij}^t = 0.1ijt$, for all ijt, initially for all the examples.

The initial allocation of the licenses, the l_{ij}^{t0}'s, were set as: $l_{ij}^{t0} = 1$, for all i, j, t. The diffusion matrix H terms, the h_{ij}^t's were set as follows: for $t = 1$: $h_{ij}^1 = 1\frac{i}{j}$, if $i \leqslant j$; and $h_{ij}^1 = 0.5\frac{i}{j}$, otherwise, for all i, j. For $t = 2$: $h_{ij}^2 = 2\frac{i}{j}$, if $i \leqslant j$; and $h_{ij}^1 = 0.1\frac{i}{j}$, otherwise, for all i, j.

The initial values for the quantities were set as follows: $q_{id}^0 = 30$, if $i \leqslant d$; and $q_{id}^0 = 50$, otherwise. The initial values for the emissions were set as follows: $e_i^t = 20$, for all i, t. The initial values for the overflows and underflows were: $\delta_{ij}^{t+} = 1$ and $\delta_{ij}^{t-} = 0$, for all i, j, t.

All other initial variables were initialized to 1.

The convergence tolerance ε was set to 0.0001 and the sequence $\{a_T\}$ was set to $0.1\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots\}$ in the Euler method for all the examples.

The algorithm was coded in FORTRAN 77. The system used was the IBM SP2 located at the Cornell Theory Center at Cornell University. We report the CPU time, exclusive of input and output and setup times, for each example.

We also report the computed quantities of production, the emissions, as well as the license prices. We do not report the licenses, the overflows and underflows, and the shadow prices due to the sizes of the vectors.

Finally, for each example, we also provide the following information, the total overflow, T^O, given by

$$T^O = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{t=1}^{r} \delta_{ij}^{t+}. \tag{58}$$

Example 1. In this example, the oligopoly consisted of 2 firms that produce 2 outputs and emit 2 pollutants, which, in turn, affect 2 receptor points. Refer to table 1 for the production cost parameters and to table 2 for the emission cost parameters.

Table 1
Example 1 – production cost parameters.

Firm i	$\phi 1_{i1}$	$\phi 1_{i2}$	$\phi 2_{i1}$	$\phi 2_{i2}$	$\phi 3_{i1}$	$\phi 3_{i2}$
1	0.12	0.18	8.0	3.0	2.0	5.0
2	0.15	0.20	6.0	3.0	4.0	7.0

Table 2
Example 1 – emission cost parameters.

Firm i	$\psi 1_i^1$	$\psi 1_i^2$	$\psi 2_i^1$	$\psi 2_i^2$
1	1.0	0.5	–40	–35
2	0.7	1.0	–35	–40

The algorithm converged in 8.00 seconds of CPU time and yielded the equilibrium production vector for firm 1 and firm 2, respectively:

$$q_1^* = (45.400, 41.192), \qquad q_2^* = (41.605, 37.173),$$

the equilibrium emission vector for firm 1 and firm 2, respectively:

$$e_1^* = (19.893, 34.621), \qquad e_2^* = (24.700, 19.932),$$

and the equilibrium prices for licenses affecting receptor point 1 and receptor point 2:

$$p_1^{1*} = 0.000, \quad p_1^{2*} = 0.472, \quad p_2^{1*} = 0.318, \quad p_2^{2*} = 0.412.$$

The total overflow in this case was $T^O = 94.25$. We then increased all penalty terms by a factor of 10. Note that this increase would reflect that the total cost faced by the firm increases. This perturbation of cost resulted in a decrease of total overflows with the new value of $T^O = 82.70$. Finally, we increased the penalty terms by another factor of 10 with the new value of $T^O = 12.88$. Hence, as one would expect, an increase in the cost associated with noncomplaint behavior led to a decrease in the actual amount of noncompliance, given by the overflows.

Example 2. In the second example, we increased the number of firms from 2 to 3. The 3 firms in the oligopoly still produce 2 outputs and emit 2 pollutants that affect 2 receptor points. Refer to table 3 for the production cost parameters and to table 4 for the emission cost parameters.

The algorithm converged in 11.28 seconds of CPU time and yielded the equilibrium production vector for firm 1, firm 2, and firm 3, respectively:

$$q_1^* = (43.484, 38.528), \quad q_2^* = (38.810, 33.197), \quad q_3^* = (29.840, 38.542),$$

the equilibrium emission vector for firm 1, firm 2, and firm 3, respectively:

$$e_1^* = (19.909, 34.579), \quad e_2^* = (24.686, 19.910), \quad e_3^* = (13.909, 22.308),$$

and the price vector for licenses affecting receptor point 1, and receptor point 2:

$$p_1^{1*} = 0.001, \quad p_1^{2*} = 0.721, \quad p_2^{1*} = 0.819, \quad p_2^{2*} = 1.157.$$

The total overflow was $T^O = 131.299$.

We then conducted an analogous simulation to that in example 1, that is, we first multiplied all the penalty terms by a factor of 10. The new total overflow value dropped to: $T^O = 112.45$. Finally, we multiplied the penalty terms by another factor of 10, yielding $T^O = 15.80$.

Table 3
Examples 2 and 3 – production cost parameters.

Firm i	$\phi 1_{i1}$	$\phi 1_{i2}$	$\phi 2_{i1}$	$\phi 2_{i2}$	$\phi 3_{i1}$	$\phi 3_{i2}$
1	0.12	0.18	8.0	3.0	2.0	5.0
2	0.15	0.20	6.0	3.0	4.0	7.0
3	0.20	0.18	9.0	3.0	3.0	5.0

Table 4
Examples 2 and 3 – emission cost parameters.

Firm i	$\psi 1_i^1$	$\psi 1_i^2$	$\psi 2_i^1$	$\psi 2_i^2$
1	1.0	0.5	–40	–35
2	0.7	1.0	–35	–40
3	1.05	1.0	–30	–45

Hence, as one would expect, the actual amounts of noncompliance lowered considerably as the cost associated with noncomplaint behavior was increased.

Example 3. In the third example, we increased the number of receptor points from 2 to 3. The 3 firms in the oligopoly still produce 2 outputs and emit 2 pollutants. Refer to table 3 for the production cost parameters and to table 4 for the emission cost parameters.

The algorithm converged in 17.04 seconds of CPU time and yielded the equilibrium production vector for firm 1, firm 2, and firm 3, respectively:

$$q_1^* = (43.484, 38.528), \quad q_2^* = (38.810, 33.197), \quad q_3^* = (29.840, 38.542),$$

the equilibrium emission vector for firm 1, firm 2, and firm 3, respectively:

$$e_1^* = (19.861, 34.176), \quad e_2^* = (24.419, 19.11), \quad e_3^* = (13.733, 22.292),$$

and the price vector for licenses affecting receptor point 1, receptor point 2, and receptor point 3, respectively:

$$\begin{aligned} p_1^{1*} &= 0.384, \quad p_1^{2*} = 0.611, \\ p_2^{1*} &= 0.425, \quad p_2^{2*} = 1.212, \\ p_3^{1*} &= 0.414, \quad p_3^{2*} = 1.311. \end{aligned}$$

The total overflow value was: $T^O = 204.397$.

As previously, we then multiplied the penalty terms by a factor of 10, which resulted in a new total overflow value of $T^O = 145.33$. We then further multiplied the terms by another factor of 10, yielding $T^O = 2.88$.

Again, as expected, the actual amounts of noncompliance, that is, the overflow value, lowered considerably as the cost associated with noncomplaint behavior was increased.

We note that a variety of additional simulations can be carried out with these examples. In particular, it would be interesting to determine what happens to equilibrium permit prices and, specifically, to noncompliance, as the demand for the products increases (or decreases). Furthermore, it would be interesting to explore what the equilibrium prices of the permits would be if there was no noncompliance, that is, if the penalties were high enough to ensure compliance. Such simulations could further suggest theoretical directions. In particular, we conjecture, in view of the variational inequalities (14) and (23) governing, respectively, the model with noncompliance and compliance, that setting the penalty terms for each firm associated with overemissions of a particular pollutant and receptor point equal to the equilibrium price for a permit associated with the particular receptor point and pollutant obtained from the model with compliance, will result in no noncompliance. We leave such investigations for future research.

6. Summary and conclusions

In this paper, we have presented both static and dynamic models for the formulation, qualitative analysis, and computation of equilibria in multiproduct, multipollutant noncompliant oligopolistic markets of pollution permits.

For the static model, we utilized the variational inequality framework for the formulation of the equilibrium conditions as well as for the qualitative anlaysis and the sensitivity analyis. For the dynamic model, we applied the theory of projected dynamical systems.

Also, we proposed an algorithm, the Euler method, that yields subproblems of very simple structure, each of which can be solved explicitly and in closed form. The algorithm is a discrete time counterpart to the continuous time adjustment process. We presented the convergence results using the stability analysis results. Finally, to illustrate both the model and the algorithm, we presented several numerical examples to provide insight into the model as well as the performance of the algorithm.

Acknowledgements

This research was supported by the National Science Foundation under grant DMS 9902471 under the Faculty Awards for Women Program. This support is gratefully acknowledged.

The authors are indebted to the three anonymous referees for their careful reading of the manuscript and many helpful suggestions.

References

[1] A.A. Cournot, *Researches into the Mathematical Principles of the Theory of Wealth*, English translation (MacMillan, London, England, 1838).

[2] P. Dupuis and A. Nagurney, Dynamical systems and variational inequalities, Annals of Operations Research 44 (1993) 9–42.

[3] D. Gabay and H. Moulin, On the uniqueness and stability of Nash equilibria in noncooperative games, in: *Applied Stochastic Control in Econometrics and Management Science*, eds. A. Bensoussan, P. Kleindorfer and C.S. Tapiero (North-Holland, Amsterdam, The Netherlands, 1980).

[4] J.D. Harford, Firm behavior under imperfectly enforceable pollution standards and taxes, Journal of Environmental Economics and Management 5 (1978) 26–43.

[5] A.G. Keeler, Noncompliant firms in transferable discharge permit markets: some extensions, Journal of Environmental Economics and Management 21 (1991) 180–189.

[6] A.S. Malik, Markets for pollution control when firms are noncompliant, Journal of Environmental Economics and Management 18 (1990) 97–106.

[7] A.S. Malik, Enforcement costs and the choice of policy instruments for controlling pollution, Economic Inquiry 30 (1992) 161–173.

[8] W.D. Montgomery, Markets in licenses and efficient pollution control programs, Journal of Economic Theory 5 (1972) 395–418.

[9] A. Nagurney, *Network Economics: A Variational Inequality Approach* (Kluwer Academic, Boston, MA, 1993).

[10] A. Nagurney and K. Dhanda, A variational inequality approach for marketable pollution permits, Computational Economics 9 (1996) 360–384.

[11] A. Nagurney, K. Dhanda and J. Stranlund, A general multiproduct, multipollutant market pollution permit model: a variational inequality approach, Energy Economics 19 (1997) 57–76.

[12] A. Nagurney, P. Dupuis and D. Zhang, Dynamical systems approach for network oligopolies and variational inequalities, Annals of Regional Science 28 (1994) 263–283.

[13] A. Nagurney and P. Ramanajum, Transportation network policy modeling with goal targets and generalized penalty functions, Transportation Science 30 (1996) 3–13.

[14] A. Nagurney, S. Thore and J. Pan, Spatial market policy modeling with goal targets, Operations Research 44 (1996) 393–406.

[15] A. Nagurney and D. Zhang, *Projected Dynamical Systems and Variational Inequalities with Applications* (Kluwer Academic, Boston, MA, 1996).

[16] J.F. Nash, Equilibrium points in n-person games, Proceedings of the National Academy of Sciences 36 (1950) 48–49.

[17] S. Thore, A. Nagurney and J. Pan, Generalized goal programming and variational inequalities, Operations Research Letters 12 (1992) 217–226.

[18] T.H. Tietenberg, *Emissions Trading, An Exercise in Reforming Pollution Policy* (Resources for the Future Inc., Washington, DC, 1985).

[19] H. van Egteren and M. Weber, Marketable permits, market power, and cheating, Journal of Environmental Economics and Management 30 (1996) 161–173.

[20] D. Zhang and A. Nagurney, On the stability of projected dynamical systems, Journal of Optimization Theory and its Applications 85 (1995) 97–124.