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ON THE OCCURRENCE OF PONZI SCHEMES IN PRESENCE OF CREDIT RESTRICTIONS PENALIZING DEFAULT.

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Abstract. This paper considers an infinite-horizon exchange economy with incomplete markets of real assets and default. Borrowers are required to constitute collateral in terms of durable goods and face credit restriction functions that depend on their past default. I show that Ponzi schemes are possible when these functions (i) are decreasing, and (ii) allow agents to simultaneously decrease their default level and increase their short-sales by a higher rate. Moreover, I prove that Ponzi schemes are ruled out for linear credit restriction functions provided that the slope of these functions is not too high.

JEL Classification: D52, D91.

Keywords: Equilibrium, Incomplete markets, Default, Collateral, Credit constraints, Ponzi schemes.

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1. Introduction.

It is well known that equilibrium existence is not compatible with the possibility of doing Ponzi schemes in economies with infinite lived agents trading financial assets. In models without default, Ponzi schemes were ruled out, and equilibrium existence was restored, either by imposing exogenous transversality-type conditions (see Magill and Quinzii (1994)) or Debt constraints (see Levine and Zame (1996). Araujo, Páscoa and Torres-Martínez (2002) prove that the seizure of collateral is a simple default-punishment mechanism that (endogenously) rules out Ponzi schemes. This is because it appears that the obligation of constituting collateral in terms of durable goods whenever an asset is sold will limit the asymptotic explosion of the debt. Firstly, short-sales become bounded, node by node. Secondly, since there are no additional default penalties, each seller delivers exactly the minimum between his debts and the value of the depreciated collateral. Therefore, the no-arbitrage condition requires borrowed values to be less than the value of the constituted collateral and therefore, bounded from above by a uniform upper bound on endowments of durable goods. In Araujo et al. (2002), collateral repossession is the unique default enforcement mechanism. Such a treatment of default is not fully convincing since default does not affect a household’s ability to borrow in the future and so does not lead to any direct reduction in consumption at the time of default.

Páscoa and Seghir (2009) provide examples illustrating that Ponzi schemes may reappear if harsh utility penalties are introduced besides collateral repossession as this may induce effective payments over collateral recollection values. In this event, loans may be larger than collateral costs and agents may prefer doing a Ponzi scheme rather than defaulting and giving up the collateral. The authors also claim that moderate penalties are compatible with equilibrium existence. Martins-da-Rocha and Vailakis (2010) propose an alternative utility penalty when all commodities serve as collateral. The additional default punishment used by Páscoa and Seghir (2009) and Martins-da-Rocha and Vailakis (2010) consists of linear and time–node additively separable utility penalties proportional to the amount of default.¹ Dubey et al. (2005) interpret these utility penalties as “the sum of third party punishment, pangs of conscience, (unmodeled) reputation losses, and (unmodeled) garnishing of future income”. As pointed out by Sabarwal (2003), with such utility penalties, which exist only in the consumer’s psyche, the lender has no legal recourse for debt recovery. Utility penalties also rule out the effect of an agent’s default on his future access to credit markets. In addition, when utility penalties are imposed, present default leads to a direct reduction in utility rather than a reduction in future consumption.

¹Utility penalties were introduced by Dubey et al. (1990) and also used, among others, by Zame (1993) and Dubey et al. (2005).
Recently, Ferreira and Torres-Martínez (2010) show that Ponzi schemes reappear when additional generic effective enforcement mechanisms are imposed (i.e. those implying payments besides the value of collateral guarantees), provided that collateral requirements are not large relative to the effectiveness of the additional mechanisms. However, the authors did not explicitly model how the market imposes additional payments on borrowers besides the value of collateral guarantees.

In this paper, I address, in a general equilibrium framework, the actual credit market practices where present default affects a household’s ability to borrow in the future, leading to a direct reduction in consumption at the time of default. More precisely, despite the seizure of the constituted collateral in case of default, defaulters face credit constraints that depend on the amount of their past default. Sabarwal (2003) uses such credit limits in a non-convex finite–horizon model with a continuum of agents. However, this paper is independent and there are many valuable differences between the two papers. First, Sabarwal (2003) introduces credit limits mainly to guarantee that short–sales are bounded, node by node, and this is sufficient to assure equilibrium existence in a finite–horizon economy. In our infinite–horizon economy, the obligation of constituting collateral in terms of durable goods of limited endowments endogenously guarantees that short–sales are bounded, node by node, but this is not sufficient to guarantee equilibrium existence in an infinite–horizon economy as agents may end up doing Ponzi schemes. Second, in order to protect the lenders against total default, Sabarwal (2003) assumes that a fraction of debtors’ income can be confiscated and given to the lenders in case of default. In our model, lenders are partially protected for such a total default as they receive at least the collateral in case of default. Moreover, in order to protect the debtors, Sabarwal (2003) introduces a bankruptcy law ensuring that a fraction of a debtor’s income (i.e.: exemption) cannot be seized. The possibility of confiscating a part of defaulters’ income together with the introduction of a bankruptcy law led to non-convexity of the model in Sabarwal (2003). In order to guarantee equilibrium existence, the author considers a continuum of agents with an atomless distribution. Credit constraints penalizing default were also used by Braido (2008) for an infinite–horizon model with a static stochastic structure to prove the existence of an ergodic Markovian equilibrium when borrowers are not required to constitute collateral. To this end, the author assumes that the credit constraints penalizing default are uniformly bounded along the event-tree. This assumption implies that short–sales are uniformly bounded and, therefore, Ponzi schemes are (exogenously) ruled out.

This paper shows that Ponzi schemes are possible in the presence of credit constraints functions provided that these functions (i) are decreasing, and (ii) allow agents to simultaneously decrease their default level and increase their short–sales by a higher rate. Moreover, I prove that Ponzi schemes are ruled out and equilibrium existence is restored for linear credit restriction functions.
provided that the slope of these functions is not too high (i.e.: when default on some asset increases, credit opportunities decrease at a lower rate).

The paper is organized as follows. The model is presented in Section 2. Section 3 is devoted to some definitions and assumptions. Section 4 illustrates the possibility of doing Ponzi schemes when collateral requirement and (general) decreasing credit constraints penalizing default coexist. Section 5 deals with linear credit restrictions functions. An Appendix is devoted to technical proofs.

2. The Model.

Stochastic Structure.
We consider a discrete time economy with infinite horizon and uncertainty. The stochastic structure of this model is described by an infinite tree with a unique root and finitely many branches at each node. Formally, let $T = \{0, 1, \cdots \}$ be the set of dates and let $F_t$ be the finite set of histories that may occur up to time $t$.

A pair $\xi = (t, \sigma)$ where $t \in T$ and $\sigma \in F_t$ is called node and $t(\xi) = t$ is the date of node $\xi$. The set $D$ consisting of all nodes is called the event-tree.

A node $\xi' = (t', \sigma')$ is said to succeed (resp. strictly) node $\xi = (t, \sigma)$ if $t' \geq t$ (resp. $t' > t$) and $\sigma' \subset \sigma$. We write $\xi' \succeq \xi$ (resp. $\xi' > \xi$).

For each $\xi \in D$, we will denote by:

- $D(\xi)$ the subtree of the nodes which succeed $\xi$,
- $D^+(\xi) = \{\xi' \in D | \xi' \succ \xi\}$ the set of the strict successors of $\xi$,
- $D_T(\xi)$ the subset of nodes of $D(\xi)$ between $t(\xi)$ and $T$,
- $\xi^+ = \{\eta \in D(\xi) | t(\eta) = t(\xi) + 1\}$ the set of immediate successors of $\xi$. The number of elements of $\xi^+$, called the branching number, is assumed to be finite.

If $\xi = (t, \sigma)$, $t \geq 1$, the unique node $\xi^- = (t - 1, \sigma')$, $\sigma \subset \sigma'$ is called the predecessor of $\xi$.

When $\xi$ is the initial node, denoted $\xi_0$, the notations are simplified to $D^+$, $D_T$.

Commodity, Financial and Demographic Structures.
At each node $\xi \in D$, a finite number $G$ of physical goods (possibly durable), indexed by $g = 1, \ldots, G$, are traded on spot markets. The structure of depreciation in the event-tree is given by a collection of $G \times G$-matrices $Y := \{Y(\xi)\}_{\xi \in D}$. As in Araujo et al. (2002), we assume that $Y(\xi)$ is a diagonal matrix, $\{\text{diag}(y(\xi, g))\}$, for each node $\xi \in D$. A commodity $g \in G$ is durable at node $\xi \in D$ if $y(\xi, g)$ is different from zero and perishes at $\xi$ otherwise. We assume that the depreciation structure is given by: $[Y(\xi)] = [\text{diag}(y(\xi, g))]_{g \in G}$ and there exists $k \in (0, 1)$ such that for each node $\xi \in D$, $\max_{g \in G} \{y(\xi, g)\} \leq k$.

At each node of the event-tree, there is a set $J(\xi)$ consisting of a finite number $\iota(\xi)$ of one-period real assets, available for intertemporal transaction and insurance. Let $A^j(\xi) \in \mathbb{R}^G_+ \setminus \{0\}$ be the
return, at node $\xi$, in quantities of the $G$ goods, of one unit of the asset $j \in J(\xi^-)$. We denote $A(\xi) = (A^j(\xi))_{j \in J(\xi^-)}$ and $A := \prod_{\xi \in D} A(\xi)$.

At each node $\xi \in D$, a commodity $g \in G$ and an asset $j \in J(\xi)$ are transacted at prices $p(\xi, g)$ and $q_j(\xi)$ respectively. Denote by $p(\xi) = (p(\xi, g), g \in G)$ and $q(\xi) = (q(\xi, j), j \in J(\xi)) \in \mathbb{R}^{|G|}$.

The demographic structure of the model is given by a finite set $I$ of infinitely-lived agents. The cardinality of $I$ will be denoted by $I$. At each node $\xi$, an agent $i \in I$ chooses:

- (i) a consumption $x^i(\xi)$ in $X^i(\xi)$. We denote by $X^i = \prod_{\xi \in D} X^i(\xi)$,
- (ii) a portfolio $z^i(\xi) := (z^i_j(\xi), j \in J(\xi))$, with $z^i(\xi) = \theta^i(\xi) - \varphi^i(\xi)$ where:
  - $\theta^i(\xi) := (\theta^i_j(\xi), j \in J(\xi)) \in \mathbb{R}^{|I|}_+$ are the quantities of assets bought by $i$ at node $\xi$,
  - $\varphi^i(\xi) := (\varphi^i_j(\xi), j \in J(\xi)) \in \mathbb{R}^{|I|}_+$ is the short-sale of assets by $i$ at node $\xi$.

At each node $\xi \in D$, aside his choices of consumption and portfolio, agent $i \in I$ chooses his default $\Delta^i(\xi) = (\Delta^i_j(\xi), j \in J(\xi^-)) \in \mathbb{R}^{|I|}$. The preferences of an agent $i \in I$ are represented by the utility function $U^i : X^i \to \mathbb{R}_+$ defined for each $x^i \in X^i$ by: $U^i(x^i) = \sum_{\xi \in D} v^i_\xi(x^i(\xi))$.

**Collateral requirement and credit constraints.**

As in Geanakoplos and Zame (1995), each seller of one unit of an asset $j \in J(\xi)$ is required to constitute a collateral $C^i_j(\xi) := (C^j_{0i}, g \in G) \in \mathbb{R}^{|G|}_+ \setminus \{0\}$, exogenously given.

Beside the seizure of his collateral, a borrower $i \in I$, whose default at a node $\xi \in D$ is $\Delta^i(\xi)$ (induced by his asset sales at $\xi^-$), is penalized by facing, on each asset $j \in J(\xi)$, narrow credit constraint functions. Formally, for each agent $i \in I$, for each node $\xi \in D$ and for each asset $j \in J(\xi)$, we define a credit constraint function $F^{i,j}_\xi : \mathbb{R}^{|I|}_+ \to \mathbb{R}$ so that $i$’s default at a node $\xi \in D$ affects the quantity of short-sales he can make at the same node as follows:

\begin{equation}
\varphi^i_j(\xi) \leq F^{i,j}_\xi(\Delta^i(\xi)).
\end{equation}

**Remark 1.** In the particular case when for each node $\xi \in D$, for each asset $j \in J(\xi)$ and for each agent $i \in I$, the function $F^{i,j}_\xi$ is constant, each agent $i$ will deliver the minimum between his debt and the value of the depreciated collateral. In such a case, agents’ behavior will be similar to that presented by Araujo et al. (2002). In addition, we obtain the same set of equilibrium allocations as in Araujo et al. (2002) when $F^{i,j}_\xi$ is a high enough constant.

In view of the anonymity of the markets, lenders do not know what the payment of each individual borrower will be. As in Dubey et al. (2005), we introduce variables representing the expected deliveries of the sellers. Formally, let $(K^j(\xi) \in [0,1], \xi \in D, j \in J(\xi^-))$ be the expected delivery rate on asset $j$ at node $\xi$.

We define the Economy $\mathcal{E}$ as follows: $\mathcal{E} := ((\omega^i, F^i, U^i), A_i, (C^i_j(\xi))_{\xi \in D, j \in J(\xi)}, Y)$.
3. Budget sets and Assumptions.

3.1. Budget sets.

Definition 1. [Budget sets]

Given \((p, q, K)\), the budget set \(B^i(p, q, K)\) of an agent \(i \in I\) is the set of \((x^i, \theta^i, \varphi^i, \Delta^i)\) in \(\mathbb{R}_+^{G \times D} \times \prod_{\xi \in D} \mathbb{R}^{(\xi)}_+ \times \prod_{\xi \in D} \mathbb{R}_+^{(\xi) \times G}\) verifying:

\[
p(\xi_0) \cdot (x^i(\xi_0) - \omega^i(\xi_0)) + p(\xi_0)C(\xi_0)\varphi^i(\xi_0) + q(\xi_0) \cdot (\theta^i(\xi_0) - \varphi^i(\xi_0)) \leq 0,
\]
and \(\forall \xi \in D \setminus \{\xi_0\}\),

\[
p(\xi)Y(\xi) \left[ x^i(\xi^-) + C(\xi^-)\varphi^i(\xi^-) \right] + \sum_{j \in J(\xi^-)} \left[ p(\xi)A^j(\xi) \left( K^j(\xi)\theta^j(\xi^-) - \varphi^j(\xi^-) \right) + \Delta^j(\xi) \right],
\]

\[
\varphi^j(\xi) \leq F^i,j(\Delta^i(\xi)), \forall j \in J(\xi).
\]

\[
\Delta^j(\xi) \leq p(\xi)A^j(\xi)\varphi^j(\xi^-) - \min\{p(\xi)A^j(\xi), p(\xi)Y(\xi)C^j(\xi^-)\} \varphi^j(\xi^-).
\]

Definition 2. [Equilibrium]

An equilibrium of \(E\) is a vector \((\overline{p}, \overline{q}, \overline{K}, (\overline{x}^i, \overline{\theta}^i, \overline{\varphi}^i, \overline{\Delta}^i)_{i \in I})\) such that \(\overline{p}(\xi) > 0\) at any node \(\xi \in D\) and verifying:

(i) For each agent \(i \in I\), \((\overline{x}^i, \overline{\theta}^i, \overline{\varphi}^i, \overline{\Delta}^i) \in \text{Argmax } U^i(x)\) over \(B^i(p, q, K)\),

(ii) \(\sum_{i \in I} [x^i(\xi_0) + C(\xi_0)\varphi^i(\xi_0)] = \sum \omega^i(\xi_0)\),

(iii) \(\sum_{i \in I} [x^i(\xi) + C(\xi)\varphi^i(\xi)] = \sum [\omega^i(\xi) + Y(\xi)x^i(\xi^-) + Y(\xi)C(\xi^-)\varphi^i(\xi^-)], \forall \xi \in D \setminus \{0\}\),

(iv) \(\sum_{i \in I} \overline{\theta}^i = \sum \overline{\varphi}^i\),

(v) \(\forall \xi \in D \setminus \{\xi_0\}, \forall j \in J(\xi^-), \sum_{i \in I} \overline{\Delta}^j(\xi) = \overline{p}(\xi)A^j(\xi) \sum_{i \in I} [\varphi^j(\xi^-) - \overline{K}^j(\xi)\overline{\theta}^j(\xi^-)]\).

Conditions (i)–(iv) are classical conditions. Condition (v) says that, at each node and for each asset, the total default made by the borrowers is equal to the total debt minus the total deliveries expected by the lenders.

Remark 2. It is possible to prove the existence of an (pure spot market) equilibrium in a trivial way when returns from asset purchases are endogenous (see Dubey et al. (2005), Páscoa and Seghir (2009), Steinert and Torres-Martinez (2007), to name a few). To overcome the problem of the absence of financial trade as a consequence of zero delivery rates, the existence of an equilibrium in
which expected delivery rates are strictly positive needs to be guaranteed. That is an equilibrium in which either there is financial trade or delivery rates are nonnull needs to be secured. On the other hand, if the credit constraint functions have nonpositive values, then there will be no financial trade in equilibrium. This brings about the following definition.

Definition 3. [Non–trivial equilibrium]

A non–trivial equilibrium \((p, q, K, (x_i, \theta_i, \phi_i, \Delta_i)_{i \in I})\) of \(E\) is an equilibrium such that for any \((\xi, j)\), we have \((\theta_j(\xi), \phi_j(\xi))\) different from zero or \(K_j(\xi) > 0\).

3.2. Assumptions. We make on \(E\) the following assumptions:

Assumption [A1]. \(\forall i \in I, \forall \xi \in D\), the function \(v_i^\xi : \mathbb{R}^G_+ \rightarrow \mathbb{R}\) is continuous, monotone\(^2\) and concave with \(v_i^\xi(0) = 0\). In addition, \(\forall i \in I, \forall \gamma \in \mathbb{R}^G_+\), \(\sum_{\xi \in D} v_i^\xi(\gamma)\) is finite.

Assumption [A2]. For each agent \(i \in I\), \(\omega^i \in \mathbb{R}^G_+^D\) and there exists \(W \in \mathbb{R}_+^+\) such that \(\forall i \in I, \forall \xi \in D, \sum_{g \in G} \omega^i(\xi, g) \leq W\).

Assumption [A3]. For each agent \(i \in I\), for each node \(\xi \in D\) and for each asset \(j \in J(\xi)\), there exists \(\Delta^i(\xi) > 0\) such that \(F^i_j(\Delta^i(\xi)) > 0\), for all \(\Delta^i(\xi) < \Delta^i(\xi)\).

Assumptions [A1] and [A2] are classical in infinite–horizon models. Assumption [A3] supposes that default is tolerable up to a certain default level beyond which the defaulter may be excluded from the credit market. This assumption guarantees the nonemptiness of the interior of the individual budget set and that short-sales are permitted.

4. Generating Ponzi schemes in the presence of credit constraints.

This section illustrates how Ponzi schemes may arise in infinite-horizon with collateral requirement when decreasing credit constraints penalizing default are introduced.

In a model with collateral requirement and linear utility penalties, Páscoa and Seghir (200) have illustrated the occurrence of Ponzi schemes by increasing short-sales at all successors of some node \(\xi\) (including node \(\xi\)) and decreasing default at all strict successors of \(\xi\). In this model, increasing short–sales at node \(\xi\) may require a simultaneous decrease (at the same node \(\xi\)) of default on assets sold at node \(\xi^-\) (as the value of what an agent is allowed to borrow is constrained according to the amount of his default). More precisely, when the credit constraints penalizing default are not binding, an agent can increase his short–sales without changing his default level (see Case 1 below). However, when the short–sales constraints are binding, an agent must to decrease his default level in order to borrow more (see Case 2 below). We prove hereafter that, in either case, an agent can always improve upon any budget feasible plan by adjusting his default and his short-sales as long as the following conditions are satisfied always in the future: (i) the credit functions penalizing

\(^2\)For each \(x, y \in \mathbb{R}^G_+: y > x \Rightarrow v_i^\xi(y) > v_i^\xi(x)\).
default are decreasing, (ii) agents can simultaneously decrease their default level and increase their short-sales by a higher rate, and (iii) the value of collateral requirements is lower than the loan value.

Formally, let \((p, q, K)\) be a system of prices and expected delivery rates and let \((x^i, \theta^i, \varphi^i, \Delta^i)\) be a collection of individual choice variables of an agent \(i \in I\) such that \((x^i, \theta^i, \varphi^i, \Delta^i) \in B^i(p, q, K)\). Let us fix a node \(\xi \in D\) and define the following set:

\[
\Theta^i(\xi) := \left\{ \sigma \in D(\xi) : \forall j \in J(\sigma), \ \varphi^i_j(\sigma) = F^i_{\sigma, j}(\Delta^i(\sigma)) \right\}.
\]

Assume that:

\[(6) \quad \forall \sigma \in D(\xi), \ \exists j \in J(\sigma) : p(\sigma)C^j_j(\sigma) - q_j(\sigma) < 0.\]

Inequality (6) requires the joint operation of constituting collateral and short-selling the asset to have a negative net price. In other words, it requires loans to exceed collateral cost. Let us distinguish the two following cases:

- **Case 1**: \(\Theta^i(\xi) = \emptyset\), i.e.: \(\forall \sigma \in D(\xi), \ \exists j_\sigma \in J(\sigma) : \varphi^i_j(\sigma) < F^i_{\sigma, j_\sigma}(\Delta^i(\sigma))\). In other words, at each successor of node \(\xi\), there exists an asset for which the credit constraints penalizing default are not binding. In such a case, it is easy to prove that agents can end up doing Ponzi schemes simply by increasing their short-sales without changing their default level and the new allocation will still be budgetary feasible as Inequality (6) holds.

- **Case 2**: \(\Theta^i(\xi) \neq \emptyset\), i.e.: \(\exists \sigma \in D(\xi) : \forall j \in J(\sigma), \ \varphi^i_j(\sigma) = F^i_{\sigma, j}(\Delta^i(\sigma))\). That is, there is some successor of node \(\xi\) for which the credit constraints penalizing default are binding for all assets. In such a case, Ponzi schemes may be generated by decreasing default and increasing short-sales at nodes \(\sigma \in \Theta(\xi)\) for which \(\Delta^i_j(\sigma) \neq 0\). Formally, let us consider the following changes on default and short-sales from node \(\xi\) onwards:

\[
(7) \quad \forall \sigma \in D(\xi), \ \tilde{\Delta}^i_j(\sigma) = \begin{cases} \Delta^i_j(\sigma) & \text{if } \sigma \notin \Theta(\xi) \\ \Delta^i_j(\sigma) - \alpha_\sigma & \text{if } \sigma \in \Theta(\xi) \end{cases}, \quad \alpha_\sigma > 0.
\]

\[
(8) \quad \forall \sigma \in D(\xi), \ \tilde{\varphi}^i_j(\sigma) = \begin{cases} \varphi^i_j(\sigma) & \text{if } \sigma \notin \Theta(\xi) \\ \varphi^i_j(\sigma) + \varepsilon_\sigma & \text{if } \sigma \in \Theta(\xi) \end{cases}, \quad \varepsilon_\sigma > 0.
\]

For each \(\sigma \in \Theta(\xi)\), when \(F^i_{\sigma, j_\sigma}\) is decreasing, one gets:

\[
F^i_{\sigma, j_\sigma}(\tilde{\Delta}^i(\sigma)) = F^i_{\sigma, j_\sigma}(\Delta^i(\sigma) - (0, \ldots, 0, \alpha_\sigma, 0, \ldots, 0) > F^i_{\sigma, j_\sigma}(\Delta^i(\sigma)) = \varphi^i_j(\sigma).
\]
Thus, one can find \( \varepsilon_\sigma > 0 \) such that \( F_{i,j}^\xi (\tilde{\Delta}^\xi (\sigma)) \geq \varphi_{j,\sigma} (\sigma) + \varepsilon_\sigma = \tilde{\varphi}_{j,\sigma} (\sigma) \). That is, \( \varepsilon_\sigma \) and \( \alpha_\sigma \) can be chosen such that \( (\tilde{\varphi}^j, \tilde{\Delta}^j) \) satisfies the credit constraint (4).

In view of Inequality (6), \( (x^i, \theta^i, \varphi^i, \tilde{\Delta}^i) \) satisfies the budget constraints (2) and (3) at nodes \( \sigma \in \Theta (\xi) \) if the following inequality holds at nodes \( \sigma \in \Theta (\xi) \):

\[
( p(\sigma) C^j (\sigma) - q^j (\sigma) ) \varepsilon_\sigma \leq -\alpha_\sigma.
\]

Note that Inequality (9) is satisfied if \( \varepsilon_\sigma \) can be chosen large enough relatively to \( \alpha_\sigma \).

Therefore, Ponzi schemes are possible and equilibrium may fail to exist if (i) collateral cost is lower than the loan value, (ii) credit restriction functions are decreasing, and (iii) agents can simultaneously decrease their default level and increase their short-sales by a higher rate. Note that Condition (ii) is not satisfied in Araujo et al. (2002), as default does not affect credit opportunities in their model, leading to constant restriction functions.

5. Equilibrium existence and Non-arbitrage condition with linear credit constraint functions.

As shown above, Ponzi schemes can be generated in the presence of decreasing credit constraints if loans exceed collateral costs, always in the future. In this section, we focus on linear credit constraint functions and utility functions that are separable in commodities. We introduce an assumption on the credit constraint functions to rule out the sufficient conditions of occurrence of Ponzi schemes.

More precisely, we assume that for each agent \( i \in I \), for each node \( \xi \in D \) and for each asset \( j \in J (\xi) \), the credit constraint function has the following form:

\[
F_{i,j}^\xi (\tilde{\Delta}^\xi (\sigma)) := a^i_j (\xi) - b^i_j (\xi) \frac{\sum_{k\in I (\xi -)} \Delta_k^i (\xi)}{p(\xi) \cdot \nu (\xi)},
\]

where \( a^i_j (\xi) > 0 \), \( b^i_j (\xi) \geq 0 \) and \( \nu (\xi) = (\nu (\xi, g), g \in G) \in \mathbb{R}^G_{++} \) is a fixed reference bundle.\(^3\)

Moreover, for each node \( \xi \in D \), let us define a path \( h (\xi) \) as the set of nodes \( \eta \in D (\xi) \) such that for each \( t > t (\xi) \), (i) there exists a unique \( \eta_t \in h (\xi) \) such that \( t (\eta_t) = t \) and (ii) \( \eta_{t+1} \in \eta_t^+ \) and we make the following assumptions:

**Assumption [A4].** For each node \( \xi \in D \), for each path \( h (\xi) \) beyond \( \xi \), there is a node \( \eta \in h (\xi) \) such that for each \( \sigma \in \eta^+ \), there exists an agent \( i \in I \), for which:

\[
b^i_j (\sigma) < \frac{a^i_j (\sigma) \sum_{g \in G} (v^i_{\sigma, g})^j \left( \sum_{i \in I} W^i (\sigma) \right)}{G v^i_0 (W)},
\]

where \( (v^i_{\sigma, g})^j \) denotes the right derivative of \( v^i_{\sigma, g} \) with respect to \( g \) and \( W^i (\sigma) := w^i (\sigma) + Y_\xi W^i (\sigma^-) \) denotes agent \( i \)'s accumulated wealth up to node \( \sigma \in D \).

---

\(^3\)If \( a^i_j (\xi) = 0 \), agents cannot sell assets short, regardless of their default level. In such a case, the only equilibrium that may exist is a (trivial) pure-spot equilibrium. Note also that Assumption [A3] is satisfied when \( a^i_j (\xi) \neq 0 \).
Assumption [A5]. \( \forall i \in I, \forall \xi \in D, \) the function \( v^i_\xi \) is separable in commodities.

Assumption [A'5]. All commodities serve as collateral.

Assumption [A4] guarantees that, along any path, there is some node \( \eta \) such that there is no incentive to pay more than the minimum between the debt and the value of the depreciated collateral at any immediate successor of \( \eta \). Together with [A5] or [A'5], it also implies that \( p(\sigma)C^j(\sigma) - q_j(\sigma) > 0 \) (see Remark 3 below).

Theorem 1. Under assumptions [A1], [A2], [A4], and either [A5] or [A'5], a non-trivial equilibrium exists and the non-arbitrage condition holds (collateral costs are higher than loans, sometime in the future).

Proof. See Appendix.

Remark 3. [On the Non-arbitrage condition]. The Kuhn-Tucker necessary conditions for agent \( i \)’s optimality guarantee the existence, at each node \( \xi \), of multipliers \( \mu^i(\xi), \lambda_j(\xi), \rho_j(\xi) \), for constraints (3), (4) and (5), respectively, together with vectors of supergradients \( (v^i_\xi)^\prime \left( x^i(\xi) + C^j(\xi)\varphi^i_j(\xi) \right) \in \partial v^i_\xi \left( x^i(\xi) + C^j(\xi)\varphi^i_j(\xi) \right) \) and \( \delta_j(\xi) \in [0, 1] \) such that:

\[
\rho_j(\eta) - \mu^i(\xi) - \lambda_j(\xi)\delta^j(\xi) = 0,
\]

\[
\mu^i(\eta)p(\eta, g) \geq (v^i_{\sigma, g})^\prime \left( x^i(\eta) + C^j(\eta)\varphi^i_j(\eta) \right).
\]

Now, applying Inequality (11) at node \( \sigma \), one gets that \( \mu^i(\sigma) \geq (v^i_{\sigma, g})^\prime \left( x^i(\sigma) + C^j(\sigma)\varphi^i_j(\sigma) \right) \geq (v^i_{\sigma, g})^\prime \left( \frac{W_j}{\epsilon_j} \right) \) (the last inequality holds as the utility function is separable in commodities and, therefore, \( (v^i_{\sigma, g})^\prime \) is non-increasing). Thus, \( \mu^i(\sigma) \geq \sum_{g \in G} \frac{(v^i_{\sigma, g})^\prime (\frac{W_j}{\epsilon_j})}{\epsilon_j} \) := \( \mu^i(\sigma) \). Moreover, using Kuhn-Tucker Theorem (see Rockafellar (1997), Theorem 28.3), one gets that \( \lambda_j(\sigma)\delta^j(\sigma) \leq v^i_j(W) \) implying that \( \lambda_j(\sigma) \leq \frac{v^i_j(W)}{\delta^j(\sigma)} := \Delta_j(\sigma) \). Now, it follows from Assumption [A4] that \( \lambda_j(\sigma)\delta^j(\sigma) < \mu^i(\sigma) \). Then, by (10), \( \rho_j(\sigma) > 0 \) and, therefore, \( \Delta_j(\sigma) = 0 \) for each \( \sigma \in \eta^+ \). That is, the optimal solution is equivalent to the optimal solution subject to the constraint \( \Delta_j(\sigma) = 0 \), for each \( \sigma \in \eta^+ \), whose first-order condition is \( p(\eta)C^j(\eta) \geq q_j(\eta) \).

Appendix.

Proof of Theorem 1. To show the existence of a nontrivial equilibrium, one can prove that in equilibrium, commodities prices are bounded from below (using the monotonicity of preferences). Then, since asset payments are greater than or equal to the minimum of the promise and the depreciated collateral, one can get a lower bound for unitary payments of borrowers. This lower bound on payments induces a lower bound on delivery rates. Thus, one can insert, in the abstract
economy, a lower bound on payments and prove that this lower bound, if properly chosen, is not binding (see Steinert and Torres-Martínez(2007) and Páscoa and Seghir (2009) for more details).

The proof of Theorem 1 is done in two main steps. The first step shows the equilibrium existence in truncated economies while the second step is devoted to asymptotic results.

**Step 1: Equilibria in truncated economies.**

Let $E^T$ be the truncated economy associated with the original economy $E$, which has the same characteristics as $E$, but where we suppose that agents are constrained to stop their exchange of goods at period $T$ and their trade of assets at period $T-1$. Formally, for each $T > 0$, let us define the following sets:

$$
\Pi^{T-1} := \left\{ (p, q) \in \mathbb{R}_{+}^{D^T \times G} \times \prod_{\xi \in D^T} \mathbb{R}^{1(\xi)} : \forall \xi : t(\xi) < T, \|p(\xi)\|_1 + \|q(\xi)\|_1 = 1, \right\},
$$

$$
K^T := [0, 1] \left( \sum_{\xi \in D^T} t(\xi) \right),
$$

and for each $i \in I$,

$$
X^{iT} = \{ (x^i(\xi), \xi \in D) \in X^i \mid \forall \xi : t(\xi) > T, x^i(\xi) = 0 \},
$$

$$
Z^{iT} = \{ (z^i(\xi), \xi \in D) \in X^i \mid \forall \xi : t(\xi) > T, \theta^i(\xi) = \varphi^i(\xi) = 0 \}.
$$

Moreover, given $(p, q, K) \in \Pi^{T-1} \times K^T$, the budget set, $B^{iT}(p, q, K)$, of an agent $i \in I$ for the truncated economy is defined by the set of $(x, z, \Delta)$ such that $x^i \in X^{iT}$, $z^i \in Z^{iT}$, (2) holds at $\xi = 0$ and (3)-(4) hold at all the other nodes. In addition, for each agent $i \in I$, the utility function $U^{iT}$ for each truncated economy $E^T$ is defined as follows: $U^{iT}(x^i, \theta^i, \varphi^i, \Delta^i) := \sum_{\xi \in D^T} v^i_\xi(\tilde{x}^i(\xi))$.

**Definition 4.** [Equilibria of the truncated economies]

An equilibrium of $E^T$ is a collection $(p^T, q^T, K^T, (\pi^T, \theta^T, \varphi^T, \Delta^T)_{i \in I})$ verifying:

(a) For each agent $i \in I$, $(\pi^T, \theta^T, \varphi^T, \Delta^T) \in \text{Argmax } U^{iT}(x)$ over $B^{iT}(p^T, q^T, K^T)$,

(b) Conditions (ii)–(v) of Definition 2 hold at $(\pi^T, \theta^T, \varphi^T, \Delta^T)$ for $\xi \in D^T$, with $\varphi^T(\xi) = 0$ when $t(\xi) = T$.

An equilibrium of $E^T$ is said to be non–trivial if it satisfies the following condition:

(c) For any $(\xi, j)$, either $(\tilde{\theta}_j(\xi), \varphi_j(\xi))$ is different from 0 or $K^T(\xi) > 0$.

**Proposition 1.** Under assumptions [A1], [A2], [A4] and [A5], each truncated economy $E^T$ has a non–trivial equilibrium $(p^T, q^T, K^T, (\pi^T, \theta^T, \varphi^T, \Delta^T)_{i \in I})$. 

Proof. The proof of Proposition 1 is analogous to the proof of the existence of a non-trivial equilibrium in Páscoa and Seghir (2009). However, there is a dissimilarity as for the non-emptiness of the interior of the budget set. In fact, as it will be explained hereafter, the credit constraint penalizing default (4) and the alteration of the decision variables in this model compared to Páscoa and Seghir (2009) makes the non–emptiness of the interior of the budget sets more problematic. To prove the non-emptiness of the interior of the budget sets, Páscoa and Seghir (2009) set short-sales equal to zero at the first period and effective payment strictly positive at the following period. Due to the presence of credit constraint functions in our model and since these functions may have negative values for some default levels, the idea used in Páscoa and Seghir (2009) would no longer hold (see Claim 2 below).

Lemma 1. Under Assumption [A2], an allocation \((x, \theta, \varphi, \Delta)\) which satisfies the conditions of Definition 4 is bounded.

Proof. Following the same idea as in Araujo et al. (2002), one gets:

\[
x^i(\xi, g) \leq W T \sum_{n=0}^{t} (Y^n G)^n := \zeta^T < +\infty, \forall g \in G,
\]

\[
\varphi_j^i(\xi) \leq \zeta^T / c_\theta(\xi) := \alpha^T(\xi) < +\infty, \forall j \in J(\xi),
\]

\[
\theta_j^i(\xi) \leq \alpha^T(\xi) < +\infty, \forall j \in J(\xi),
\]

where \(Y^T := \max\{Y(\xi)_{g, g', (\xi, g, g') \in D^T \times G \times G}\} \) and \(c^j(\xi) = \min\{C^j_2(\xi) : C^j_2(\xi) > 0\}\). For each node \(\xi \in D^T\), let us define \(\chi^T(\xi) = \max\{\chi^T(\xi), \alpha^T(\xi)\}\) and \(\chi^T = \max_{\xi \in D^T} \chi^T(\xi)\). Now, for each \(i \in I\), let us define:

\[
B_{\chi^T}(p, q, K, \chi) = \left\{ (x, \theta, \varphi, \Delta) \in B_{\chi^T}(p, q, K) \ \mid \begin{array}{l}
x^i(\xi, g) \leq 2\chi^T, \\
\theta_j^i(\xi) \leq 2\chi^T, \\
\varphi_j^i(\xi) \leq 2\chi^T,
\end{array} \right\}
\]

\^The proof of non–triviality in Páscoa and Seghir (2009) can be easily adapted to this model. Indeed, one can easily show that the delivery rates can be set greater or equal to \(\min\{p(\zeta) A^j(\zeta), p(\zeta) (Y(\xi) C^j(\xi))\} \) (where \(p\) is different from zero at equilibrium as preferences are monotone), which is bounded from below by \(\min\{1, \frac{m(\zeta) c^j(\xi) A^j(\zeta)}{\chi(\xi)}\}\), with: \(m(\zeta) = \min\{y(\xi, g) : y(\xi, g) > 0\}\), \(c^j(\xi) = \min\{C^j_2(\xi) : C^j_2(\xi) > 0\}\), \(\chi(\xi) = \max\{A^j_2(\xi), g \in G\}\) and \(\delta_j(\zeta) = \sum_{g \in S(\xi, j)} p(\xi, g)\), where \(S(\xi, j) = \{g \in G : y(\xi, g) > 0\}\) and \(A^j_2(\zeta) > 0\). We omit the proof of non–triviality of the equilibrium as the similarities with the proof in Páscoa and Seghir (2009) are substantial.
Let $E^T(\chi)$ be the compactified economy which has the same characteristics as $E^T$ except for the budget constraints which are now defined by the sets $B^T(p, q, K)$.

**Definition 5.** An equilibrium of the compactified economy $E^T(\chi)$ is a vector $(\overline{p}^T, \overline{q}^T, \overline{K}^T, (\overline{\pi}^T, \overline{\theta}^T, \overline{\varphi}^T, \overline{\Delta}^T)_{i \in I})$ verifying conditions (b) and (c) of Definition 4 and such that:

(i') $\forall i \in I$, $(\overline{\pi}^T, \overline{\theta}^T, \overline{\varphi}^T, \overline{\Delta}^T) \in \text{Argmax } U^T_i(x)$ over $B^T(p, q, K, \chi)$.

**Lemma 2.** Under assumptions [A1], [A2], [A4] and [A5], each compactified economy $E^T(\chi)$ has a non-trivial equilibrium $(\overline{p}^T, \overline{q}^T, \overline{K}^T, (\overline{\pi}^T, \overline{\theta}^T, \overline{\varphi}^T, \overline{\Delta}^T)_{i \in I})$.

**Proof.** Note that for each $i \in I$, $B^T_i$ is upper semicontinuous with nonempty closed convex values. For each $(p, q, K) \in \Pi^{T-1} \times [0, 1]$ and each agent $i \in I$, let us define the set $B^T(p, q, K, \chi)$ by replacing all the inequalities in $B^T(p, q, K, \chi)$ by strict inequalities.

**Claim 2.** $\forall i \in I$, $\forall (p, q, K) \in \Pi^{T-1} \times [0, 1]$ $(\sum_{x \in \Pi^T} i(x))$ $B^T(p, q, K, \chi) \neq 0$.

**Proof.** The proof is done by upward induction as follows:

- At node $\xi = \xi_0$,
  - If $p(\xi_0) \neq 0$, since $\omega^i(\xi_0) \gg 0$, one can choose $x^i(\xi_0) \gg 0$ and $\varphi^i(\xi_0) > 0$, $\varphi^i(\xi_0)$ small enough, such that $p(\xi_0) \cdot \left[ x^i(\xi_0) + C(\xi_0)\varphi^i(\xi_0) \right] < p(\xi_0) \cdot \omega^i(\xi_0)$. Letting $\theta^i(\xi_0) = 0$, one gets that the constraints of the period 0 are satisfied strictly.
  - If $p(\xi_0) = 0$ (then $q(\xi_0) \neq 0$), one can choose $\theta^i(\xi_0) = 0$ and $\varphi^i(\xi_0) \gg 0$, $\varphi^i(\xi_0)$ small enough, such that $q_i(\xi_0) \cdot \varphi^i(\xi_0) > 0$ and the constraints of the period 0 will be satisfied strictly.

- At each $\xi \in \xi_0^+$,
  - If $p(\xi) \neq 0$, since $[\omega^i(\xi) + Y(\xi)x^i(\xi_0)] \gg 0$, one can choose $x^i(\xi) \gg 0$, $\varphi^i(\xi) > 0$, $\varphi^i(\xi)$ small enough and $\Delta^i(\xi) < \Delta^i(\xi)$ such that
    \begin{equation}
    p(\xi) \cdot \left[ x^i(\xi) + C(\xi)\varphi^i(\xi) \right] < p(\xi)Y(\xi) \left[ x^i(\xi_0) + C(\xi_0)\varphi^i(\xi_0) \right]
    \end{equation}
    and
    \begin{equation}
    \varphi^i_j(\xi) < F^{i,j}_\xi \left( \Delta^i(\xi) \right).
    \end{equation}
    Since $a^i_j(\xi) > 0$, one can choose $\varphi^i(\xi)$ small enough to satisfy Inequality (15) and this $\varphi^i(\xi)$ is compatible with the default $\Delta^i$ satisfying (16). Letting $\theta^i(\xi) = 0$, one gets that the constraints of node $\xi$ are satisfied strictly.
  - If $p(\xi) = 0$ (then $q(\xi) \neq 0$), one can choose $\varphi^i(\xi) > 0$ and $\Delta^i(\xi) > 0$ and $\varphi^i_j(\xi) < F^{i,j}_\xi \left( \Delta^i(\xi) \right)$ (as $a^i_j(\xi) > 0$). Take $\theta^i(\xi) = 0$.

- The same ideas can be used until the period $T - 1$. 

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At node $\xi \in D_T$ (i.e. $t(\xi) = T$). Since $p(\xi) \neq 0$, one can choose $x^i(\xi) \in X^i(\xi)$ such that $p(\xi) \cdot x^i(\xi) < p(\xi) \cdot [\omega^i(\xi) + Y(\xi)x^i(\xi^-)]$.

Claim 3. \(\forall i \in I, B^{iT} \text{ is lower semicontinuous.}\)

Proof. It follows from the convexity and the non-emptiness of $B^{iT}(p, q, K, \chi)$ for each $p, q, K \in \Pi^T \times \prod_{\xi \in D_T} J(\xi)$ that $B^{iT}(p, q, K, \chi) = \overline{B^{iT}(p, q, K, \chi)}$. The Claim follows from the fact that $B^{iT}$ is lower semicontinuous \(\square\)

The end of the proof of Proposition 1 follows the same techniques as in Páscoa and Seghir (2009) using Kakutani fixed point theorem and convexity arguments.

Step 2: Asymptotic Results.

Under assumptions [A1], [A2], [A4] and [A5], one has for each node $\xi \in D_T$:

\[ \sum_{(i,g) \in I \times G} [\pi^iT(\xi,g) + \sum_{j \in J(\xi)} C^j_g(\xi) \pi^iT_j(\xi)] \leq W^T \sum_{n=0}^{+\infty} k^n = \frac{W^T}{1-k} < +\infty, \]

\[ \sum_{i \in I} \theta^iT_j(\xi) \leq \frac{1}{c^j(\xi)} \frac{W^T}{1-k}, \]

\[ \sum_{i \in I} \phi^iT_j(\xi) \leq \frac{1}{c^j(\xi)} \frac{W^T}{1-k}, \]

where $c^j(\xi) = \min\{C^j_g(\xi) : C^j_g(\xi) > 0\}$. In view of conditions (17)–(20) and the countability of $D$, we get, via a diagonalization procedure as in Araujo et al. (2002), a sequence $\{T_k\}_{k \in \mathbb{N}}$ such that \((\pi^{Tk}, \theta^{Tk}, \phi^{Tk}, \Delta^{Tk})\) which converges, at each node, to some \((\pi^i, \theta^i, \phi^i, \Delta^i)\) in $B^{iT}(p, q, K)$.

Proposition 2. For each agent $i \in I$, the cluster point $(\pi^i, \theta^i, \phi^i, \Delta^i)$ is optimal in $B^{iT}(p, q, K)$.

Proof. Suppose that there exists an agent $i$ and a plan $(\tilde{\pi}^i, \tilde{\theta}^i, \tilde{\phi}^i, \tilde{\Delta}^i) \in B^{iT}(p, q, K)$ such that $U^i(\tilde{\pi}^i) > U^i(\pi^i) > 0$. Then, there is $T$ such that for every $T > T$, $\sum_{\xi \in D_T} v^i_T(\tilde{\pi}^i) > U^i(\pi^i)$. 
Let us fix $\hat{T} > T$ such that for each node $\sigma$ : $t(\sigma) = \hat{T} + 1$ the credit constraint functions satisfy Assumption [A4]. For each plan $y := (y(\xi), \xi \in D)$ let us define the following correspondence: $\psi^\hat{T}$ and $\beta^\hat{T}$:

$$\psi^\hat{T}(y) := \{(x(\xi), \xi \in D^T) \mid U^i(x) > U^i(y)\}.$$ 

Moreover, for each price and expected delivery rate process $(p, q, K)$, let us define the following correspondence:

$$\beta^\hat{T}(p, q, K) = \{(x(\xi), \xi \in D^T) : \exists (\theta(\xi), \varphi(\xi), \Delta(\xi), \xi \in D^T) \text{ s. t. } (x(\xi), \theta(\xi), \varphi(\xi), \Delta(\xi)) \text{ satisfies the budget constraints (3), (4) and (5) of the original economy } \mathcal{E} \text{ at } (p, q, K)\}.$$ 

Since $\hat{x} \in \beta^\hat{T}(p, q, K) \cap \psi^\hat{T}(\pi)$ and $\beta^\hat{T}(p, q, K) \cap \psi^\hat{T}(\pi)$ is lower semicontinuous with respect to the product topology on $L^\infty(D)$ (recall that $U^i$ is weak star upper semicontinuous and apply Hildenbrand (1974), p.35, Prob.6 (1)), one gets the existence of $T^*$ and a sequence $\tilde{x}^T$ converging, node by node, to $\hat{x}$ such that $\forall T \geq T^*$, $\tilde{x}^T \in \beta^\hat{T}(\tilde{p}^T, \tilde{q}^T, \tilde{K}^T) \cap \psi^\hat{T}(\tilde{\pi}^T)$. 

With no loss of generality, one can assume that $T^* > \hat{T}$. Take $T = T^*$ to get that $U^i(\tilde{x}^T) > U^i(\pi_T^T)$ and the existence of $(\tilde{\theta}^T, \tilde{\varphi}^T, \tilde{\Delta}^T)$ such that $(\tilde{x}^T, \tilde{\theta}^T, \tilde{\varphi}^T, \tilde{\Delta}^T)$ satisfies the budget constraints till $\hat{T}$ at $(\tilde{p}^T, \tilde{q}^T, \tilde{K}^T)$. 

Let $\epsilon > 0$ and $\alpha \in ]0, 1[$ and let us define the following changes:

$$x^T_g(\xi) = \begin{cases} \frac{\tilde{x}^T_g(\xi, g)}{\tilde{\pi}^T(\xi, g)} & \text{if } t(\xi) \leq \hat{T} - 1 \\ \tilde{x}^T_g(\xi, g) + \frac{\epsilon}{\tilde{\pi}^T(\xi, g)} & \text{if } t(\xi) = \hat{T} \\ 0 & \text{if } t(\xi) > \hat{T} \end{cases},$$

$$\varphi^T_j(\xi) = \begin{cases} \frac{\tilde{\varphi}^T_j(\xi)}{\tilde{\Delta}^T(\xi)} - \frac{\epsilon}{\tilde{\Delta}^T(\xi) \tilde{\pi}_u} & \text{if } t(\xi) = \hat{T} \\ 0 & \text{if } t(\xi) > \hat{T} \end{cases},$$

Clearly, $(x^T, \theta^T, \varphi^T, \Delta^T)$ satisfies the budget constraints up to time $\hat{T} - 1$ of the truncated economy $\mathcal{E}^T$. Moreover, $(x^T, \theta^T, \varphi^T, \Delta^T)$ satisfies the credit constraint (4) at $\hat{T}$ node $\xi$. In addition, Assumption [A4] guarantees that by choosing this new default-short-sales vector, agent $i$ can increase his consumption, from $\tilde{x}^T$ to $x^T$, at node $\xi$, which contradicts the optimality of $(\pi^T, \tilde{\theta}^T, \tilde{\varphi}^T, \tilde{\Delta}^T)$ in $B^i(\tilde{p}^T, \tilde{q}^T, \tilde{K}^T)$. 

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