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Fixed points of abelian actions on S2

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Abstract

We prove that if $\mathcal{F}$ is a finitely generated abelian group of orientation preserving C^1 diffeomorphisms of $\mathbb{R}^2$ which leaves invariant a compact set then there is a common fixed point for all elements of $\mathcal{F}$. We also show that if $\mathcal{F}$ is any abelian subgroup of orientation preserving C^1 diffeomorphisms of S^2 then there is a common fixed point for all elements of a subgroup of $\mathcal{F}$ with index at most two.

1 Introduction and Notation

The *global fixed point set*, denoted $\mathrm{Fix}(\mathcal{F})$, of a group $\mathcal{F}$ of self maps of a space S is the set of points that are fixed by every $f \in \mathcal{F}$. Our main results are for $S = S^2$ or $S = \mathbb{R}^2$ and for $\mathcal{F}$ a subgroup of the group $\mathrm{Diff}^1_+(S)$ of orientation preserving C^1 diffeomorphisms of S.

The Lefschetz theorem implies that every element of $\mathrm{Diff}^1_+(S^2)$ has fixed points. Bonatti [1] proved that if $\mathcal{F}$ is a finitely generated abelian subgroup of $\mathrm{Diff}^1_+(S^2)$ and if a generating set for $\mathcal{F}$ is sufficiently C^1-close to the identity, then $\mathrm{Fix}(\mathcal{F}) \neq \emptyset$. On the other hand, the diffeomorphisms R_x and R_y of S^2 determined by rotation by π about the x-axis and about the y -axis of $\mathbb{R}^3$ commute but have no common fixed points.

If $f, g \in \mathrm{Homeo}_+(S^2)$ commute and f_t, g_t are isotopies from $f = f_1$ and $g = g_1$ to $id = f_0 = g_0$ define $w(f,g) \in \pi_1(\mathrm{Homeo}_+(S^2), \mathrm{identity}) \cong \mathbb{Z}/2\mathbb{Z}$ to be the element determined by the loop $\gamma(t) = [f_t, g_t]$ in $\mathrm{Homeo}_+(S^2)$ with basepoint at the identity. We show below that $w(f,g)$ is independent of the choice of the isotopies f_t and g_t and that if an abelian subgroup $\mathcal{F}$ of $\mathrm{Homeo}_+(S^2)$ has a global fixed point then $w(f,g) = 0$ for all $f, g \in \mathcal{F}$.

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A result of the second author [10] shows that if f and g are commuting orientation preserving diffeomorphisms of S^2 and $w(f,g) = 0$ then f and g have a common fixed point. Our main result concerning S^2 is a generalization to arbitrary abelian subgroups of $\mathrm{Diff}^1_+(S^2)$.

Theorem 1.1. *For any abelian subgroup $\mathcal{F}$ of $\mathrm{Diff}^1_+(S^2)$ there is a subgroup $\mathcal{F}'$ of index at most two such that $\mathrm{Fix}(\mathcal{F}') \neq \emptyset$. Moreover, $\mathrm{Fix}(\mathcal{F}) \neq \emptyset$ if and only if $w(f,g) = 0$ for all $f, g \in \mathcal{F}$.*

Druck, Fang and Firmo [5] generalized Bonatti's result to include nilpotent groups. We conjecture that Theorem 1.1 will also generalize to the nilpotent case.

We derive Theorem 1.1 as a consequence of a related result about abelian subgroups of $\mathrm{Diff}^1_+(R^2)$. The Brouwer plane translation theorem implies that if $f \in \mathrm{Homeo}_+(R^2)$ preserves a compact set then $\mathrm{Fix}(f) \neq \emptyset$. We generalize this as follows.

Theorem 1.2. *If $\mathcal{F}$ is a finitely generated abelian subgroup of $\mathrm{Diff}^1_+(\mathbb{R}^2)$ and if there is a compact $\mathcal{F}$-invariant set $C \subset \mathbb{R}^2$, then $\mathrm{Fix}(\mathcal{F})$ is non-empty.*

We also prove the following corollary concerning diffeomorphisms of closed two disk $\mathbb{D}^2$.

Corollary 1.3. *If $\mathcal{F}$ is an abelian subgroup of $\mathrm{Diff}^1_+(\mathbb{D}^2)$ then $\mathrm{Fix}(\mathcal{F})$ is non-empty.*

Related results were obtained by Hirsch [13].

Theorem 1.2 has applications to other surfaces because the plane is the universal cover of all surfaces but the unpunctured sphere. This is the reason that the proof of Theorem 1.1 makes use of Theorem 1.2. We also have the following result, which generalizes Corollary 0.5 of [10].

Theorem 1.4. *Suppose that M is a closed orientable surface of negative Euler characteristic, that $\mathcal{F}$ is a finitely generated abelian subgroup of $\mathrm{Diff}^1_+(M)$ and that $f \in \mathcal{F}$. Let $\phi : M \to M$ be a Thurston normal form for f. If there is a subsurface M_i of M on which ϕ restricts to a pseudo-Anosov homeomorphism and if $z \in \mathrm{Fix}(\phi|_{\mathrm{Int}(M_i)})$ then there exists $x \in \mathrm{Fix}(\mathcal{F})$ such that the f-Nielsen class of x corresponds, via the isotopy between f and ϕ, to the ϕ-Nielsen class of z.*

Theorem 1.4 is in fact a special case of a result on relative Nielsen classes. The full statement is Proposition 7.1.

We make use of the obvious fact that if $\mathcal{F}$ is a subgroup of $\mathrm{Homeo}_+(S)$ generated by $\{f_i\}$ then $\mathrm{Fix}(\mathcal{F}) = \cap \mathrm{Fix}(f_i)$. If $\mathcal{F}$ is generated by $f_1, \dots, f_n$ then we may write $\mathrm{Fix}(\mathcal{F}) = \mathrm{Fix}(f_1, \dots, f_n)$ and for convenience we define $\mathrm{Fix}(\emptyset) = S$.

2 Normal Form

In this section, we assume that S is a finitely punctured surface.

We defined a normal form for an element $f \in \mathrm{Diff}^1_+(S)$ relative to its fixed point set in [7]. Given the Thurston classification theorem [16], the existence of a normal form for f follows from the existence of a subsurface $W \subset S$ that contains all but finitely many elements of $\mathrm{Fix}(f)$ and such that f is isotopic rel $\mathrm{Fix}(f)$ to a homeomorphism θ that restricts to the identity on W. We are now interested in abelian subgroups of $\mathrm{Diff}^1_+(S)$ and it is natural to work relative to their global fixed point sets. Our main results are contained in Lemma 2.6, Lemma 2.7 and Lemma 2.8 which allow us to choose W in a canonical way.

Assume that $\mathcal{F}$ is a finitely generated subgroup of $\mathrm{Diff}^1_+(S)$ and that B and C are $\mathcal{F}$-invariant compact subsets of S such that:

- $B \subset \mathrm{Fix}(\mathcal{F})$.
- $C \cap \mathrm{Fix}(\mathcal{F}) = \emptyset$.
- $\mathcal{F}$ is abelian up to isotopy rel $B \cup C$.

Remark 2.1. In our applications $\mathcal{F}$ will be abelian but it is easier to work in the category of 'abelian up to relative isotopy'.

Definition 2.2. *$\mathcal{W}(\mathcal{F}, B, C)$ is the set of compact subsurfaces $W \subset S$ such that:*

($\mathcal{W}$-1) *∂W has finitely many components, each of which is contained in and is essential in $M := S \setminus (B \cup C)$.*

($\mathcal{W}$-2) *W contains all but finitely many points of B and every component of W intersects B in an infinite set.*

($\mathcal{W}$-3) *for all $f \in \mathcal{F}$ the following is satisfied:*

($\mathcal{W}$-3f) *there exists $\phi : S \to S$ isotopic to f rel $B \cup C$ such that $\phi|_W$ is the identity.*

Remark 2.3. Property ($\mathcal{W}$-3) implies that each $W \in \mathcal{W}(\mathcal{F}, B, C)$ is disjoint from C.

Remark 2.4. If W_i satisfies ($\mathcal{W}$-1), ($\mathcal{W}$-2) and ($\mathcal{W}$-3f_i) for $i = 1, 2$ and if ∂W_1 and ∂W_2 intersect transversely, then the subsurface W obtained from $W_1 \cap W_2$ by removing any components that intersect B in a finite set, satisfies ($\mathcal{W}$-1), ($\mathcal{W}$-2), ($\mathcal{W}$-3f_1) and ($\mathcal{W}$-3f_2).

Property ($\mathcal{W}$-2) implies that $\mathcal{W}(\mathcal{F}, B, C)$ is empty if B is finite. The smoothness of $\mathcal{F}$ is used only in the following lemma.

Lemma 2.5. *If B is infinite then $\mathcal{W}(\mathcal{F}, B, C)$ is non-empty.*

Proof. Lemma 4.1 of [10] and the isotopy extension theorem (see the proof of Theorem 1.2 of [7]) imply that there is a subsurface $W(f)$ satisfying ($\mathcal{W}$-1), ($\mathcal{W}$-2) and ($\mathcal{W}$-3f) for any single $f \in \mathcal{F}$. Since $\mathcal{F}$ is finitely generated, Remark 2.4 produces $W \in \mathcal{W}(\mathcal{F}, B, C)$. □

There is a partial order on $\mathcal{W}(\mathcal{F}, B, C)$ defined by : $W_1 < W_2$ if and only if W_1 is isotopic rel $B \cup C$ to a subsurface of W_2 but is not isotopic rel $B \cup C$ to W_2. Our next lemma gives sufficient conditions for the existence of a maximal element of $\mathcal{W}(\mathcal{F}, B, C)$. Before stating it we recall the idea of Nielsen equivalence for fixed points of isotopic homeomorphisms.

Let $h : N \to N$ be a homeomorphism of a connected surface N and let $x, y \in \text{Fix}(h)$. We say that *x is Nielsen equivalent to y* if there is an arc $\alpha \subset N$ connecting x to y such that $h(\alpha)$ is homotopic to α relative to endpoints. Equivalently some, and hence every, lift $\tilde{h} : \tilde{N} \to \tilde{N}$ to the universal cover that fixes a lift $\tilde{x}$ of x also fixes a lift $\tilde{y}$ of y. The Nielsen equivalence class of $x \in \text{Fix}(f)$ is denoted $N(f, x)$.

If $\tilde{h}$ is a lift of h and $\text{Fix}(\tilde{h}) \neq \emptyset$ then the projection of $\text{Fix}(\tilde{h})$ into N is an entire Nielsen class μ of $\text{Fix}(h)$. We say that $\tilde{h}$ is a *lift for* μ and that μ is *the Nielsen class determined by* $\tilde{h}$. Another lift of h is also a lift for μ if and only if it equals $T\tilde{h}T^{-1}$ for some covering translation T.

Given an isotopy h_t from h_0 to h_1 and a Nielsen class μ_0 for h_0, let $\tilde{h}$ be a lift for μ_0 and let $\tilde{h}_t$ be the lift of h_t that begins with $\tilde{h}_0$. The Nielsen class μ_t for h_t determined by $\tilde{h}_t$ is independent of the choice of $\tilde{h}_0$. We say that *μ_t is the Nielsen class determined by μ and h_t*. If $x \in \text{Fix}(h_s)$ is an element of μ_s and $y \in \text{Fix}(h_t)$ is an element of μ_t then we write $N(h_s, x) \sim N(h_t, y)$. In particular, $N(h_t, x) \sim N(h_t, y)$ if and only if $N(h_t, x) = N(h_t, y)$.

Lemma 2.6. *Suppose that B is infinite and that either of the following is satisfied.*

1. *C is finite.*

2. *$\mathcal{F} = \langle f \rangle$, $\text{Fix}(f)$ is compact and B is a union of Nielsen classes of $\text{Fix}(f)$. In particular, B could be all of $\text{Fix}(f)$.*

Then $\mathcal{W}(\mathcal{F}, B, C)$ has maximal elements.

Proof. If $W_1 \in \mathcal{W}(\mathcal{F}, B, C)$ is not contained in a maximal element of $\mathcal{W}(\mathcal{F}, B, C)$ then there is an infinite increasing sequence $W_1 \subset W_2 \subset \ldots$ of non-isotopic elements of $\mathcal{W}(\mathcal{F}, B, C)$. We may assume that $W_l \cap B$ and the number of components of W_l are independent of l. The number of components of $S \setminus W_l$ is unbounded. We may therefore choose l so that some complementary component of W_l is a disk D that is disjoint from B and from any chosen finite subset of C. Since ∂D is essential in M, it must be that C is infinite.

We now assume that $\mathcal{F} = \langle f \rangle$ and that B is a union of Nielsen classes of $\mathrm{Fix}(f)$. Let $C_D = C \cap D$, let U be the component of M that contains ∂D and let $U_D = U \cup D$. In other words, U_D is U with the punctures in D filled in. Choose a component $\tilde{D}$ of the full pre-image of D in the universal cover $\tilde{U}_D$ of U_D. Then $\tilde{D}$ is a disk and C_D lifts to a compact subset of $\tilde{D}$ that is invariant under a lift $\widetilde{f|_{U_D}}$ of $f|_{U_D}$. By the Brouwer translation theorem, there is at least one fixed point $\tilde{x} \in \widetilde{f|_{U_D}}$. Let $x \in U_D$ be the image of $\tilde{x}$ and note that $x \notin B$. To complete the proof we will show that $N(f, x)$ contains an element of B which will contradict the assumption that B is a union of Nielsen classes.

There exists $\phi : S \to S$ isotopic to f rel $B \cup C$ such that $\phi|_{W_l}$ is the identity. The isotopy from ϕ to f lifts to an isotopy between $\widetilde{f|_{U_D}}$ and a lift $\widetilde{\phi|_{U_D}}$ of $\phi|_{U_D}$. This lift setwise preserves $\tilde{D}$ and so has a fixed point $\tilde{y} \in \tilde{D}$. Let $y \in \mathrm{Fix}(\phi) \cap D$ be its projected image and let Y be the component of W_l containing ∂D. Choose $b \in B \cap Y$ and $z \in \partial D$. Then

$$N(f, b) \sim N(\phi, b) = N(\phi, z) = N(\phi, y) \sim N(f, x)$$

where the first relation follows from the fact that the isotopy between ϕ and f is relative to B, the second from the fact that $Y \subset \mathrm{Fix}(\phi)$, the third from the fact that D is a ϕ-invariant disk and the last from the definition of $\sim$. $\square$

Lemma 2.7. *Suppose that W is a maximal element of $\mathcal{W}(\mathcal{F}, B, C)$ and that σ is either a simple closed curve in M or a simple arc whose interior is in M and whose endpoints are in B. Suppose further that the isotopy class of σ rel $B \cup C$ is fixed by each $f \in \mathcal{F}$. Then σ is isotopic rel $B \cup C$ to a closed curve or arc that is disjoint from ∂W.*

Proof. All isotopies in this proof are rel $B \cup C$. After performing an isotopy, we may assume without loss that σ intersects ∂W transversely and that no component of $S \setminus (\sigma \cup \partial W)$ is a disk in the complement of $B \cup C$ whose boundary consists of an arc in σ and an arc in ∂W. We assume that $\sigma \cap \partial W \neq \emptyset$ and argue to a contradiction.

Let W' be the essential subsurface obtained from a regular neighborhood of $W \cup \sigma$ by adding in all contractible components of its complement. Note that if σ is an arc ending in a point of $b \in B$ not in W then b is an isolated point of B since all but finitely many points of B are in the interior of W. Now $W \subset W'$ and W is not isotopic to W'. We will complete the proof by showing that $W' \in \mathcal{W}(\mathcal{F}, B, C)$ in contradiction to the maximality of W.

Fix $f \in \mathcal{F}$. Write σ as an alternating concatenation of subpaths α_i and β_i where $\alpha_i \subset W$ and $\beta_i \cap W = \partial \beta_i$. By Lemma 3.5 of [11] we may assume, after an isotopy, that $f|_{\partial W \cup \sigma}$ is the identity. In particular, $f(W) = W$. There is a further isotopy f_t from $f_0 = f$ to $f_1 = f'$ such that $f_t(W) = W$ for all t and such that $f'|_W$ is the identity. If C is a component of W that contains some α_i then one may assume that $f_t|_{\partial C}$ is the identity for all t. This follows from the fact that both f and f' pointwise fix α_i which implies that the isotopy f_t has no net rotation about ∂C. We may therefore assume that f_t pointwise fixes each β_j for all t and after an obvious modification we may assume that $f_1|_{W'}$ is the identity. This proves that $W' \in \mathcal{W}(\mathcal{F}, B, C)$ as desired. $\square$

Lemma 2.8. *Any two maximal elements of $\mathcal{W}(\mathcal{F}, B, C)$ are isotopic rel $B \cup C$.*

Proof. Suppose that W_1 and W_2 are maximal elements of $\mathcal{W}(\mathcal{F}, B, C)$. We may assume by Lemma 2.7 that components of ∂W_1 and ∂W_2 are disjoint or equal. Moreover, we may assume that a component of ∂W_1 is isotopic rel $B \cup C$ to a component of ∂W_2 only if they are equal or bound an open annulus in $S \setminus (W_1 \cup W_2 \cup B \cup C)$.

If $W_1 \neq W_2$ then the interior of one of them contains points not in the interior of the other. We will show this leads to a contradiction. So assume without loss of generality that there is a non-empty component of $int(W_2) \setminus W_1$ and let Y denote its closure. Then Y is a compact surface with boundary, is disjoint from C and contains at most finitely many elements of B.

Let W_2^0 denote the component of W_2 which contains Y. Since $W_2^0 \cap B$ is infinite, at least one component of $W_2^0 \setminus Y$ must intersect B in an infinite set. Call the closure of such a component Z and note that $Y \cap Z \subset \partial W_1$. If $Y \cap B \neq \emptyset$ then there is an arc $\sigma \subset W_2$ connecting an element of B in Y to an element of B in $Z \cap W_1$. This contradicts Lemma 2.7 and the fact that σ is isotopic rel $B \cup C$ to its image under any element of $\mathcal{F}$. We conclude that $B \cap Y = \emptyset$. A similar argument shows that Z is the only component of $W_2^0 \setminus Y$ which intersects B, since otherwise there would be an arc joining points of B in different components of $W_2^0 \setminus Y$ and crossing a boundary component of Y transversely in a single point. This would also contradict Lemma 2.7.

We next assume Y is not an annulus. There must be at least one common boundary component of Z and Y. If there is more than one then there is a simple closed curve σ in $int(Y \cup Z) \subset int(W_2^0)$ which has non trivial intersection number with two of these common components. Since $\sigma \subset W_2^0$ it is isotopic rel $B \cup C$ to its image under any element of $\mathcal{F}$. But it cannot be isotoped to be disjoint from $Y \cap Z$ which contradicts Lemma 2.7.

We may therefore assume that $Z \cap Y$ consists of a single common boundary component; call it X. There is then a simple arc σ in $int(Y \cup Z) \subset int(W_2^0)$ joining points of B, crossing X transversely twice and representing a non-trivial element of $H_1(Z \cup Y, Z)$. We now have compact surfaces $Y \subset W_2^0 \subset S$ and we wish to lift to the universal covering space $\tilde{S}$. More precisely, choose a component $\tilde{Y}$ of the complete lift of Y and a component $\tilde{W}$ of the complete lift of W_2^0 such that $\tilde{Y} \subset \tilde{W} \subset \tilde{S}$. Note that $\tilde{Y}$ separates $\tilde{W}$ and the fact that σ represents a non-trivial element of $H_1(Z \cup Y, Z)$ means that it has a lift $\tilde{\sigma}$ to a simple arc in $\tilde{W}$ with endpoints separated by $\tilde{Y}$. If σ' is a simple arc in W_2^0 isotopic in S to σ rel $B \cup C$ then it has a lift $\tilde{\sigma}'$ with the same endpoints as $\tilde{\sigma}$. It follows that $\tilde{\sigma}' \cap \tilde{Y} \neq \emptyset$ and $\sigma' \cap Y \neq \emptyset$ contradicting Lemma 2.7. This completes the proof when Y is not an annulus.

If Y is an annulus then each of its boundary components is a boundary component of W_1, W_2 or both. If one is in ∂W_1 and the other is in ∂W_2 we would contradict the assertion above that such boundary components can be isotopic rel $B \cup C$ only if the open annulus they bound lies in $S \setminus (W_1 \cup W_2 \cup B \cup C)$. If both components of ∂Y are contained in ∂W_2 then Y would be a component of W_2 which is a contradiction. Hence both components of ∂Y must be in ∂W_1 and disjoint from ∂W_2. In this case $W_2^0 \setminus int(Y)$ can have no component which is an unpunctured annulus.

Let Z' be a component of $W_2^0 \setminus int(Y)$ which is different from Z. It is disjoint from B and is not an annulus. In this case too there is an arc σ with endpoints in B whose interior lies in $W_2^0 \setminus B$ which crosses X transversely twice and represents a non-trivial element of $H_1(Z \cup Y \cup Z', Z)$. The same covering space argument given above shows if σ' is a simple arc in W_2^0 isotopic in S to σ rel $B \cup C$ then $\sigma' \cap X \neq \emptyset$. This again contradicts Lemma 2.7. $\square$

The following result is proved in [6] (see Théorème III of Exposé 12) at least in the case that S has no punctures. The proof we give here is quite different and very much in the spirit of extracting information from the Nielsen classes of fixed points of iterates which we continue in later sections. More details about Nielsen classes (including definitions) are given at the beginning of Section 4 and in references cited there. For the following result we use only the fact that interior fixed points of pseudo-Anosov homeomorphisms are unique in their Nielsen class.

Lemma 2.9. *Let S be a surface, perhaps finitely punctured, with negative Euler characteristic and let f be a pseudo-Anosov homeomorphism of S. If g is another pseudo-Anosov homeomorphism of S which is homotopic to f then there exists a unique homeomorphism $h : S \to S$ which is homotopic to the identity and satisfies $h \circ f = g \circ h$.*

Proof. We consider the Poincaré disk model for the hyperbolic plane $\mathbb{H}$. In this model, $\mathbb{H}$ is identified with the interior of the unit disk and geodesics are segments of Euclidean circles and straight lines that meet the boundary in right angles. A choice of complete hyperbolic structure on S provides an identification of the universal cover $\tilde{S}$ of S with $\mathbb{H}$. Under this identification covering translations become isometries of $\mathbb{H}$ and geodesics in S lift to geodesics in $\mathbb{H}$. The compactification of the interior of the unit disk by the unit circle induces a compactification of $\mathbb{H}$ by the 'circle at infinity' S_∞. Geodesics in $\mathbb{H}$ have unique endpoints on S_∞. Conversely, any pair of distinct points on S_∞ are the endpoints of a unique geodesic.

Suppose that $F : S \to S$ is an orientation preserving homeomorphism of S. We can use the identification of $\mathbb{H}$ with $\tilde{M}$ and write $\tilde{F} : \mathbb{H} \to \mathbb{H}$ for lifts of $F : S \to S$ to the universal cover. A fundamental result of Nielsen theory is that every lift $\tilde{F} : \mathbb{H} \to \mathbb{H}$ extends uniquely to a homeomorphism (also called) $\tilde{F} : \mathbb{H} \cup S_\infty \to \mathbb{H} \cup S_\infty$ and the restriction of this homeomorphism to S_∞ depends only on the homotopy class of F and the choice of lift $\tilde{F}$. Using the identification above we will typically consider $\tilde{F}$ to be a homeomorphism of $\tilde{S} \cup S_\infty$. The homeomorphism F induces an isomorphism (which we will denote by $F_\#$) of the fundamental group of S or equivalently of the group of covering translations of $\tilde{S}$.

Let X denote the compactified universal covering space of S, i.e $X = \tilde{S} \cup S_\infty$. Let $\tilde{f} : X \to X$ be the extension of a lift of f and let $\tilde{g} : X \to X$ be the extension of the lift of g obtained by lifting the homotopy from f to g starting at $\tilde{f}$. If x is a point of period p of f and $\tilde{x}$ is a lift of x then there is a unique covering translation T_0 such that $T_0 \circ \tilde{f}^p(\tilde{x}) = \tilde{x}$. There is a unique point $\tilde{y} \in \tilde{S}$ which satisfies $T_0 \circ \tilde{g}^p(\tilde{y}) = \tilde{y}$. This is because f^p and g^p are homotopic pseudo-Anosov homeomorphisms and for any pseudo-Anosov homeomorphism any interior fixed point is the unique point in an essential Nielsen class.

The assignment $\tilde{x} \mapsto \tilde{y}$ defines a bijection $\tilde{h} : \widetilde{\mathrm{Per}}(f) \to \widetilde{\mathrm{Per}}(g)$ from the lifts of all periodic points of f to the lifts of all periodic points of g. This function satisfies $\tilde{h} \circ \tilde{f} = \tilde{g} \circ \tilde{h}$ because

$$f_\#^{-1}(T_0) \circ \tilde{f}^p(\tilde{f}(\tilde{x})) = f_\#^{-1}(T_0) \circ \tilde{f} \circ \tilde{f}^p(\tilde{x}) = \tilde{f} \circ T_0 \circ \tilde{f}^p(\tilde{x}) = \tilde{f}(\tilde{x}).$$

so $\tilde{f}(\tilde{x})$ is the unique fixed point in $\tilde{S}$ of $f_\#^{-1}(T_0) \circ \tilde{f}^p$. Similarly $\tilde{g}(\tilde{y})$ is the unique fixed point of $g_\#^{-1}(T_0) \circ \tilde{g}^p$. Since $g_\# = f_\#$ we conclude that $\tilde{h}(\tilde{f}(\tilde{x})) = \tilde{g}(\tilde{y}) = \tilde{g}(\tilde{h}(\tilde{x}))$. A similar computation shows that $T \circ \tilde{h} = \tilde{h} \circ T$ for any covering translation T.

Let $\mathcal{F}^u(f)$ and $\mathcal{F}^s(f)$ denote the f-invariant expanding and contracting foliations and let $\tilde{\mathcal{F}}^u(f)$ and $\tilde{\mathcal{F}}^s(f)$ denote their lifts to the universal covering space $\tilde{S}$. The leaves of these foliations have well defined ends in S_∞. (See [6] for more details on the properties of these foliations.) Note that any leaf of $\tilde{\mathcal{F}}^u(f)$ intersects a leaf of $\tilde{\mathcal{F}}^s(f)$ in at most one point and there is a point of intersection precisely if the ends of the two leaves are linked in S_∞. By linked we mean that the ends (there may be more than two) of one leaf separate the ends of the other. Note also that the ends of the leaf $\tilde{W}^u(\tilde{x}, f)$ containing $\tilde{x}$ (the fixed point of $T_0 \circ \tilde{f}^p$) are precisely the attracting fixed points of the restriction of $T_0 \circ \tilde{f}^p$ to S_∞ and hence they are also the ends of the leaf $\tilde{W}^u(\tilde{h}(\tilde{x}), g)$. This implies that linking properties of the ends of leaves through points of $\widetilde{\mathrm{Per}}(f)$ are preserved by $\tilde{h}$ as are the order properties of those leaves. By this we mean if such a leaf in $\mathcal{F}^u(f)$ separates two others then the corresponding leaves in $\mathcal{F}^u(g)$ will have the corresponding separation properties.

From this it is straightforward to see that $\tilde{h}$ may be extended uniquely to a bijection from a dense set of lifted heteroclinic points in $\tilde{W}^u(\tilde{x}, f)$ to lifted heteroclinic points of $\tilde{W}^u(\tilde{h}(\tilde{x}), g)$ by defining $\tilde{h}(\tilde{W}^u(\tilde{x}, f) \cap \tilde{W}^s(\tilde{z}, f)) = \tilde{W}^u(\tilde{h}(\tilde{x}), g) \cap \tilde{W}^s(\tilde{h}(\tilde{z})g)$ for all $\tilde{z} \in \widetilde{\mathrm{Per}}(f)$ for which $\tilde{W}^u(\tilde{x}, f) \cap \tilde{W}^s(\tilde{z}, f)$ is non-empty.

Moreover, the function $\tilde{h}$ is order preserving on the dense subset of $\tilde{W}^u(\tilde{x}, f)$ on which it is defined and hence extends uniquely to a continuous order preserving function $\tilde{h} : \tilde{W}^u(\tilde{x}, f) \to \tilde{W}^u(\tilde{h}(\tilde{x}), g)$. Doing the same construction for all $\tilde{z} \in \widetilde{\mathrm{Per}}(f)$ we extend $\tilde{h}$ to the union of all unstable leaves through points of $\widetilde{\mathrm{Per}}(f)$. It is clear that $\tilde{h}$ so defined respects those stable foliation leaves which contain points of $\widetilde{\mathrm{Per}}(f)$ and the restriction of $\tilde{h}$ to the lifted heteroclinic points in a single stable leaf is order preserving. Hence there is a unique extension of $\tilde{h}$ to the union of all stable and unstable leaves which contain a point of $\widetilde{\mathrm{Per}}(f)$. Finally the local product structure of the foliations is preserved and hence there is a unique continuous extension of $\tilde{h}$ to $\tilde{h} : \tilde{S} \to \tilde{S}$.

If $x \in \widetilde{\mathrm{Per}}(f)$ the ends of the leaf $\tilde{W}^u(\tilde{x}, f)$ coincide with the ends of the leaf $\tilde{W}^u(\tilde{h}(\tilde{x}), g)$ since both are the attracting fixed points of $\tilde{f} = \tilde{g}$ on S_∞. Thus setting $\tilde{h} = id$ on S_∞ extends it continuously to all of X. We have $\tilde{h} \circ \tilde{f} = \tilde{g} \circ \tilde{h}$ and $T \circ \tilde{h} = \tilde{h} \circ T$ for every covering translation T, because these equations hold on the dense subset $\widetilde{\mathrm{Per}}(f)$.

Thus $\tilde{h}$ is the lift of a homeomorphism $h : S \to S$. which satisfies $h \circ f = g \circ h$ and which is isotopic to the identity since $\tilde{h}$ commutes with covering translations. If k were another such conjugacy from f to g then its identity lift $\tilde{k}$ would have to agree with $\tilde{h}$ on the dense set $\widetilde{\mathrm{Per}}(f)$ and hence they would agree everywhere. $\square$

The following result in the case $n = 2$ is essentially contained in Lemmas 2.2 and 2.3 of [10].

Lemma 2.10. *Let S be a surface, perhaps finitely punctured, with negative Euler characteristic and $A = \langle \beta_1, \dots, \beta_n \rangle$ an abelian subgroup of $\mathrm{MCG}^+(S)$. Suppose that either some element of A is pseudo-Anosov, or A is finite. Then there is a homomorphism $\phi : A \to \mathrm{Homeo}^+(S)$ such that $\phi(g)$ is a canonical Thurston representative in the mapping class g. Moreover if A contains a pseudo-Anosov element then $\phi(A)$ is the direct sum of an infinite cyclic group with a finite group.*

Proof. In the case that $A = \langle \beta_1, \dots, \beta_n \rangle$ is finite, Kerchoff's solution of the Nielsen realization problem([14]) implies that there is a Riemannian metric on S and homomorphism ϕ from A to the isometries of this metric such that for each $\beta \in A$, $\phi(\beta)$ is in the mapping class β.

If A contains an irreducible pseudo-Anosov element β_0, we will prove the result when A is the centralizer of β_0. Let f_0 be a pseudo-Anosov diffeomorphism with β_0 equal to the isotopy class $[f_0]$ of f_0. Suppose β is in the centralizer of β_0 and h is a diffeomorphism with $\beta = [h]$. Then $\beta_0 = [h^{-1} \circ f_0 \circ h]$ so Lemma 2.9 implies there is a unique homeomorphism $k : S \to S$, homotopic to the identity, such that

$$(h \circ k)^{-1} \circ f_0 \circ (h \circ k) = f_0.$$

In other words there is a unique choice of h representing β that commutes with f_0. We define $\phi : A \to \mathrm{Homeo}^+(S)$ by letting $\phi(\beta)$ be this unique h. The uniqueness property implies that ϕ is a homomorphism. It is clear that $\beta = [\phi(\beta)]$.

To see that $h = \phi(\beta)$ is a Thurston canonical form we observe that it preserves both the stable and unstable foliations of f_0. These foliations possess transverse invariant measures μ_s and μ_u which are unique up to a scalar multiple (See Exposé 12 of Fathi, Laudenbach, and Poenaru [6]). Since h carries one such measure to another there is a unique positive constant $\alpha_u(h)$ such that $h_*(\mu_u) = \alpha_u(h)\mu_u$. Similarly there is a unique positive constant $\alpha_s(h)$ such that $h_*(\mu_s) = \alpha_s(h)\mu_s$. The fact that h is homeomorphism implies that the area of $h(S)$ equals the area of S so $\alpha_s(h)\alpha_u(h) = 1$.

The functions α_s and α_u are homomorphisms to $\mathbb{R}^+$, the positive reals under multiplication. If $\alpha_u(h) = 1$ and h has an interior fixed point and fixes a branch of its unstable manifold then it is clear that $h = id$ since this branch is dense. Since any h must permute fixed points of f_0 and the branches of their unstable manifolds we must have $h^k = id$ for some $k > 0$. Thus the kernel of α_u is contained in the torsion subgroup of $\phi(A)$ and in fact equal to it since the image of α_u has no torsion. If $\alpha_u(h) \neq 1$ then h is pseudo-Anosov with the same foliations as f_0 (perhaps with stable and unstable switched). To complete the proof we need only the fact that $\alpha_u(\phi(A))$ is cyclic. But this follows from the fact that the image of α_u must be discrete. If this were not the case there would be pseudo-Anosov homeomorphisms with expanding and contracting constants arbitrarily close to one. Such a homeomorphism which preserved branches of stable and unstable leaves at an interior fixed point would be C^0 close to the identity. This is not possible since no pseudo-Anosov homeomorphism can be homotopic to the identity. $\square$

We now apply these results to produce normal forms for an abelian subgroup $\mathcal{F}$ of $\mathrm{Diff}^1_+(S)$ assuming that $\mathrm{Fix}(\mathcal{F}) = \emptyset$. Our applications in this paper are in genus 0 and we use normal forms in proofs by contradiction, ultimately showing that $\mathrm{Fix}(\mathcal{F}) \neq \emptyset$. Thus the proposition below is less important for future applications than the preceding results on $\mathcal{W}(\mathcal{F}, B, C)$, which allow us to produce normal forms.

Proposition 2.11. *Suppose that S is a finitely punctured surface, that $f_1, \dots, f_n \in \mathrm{Diff}^1_+(S)$ generate an abelian subgroup $\mathcal{F}$ and that $\mathrm{Fix}(\mathcal{F}) = \emptyset$. Suppose further that:*

- *$K \subset \mathrm{Fix}(f_1, \dots, f_{n-1})$ is compact and $\mathcal{F}$-invariant.*
- *$L \subset \mathrm{Fix}(f_n)$ is compact and $\mathcal{F}$-invariant.*
- *If L is infinite then $\mathrm{Fix}(f_n)$ is compact and $L = \mathrm{Fix}(f_n)$.*
- *If $(K \cup L)$ is finite then $\chi(M) < 0$ where $M := S \setminus (K \cup L)$.*

Then there is a finite set R of disjoint simple closed curves (called reducing curves*) in M and for $1 \leq j \leq n$ there are homeomorphisms $\theta_j : S \to S$ isotopic to f_j rel $K \cup L$ such that:*

(1) *θ_j permutes disjoint open annulus neighborhoods in S of the elements of R for $1 \leq j \leq n$.*

Denote the union of the annular neighborhoods by $\mathcal{A}$, let $\{S_i\}$ be the components of $S \setminus \mathcal{A}$, let $X_i = (K \cup L) \cap S_i$ and let $M_i := S_i \setminus X_i$.

(2) *If X_i is infinite then either $X_i \subset L$ and $\theta_n|_{S_i}$ is the identity or $X_i \subset K$ and $\theta_j|_{S_i}$ is the identity for $1 \leq j \leq n - 1$.*

(3) *If X_i is finite then M_i has negative Euler characteristic and, for $1 \le j \le n$, $\theta_j^{r_{ij}}|_{M_i}$ is either periodic or pseudo-Anosov, where r_{ij} is the smallest positive integer such that $\theta_j^{r_{ij}}(M_i) = M_i$. Moreover, the $\theta_j^{r_{ij}}|_{M_i}$'s generate an abelian subgroup that is either finite or virtually cyclic.*

Proof. During this proof $1 \le j \le n-1$.

If L is finite, define $W_L = \emptyset$. Otherwise, let W_L be a maximal element of $\mathcal{W}(\langle f_n \rangle, L, K)$. By definition, there is a diffeomorphism $\theta_n : S \to S$ that is isotopic to f_n rel $K \cup L$ and that restricts to the identity on W_L. Since f_j preserves L and K and commutes with f_n it follows that $f_j(W_L)$ is a maximal element of $\mathcal{W}(\langle f_n \rangle, L, K)$. Corollary 2.8 therefore implies that f_j preserves W_L up to isotopy rel $K \cup L$. Choose diffeomorphisms $\theta_j : S \to S$ that are isotopic to f_j rel $K \cup L$ and that preserve W_L.

Denote the finitely punctured subsurface $S \setminus W_L$ by S' and the subset of L that is not contained in W_L by L'. If K is finite, define $W_K = \emptyset$. Otherwise, define W_K to be a maximal element of $\mathcal{W}(\langle \theta_{1|S'}, \dots, \theta_{n-1|S'} \rangle, K, L')$. After another isotopy rel $K \cup L$ we may assume that each θ_j restricts to the identity on W_K and that W_K is θ_n-invariant.

A partial set of reducing curves R' is defined from $\partial W_K \cup \partial W_L$ by removing any pair of curves that cobound an unpunctured annulus and replacing them with a core curve of that annulus. We may assume without loss that (1) and (2) are satisfied with respect to R'. Let S'' be the complement in S of $W_L \cup W_K$ and the annuli associated to R'. Then $L \cup K$ intersects S'' in a finite set and its complement in S'' is denoted N. To complete the proof we must show that R' can be extended by adding in simple closed curves in N to maintain (1) and to arrange (3). The existence of reducing curves for the induced action of $\mathcal{F}$ on N is well known; see for example Lemma 2.2 of [10]. These new curves divide N into irreducible subsurfaces and (3) is then a consequence of Lemma 2.10. $\square$

The set R of Proposition 2.11 is not uniquely determined. For example, if $\theta_j|_{M_i}$ is the identity for all $1 \le j \le n-1$ and if θ_n preserves an essential non-peripheral simple closed curve $\alpha \subset M_i$ then one can add γ to R. Lemma 2.12 below produces such curves α and is applied in the proof of Lemma 6.3.

Suppose that M is obtained from a closed surface by puncturing at a possibly infinite set. Choose a complete hyperbolic metric of finite volume for M. It is not strictly necessary to work with such a metric but it simplifies statements and proofs because the free conjugacy class of any essential non-peripheral closed curve is represented by a unique closed geodesic. In particular, a homeomorphism $f : M \to M$ induces a bijection $f_\#$ of the set of closed geodesics in M.

For any finite collection $\Sigma = \{\sigma_1, \dots, \sigma_l\}$ of closed geodesics in M define the *augmented regular neighborhood* $N(\Sigma)$ to be the union of a regular neighborhood of $\sigma_1 \cup \cdots \cup \sigma_l$ with all contractible components of its complement. The isotopy class of $N(\Sigma)$ is well defined.

The following lemma is similar to Lemma 2.2 of [12]. In this paper we apply it only with $l = 1$ and with M having genus zero.

Lemma 2.12. *If $f : M \to M$ is an orientation preserving homeomorphism that preserves the homotopy class of σ_i for $1 \le i \le l$ then f is isotopic to a homeomorphism $F : M \to M$ that preserves $N(\Sigma)$ and such that $F|_{N(\Sigma)}$ has finite order.*

Proof. There is no loss in assuming that $\sigma_1 \cup \dots \cup \sigma_l$, and hence $N(\Sigma)$, is connected. The lemma is clear if $N(\Sigma)$ is an annulus so we may assume that each element of Σ is non-peripheral in $N(\Sigma)$.

Define Σ^c to be the set of simple closed geodesics $\gamma \subset M$ that are disjoint from each σ_i. Then Σ^c is $f_\#$-invariant. We claim that, up to isotopy, $N(\Sigma)$ is the unique essential subsurface $N \subset M$ that has finite type, that is a closed subset of M and that satisfies

(∗) $\gamma \in \Sigma^c$ if and only if there is a representative of the isotopy class of N that is disjoint from γ.

To see that $N(\Sigma)$ satisfies (∗), suppose that $\gamma \in \Sigma^c$. Then γ is contained in a non-contractible component of the complement of $\sigma_1 \cup \dots \cup \sigma_l$. For an appropriate choice of regular neighborhood of $\sigma_1 \cup \dots \cup \sigma_l$, the augmented regular neighborhood is disjoint from γ. Conversely, if there is a representative of the isotopy class of N that is disjoint from γ then γ is disjoint from a representative of the isotopy class of each σ_i. Since γ and σ_i are geodesics, they must be disjoint or equal. Since $N(\Sigma)$ is not an annulus, γ must be disjoint from each σ_i.

To prove uniqueness up to isotopy, suppose that N_1 and N_2 satisfy (∗). After an isotopy, we may assume without loss that $\partial N_1 \cap N_2 = \emptyset$. Since $N_1 \cap N_2 \neq \emptyset$ (because N_2 must intersect any essential non-peripheral simple closed curve in N_1) it follows that $N_1 \subset N_2$. The symmetric argument shows that N_2 is isotopic into N_1 and so $N_1 = N_2$.

This completes the proof of the claim. Since $N(\Sigma)$ and $f(N(\Sigma))$ both satisfy (∗), f is isotopic to a homeomorphism F that preserves $N(\Sigma)$.

It remains to prove that the isotopy class of $F|_{N(\Sigma)}$ has finite order. Since σ_i is $f_\#$-invariant, θ can not be pseudo-Anosov and σ_i can not intersect any reducing curve in the Thurston canonical form for θ. It follows that there are no reducing curves and the Thurston classification theorem completes the proof. □

3 The invariant $w(f, g)$.

In this section we record some properties of a well known $\mathbb{Z}/2\mathbb{Z}$ invariant $w(f, g)$ associated to a pair of commuting elements of $f, g \in \text{Homeo}_+(S^2)$. Suppose $f, g \in \text{Homeo}_+(S^2)$ commute and f_t, g_t are isotopies from $f = f_1$ and $g = g_1$ to $id = f_0 = g_0$. We do not assume that f_t and g_t commute for $t \neq 0, 1$. The invariant $w(f, g) \in$

$\pi_1(\mathrm{Homeo}_+(S^2))$ is the element determined by the loop $\gamma(t) = [f_t, g_t] = f_t^{-1}g_t^{-1}f_tg_t$ in $\mathrm{Homeo}_+(S^2)$ with basepoint id. We recall that $\mathrm{Homeo}_+(S^2)$ is homotopy equivalent to $SO(3)$ [15] and hence that that $\pi_1(\mathrm{Homeo}_+(S^2)) \cong \mathbb{Z}/2\mathbb{Z}$.

Lemma 3.1. *The invariant $w(f,g) \in \pi_1(\mathrm{Homeo}_+(S^2))$ is also represented by the loops $\alpha(t) = [f, g_t]$ and $\beta(t) = [f_t, g]$ in $\mathrm{Homeo}_+(S^2)$ with basepoint id. As a consequence it is independent of the choice of isotopies f_t and g_t*

Proof. Consider the function $\phi(s,t) = [f_s, g_t]$ defined on the square $0 \le s \le 1,\ 0 \le t \le 1$ with values in $\mathrm{Homeo}_+(S^2)$. The loop $\gamma(t) = \phi(t,t)$ is homotopic to

$$\alpha_0(t) = \begin{cases} \phi(2t, 0) \text{ when } 0 \le t \le 1/2, \\ \phi(1, 2t-1) \text{ when } 1/2 \le t \le 1. \end{cases}$$

But $\alpha_0(t)$ is just a reparametrization of α which is constant on the interval $0 \le t \le 1/2$, so the loop $\gamma(t)$ is homotopic to $\alpha(t)$. The fact that $\gamma(t)$ is homotopic to $\beta(t)$ is proved similarly. ☐

Proposition 3.2. *Suppose f, g, and h are commuting elements of* $\mathrm{Homeo}_+(S^2)$*. Then $w(fg, h) = w(f,h) + w(g,h)$ and $w(f,g) = w(g,f)$.*

Proof. Consider the function $\phi(s,t) = [f_sg_t, h]$ defined on the square $0 \le s \le 1,\ 0 \le t \le 1$ with values in $\mathrm{Homeo}_+(S^2)$. The loop $\phi(t,t)$ represents $w(fg,h)$ by Lemma 3.1. This loop is homotopic to the loop

$$\alpha(t) = \begin{cases} \phi(2t, 0) \text{ when } 0 \le t \le 1/2, \\ \phi(1, 2t-1) \text{ when } 1/2 \le t \le 1. \end{cases}$$

But $\alpha(t)$ is the concatenation of the two loops

$$\phi(2t, 0) = g_0^{-1}f_{2t}^{-1}h^{-1}f_{2t}g_0h = f_{2t}^{-1}h^{-1}f_{2t}h = [f_{2t}, h],$$

where $0 \le t \le 1/2$, and

$$\phi(1, 2t-1) = g_{2t-1}^{-1}f_1^{-1}h^{-1}f_1g_{2t-1}h = g_{2t-1}^{-1}h^{-1}g_{2t-1}h = [g_{2t-1}, h],$$

where $1/2 \le t \le 1$. The first of these loops represents $w(f,h)$ and the second represents $w(g,h)$. Similarly one can show $w(h, fg) = w(h,f) + w(h,g)$.

To see the symmetry of $w(f,g)$ observe that for any h we trivially have $w(h,h) = 0$. Hence expanding $w(fg, fg)$ using the bilinearity just established we get

$$0 = w(fg, fg) = w(f,f) + w(f,g) + w(g,f) + w(g,g) = w(f,g) + w(g,f).$$

Since w takes values in $\mathbb{Z}/2\mathbb{Z}$ this implies $w(f,g) = w(g,f)$. ☐

The universal covering space of $\mathrm{Homeo}_+(S^2)$ which we denote $\widetilde{\mathrm{Homeo}}_+(S^2)$ is itself a group. Since $\pi_1(\mathrm{Homeo}_+(S^2)) \cong \mathbb{Z}/2\mathbb{Z}$ this is a two-fold covering.

Proposition 3.3. *If $f, g \in \mathrm{Homeo}_+(S^2)$ commute then $w(f,g) = 0$ if and only if there are lifts $\tilde{f}, \tilde{g} \in \widetilde{\mathrm{Homeo}}_+(S^2)$ of f and g respectively such that $\tilde{f}\tilde{g} = \tilde{g}\tilde{f}$.*

Proof. Let $\tilde{f}, \tilde{g} \in \widetilde{\mathrm{Homeo}}_+(S^2)$ be lifts of f and g respectively. By definition $\tilde{f}$ is a homotopy class of paths from id to f in $\mathrm{Homeo}_+(S^2)$ relative to the endpoints. We let f_t be a representative path for $\tilde{f}$ and g_t a representative path for $\tilde{g}$. Then $w(f,g)$ is the homotopy class of the loop $\alpha(t) = [f_t, g_t]$ in $\mathrm{Homeo}_+(S^2)$. This loop is null homotopic if and only if α lifts to a closed loop $\tilde{\alpha}(t)$ in $\widetilde{\mathrm{Homeo}}_+(S^2)$ with endpoints at $\widetilde{id}$, the identity element of the group $\widetilde{\mathrm{Homeo}}_+(S^2)$. But this occurs if and only if $[\tilde{f}, \tilde{g}] = \widetilde{id}$, i.e., if and only if $\tilde{f}\tilde{g} = \tilde{g}\tilde{f}$. □

We recall the following result from Lemma 1.2 of [10].

Lemma 3.4. *If $f \in \mathrm{Homeo}_+(S^2)$ is conjugate to a non-trivial rotation and $g \in \mathrm{Homeo}_+(S^2)$ commutes with f and permutes the elements of $\mathrm{Fix}(f)$ then $w(f,g) = 1$.*

Suppose that N is a genus zero surface and that $h : N \to N$ is a homeomorphism. Let $\bar{N}$ be the closed surface obtained from the interior of N by compactifying each of its ends with a point. Then $\bar{N}$ is naturally identified with S^2 and there is an induced homeomorphism $\bar{h} : S^2 \to S^2$. We allow the possibility that $N = S^2$ in which case $\bar{h} = h$.

The following result is essentially Lemma 3.1 of [10]. We include a proof for the reader's convenience.

Lemma 3.5. *Assume the notation of Proposition 2.11 with $S = \mathbb{R}^2$ or $S = S^2$. Denote $\langle \theta_1, \dots, \theta_n \rangle$ by Θ. Suppose that M_i has finite type and is Θ-invariant, that $f \in \mathcal{F}$ corresponds to $\psi \in \Theta$ and that $g \in \mathcal{F}$ corresponds to $\phi \in \Theta$. Then $w(\bar{f}, \bar{g}) = w(\overline{\psi|_{M_i}}, \overline{\phi|_{M_i}})$.*

Proof. Denote $\psi|_{M_i}$ by ψ_i and $\phi|_{M_i}$ by ϕ_i. Let X be a finite set of punctures in M, one in each component of the complement of M_i, and let $\bar{X}$ be the corresponding set in $\bar{S} = S^2$. Choose isotopies $\bar{f}_t$ and $\bar{g}_t$ rel $\bar{X}$, $t \in [1,2]$, such that:

(1) $\bar{f}_1 = \bar{f}$ and $\bar{f}_2 = \bar{\psi}$.

(2) $\bar{g}_1 = \bar{g}$ and $\bar{g}_2 = \bar{\phi}$.

Let M_i' be the complement in M_i of a regular neighborhood of ∂M_i. Choose isotopies $\bar{f}_t$ and $\bar{g}_t$ rel M_i', $t \in [2,3]$, such that:

(3) $\bar{f}_2 = \bar{\psi}$ and $\bar{f}_3 = \overline{\psi_i}$.

(4) $\bar{g}_2 = \bar{\phi}$ and $\bar{g}_3 = \overline{\phi_i}$.

Define $\bar{h}_t = [\bar{f}_t, \bar{g}_t]$ for $t \in [1,3]$. By the third item of Proposition 2.11, $\bar{h}_t$ is a loop in $\text{Homeo}_+(S^2)$ with basepoint id and it suffices to show that this loop represents the trivial element of $\pi_1(\text{Homeo}_+(S^2)$. Moreover $\bar{h}_2|_{M_i}$ is the identity. Choose an isotopy $\bar{q}_t$ rel $M_i' \cup X$, $t \in [0,1]$, such that:

(5) $\bar{q}_0 = \bar{h}_2$ and $\bar{q}_1 =$ identity.

Define loops in $\text{Homeo}_+(S^2)$ by

$$\bar{\alpha}(t) = \begin{cases} \bar{h}_t \text{ when } 1 \le t \le 2, \\ \bar{q}_{t-2} \text{ when } 2 \le t \le 3 \end{cases}$$

and

$$\bar{\beta}_t = \begin{cases} \bar{q}_{2-t} \text{ when } 1 \le t \le 2, \\ \bar{h}_t \text{ when } 2 \le t \le 3. \end{cases}$$

Then $\bar{h}_t$ and the concatenation of $\bar{\alpha}_t$ and $\bar{\beta}_t$ determine the same element of $\pi_1(\text{Homeo}_+(S^2))$. Since $\bar{\alpha}_t$ and $\bar{\beta}_t$ are both relative to at least three points, Lemma 1.1 of [10] implies that both $\bar{\alpha}_t$ and $\bar{\beta}_t$ determine the trivial element of $\pi_1(\text{Homeo}_+(S^2))$. $\square$

The following lemma will be used repeatedly in the proofs of Theorem 1.2 and Theorem 1.1.

Lemma 3.6. *Assume that M is a genus zero surface with finite type and that Θ is a finite or virtually cyclic abelian subgroup of* $\text{Homeo}_+(M)$*, each element of which either has finite order or is pseudo-Anosov. Then there is a subgroup Θ' of Θ with index at most two such that the sum of the number of Θ'-fixed punctures, the number of Θ'-fixed points in the interior of M and the number of Θ'-invariant components of ∂M is at least two. Moreover, if one of the following conditions is satisfied*

- *there is at least one Θ-fixed puncture or Θ-invariant component of ∂M*
- *$w(\phi, \psi) = 0$ for all $\phi, \psi \in \Theta$.*
- *Θ is cyclic*

then one may choose $\Theta' = \Theta$:

Proof. Let $\{\theta_j\}$ be a set of generators for Θ and let $\bar{\Theta}$ be the abelian subgroup of $\text{Homeo}_+(\bar{M} = S^2)$ determined by Θ. Suppose at first that some non-trivial $\bar{\theta} \in \bar{\Theta}$ has finite order and so is conjugate to a rotation (see for example Remark 2.4 of [10]). Since $\bar{\Theta}$ is abelian, it acts on the two point set $\text{Fix}(\bar{\theta})$. Let $\bar{\Theta}'$ be the subgroup of $\bar{\Theta}$ for which this action is trivial. If the first bulleted item is satisfied then $\bar{\Theta}' = \bar{\Theta}$ because a permutation of a two point set that fixes at least one point fixes both points. Similarly if the second bulleted item is satisfied then Lemma 3.4 implies that

$\bar{\Theta}' = \bar{\Theta}$. If the third bulleted item is satisfied then it is obvious that $\bar{\Theta}' = \bar{\Theta}$. The lemma follows in this case from the fact that a point in $\bar{M}$ corresponds to either a puncture in M, a boundary component of M of a point in the interior of M.

If every non-trivial element of $\bar{\Theta}$ has infinite order then $\bar{\Theta}$ is an infinite cyclic group (see, for example, Lemma 2.3 of [11]) generated by a pseudo-Anosov homeomorphism $\bar{\psi}$. The fixed points of $\bar{\psi}$ have index at most one so there are at least two of them and each is fixed by $\bar{\theta}$ for all θ. The proof now concludes as in the previous case. $\square$

Lemma 3.7. *Let α_t be a loop in $GL(2,\mathbb{R})$ with $\alpha_0 = \alpha_1 = I$. For $y \in \mathbb{R}^2$ let $ev_y : GL(2,\mathbb{R}) \to \mathbb{R}^2$ be the evaluation map defined by $ev_y(A) = Ay$. Suppose that there is $x \in \mathbb{R}^2 \setminus \{0\}$ such that the loop $ev_x(\alpha_t) = \alpha_t x$ is inessential in $\mathbb{R}^2 \setminus \{0\}$. Then α_t is inessential in $GL(2,\mathbb{R})$.*

Proof. There is a deformation retraction $\mathcal{R}$ of $GL(2,\mathbb{R})$ onto $SO(2)$, the group of rotations of the plane. This follows from the matrix polar decomposition according to which any real matrix A can be written uniquely as PO with P symmetric positive definite and O orthogonal (see [17] pp. 131-6). Since this decomposition is continuous and the symmetric positive definite matrices are convex this gives $\mathcal{R}$.

Let $\beta_t = \mathcal{R} \circ \alpha_t$ so β_t is loop in $SO(2) \subset GL(2,\mathbb{R})$ and the loop $ev_x(\beta_t) = \beta_t x$ is is inessential. The evaluation map $ev_x : SO(2) \to \mathbb{R}^2$ is a homeomorphism from $SO(2)$ onto the circle of radius $r = \|x\|$ in $\mathbb{R}^2$. It follows that β_t is an inessential loop in $SO(2)$ and hence that α_t is inessential in $GL(2,\mathbb{R})$. $\square$

Proposition 3.8. *Suppose f and g are commuting elements of $\mathrm{Diff}^1_+(\mathbb{R}^2)$ and $\bar{f}$ and $\bar{g}$ are their extensions to S^2, the one-point compactification of $\mathbb{R}^2$. Then $w(\bar{f}, \bar{g}) = 0$.*

Proof. Clearly it suffices to prove that the loop $h_t = [f_t, g_t]$ in $\mathrm{Diff}^1_+(\mathbb{R}^2)$ is contractible.

Choose balls $B_1 \subset B_2$ in $\mathbb{R}^2$ centered at the origin and a function $H : [0,1]^2 \to \mathrm{Diff}^1_+(\mathbb{R}^2)$ such that

1. $h_t(0) \in B_1$ for all $0 \le t \le 1$.
2. $H(s,t)(x) = x - sh_t(0)$ for all $x \in B_1$ and all $0 \le s, t \le 1$, and
3. $H(s,t)(x) = x$ for all x outside B_2 and all $0 \le s, t \le 1$.

Then if we define $\theta_t = H(1,t) \circ h_t$, the loop θ_t in $\mathrm{Diff}^1_+(\mathbb{R}^2)$ is homotopic to h_t (the homotopy being given by $H(s,t) \circ h_t,\ 0 \le s \le 1$. Moreover, $\theta_t(0) = 0$ for all $t \in [0,1]$ and $\theta_t(x) = h_t(x)$ for x outside B_2. To prove our result it suffices to show θ_t is an inessential loop in $\mathrm{Diff}^1_+(\mathbb{R}^2)$.

To do this we will apply a result of D. Calegari (see Section 4.1 and Theorem D from Section 4.8 of [3]). This result asserts that if G denotes the group generated by f and G_∞ is the group of germs at ∞ of G then there is a split exact sequence

$$0 \to \mathbb{Z} \to \hat{G}_\infty \to G_\infty \to 0$$

where $\hat{G}_\infty$ is the group of all lifts of f and g to the universal cover $\tilde{A}$ of the germ A of a neighborhood of ∞. The fact that this sequence splits implies there are lifts $\tilde{f}, \tilde{g} \in \tilde{G}_\infty$ of the germs of f and g respectively such that $[\tilde{f}, \tilde{g}] = id$.

This implies that if N is a sufficiently small punctured disk neighborhood of ∞ and $x_0 \in N$ then $h_t(x_0)$ is an inessential loop in N. Moreover if N is sufficiently small then $\theta_t(x_0)$ is an inessential loop in N since it agrees with $h_t(x_0)$.

We now perform a further homotopy from the loop θ_t in $\mathrm{Diff}^1_+(\mathbb{R}^2)$ to the loop $\phi_t = \Phi(0,t)$ where $\Phi(s,t) \in \mathrm{Diff}^1_+(\mathbb{R}^2)$ is defined by

$$\Phi(s,t)(x) = \begin{cases} \frac{\theta_t(sx)}{s} \text{ when } 0 < s \leq 1, \text{ and} \\ D\theta_t(0)(x) \text{ when } s = 0, \end{cases}$$

It now suffices to show the loop ϕ_t is inessential in $\mathrm{Diff}^1_+(\mathbb{R}^2)$. We observe that the loop $\phi_t(x_0)$ is homotopic to $\theta_t(x_0)$ in $\mathbb{R}^2 \setminus \{0\}$ and therefore inessential. We also note that $\phi_t = D\theta_t(0) \in GL(2,\mathbb{R})$. Hence Lemma3.7 implies the loop ϕ_t is inessential in $\mathrm{Diff}^1_+(\mathbb{R}^2)$. ∎

As an immediate corollary we have the following.

Corollary 3.9. *Suppose f and g are commuting elements of $\mathrm{Diff}^1_+(S^2)$ which have a common fixed point x_0. Then $w(f,g) = 0$.*

4 Nielsen Classes

We recall some definitions from Nielsen fixed point theory that are needed in the proof of Proposition 4.2. Further details can be found for example in Brown [2].

Let $h : N \to N$ be a homeomorphism of a connected surface N. In the paragraphs preceding Lemma 2.6 we recalled the definition of Nielsen equivalence and the relationship between Nielsen classes and and lifts of h to the universal cover of N.

We say that a Nielsen class μ for $h : N \to N$ is *compactly covered* if $\mathrm{Fix}(\tilde{h})$ is compact for some, and hence every, lift $\tilde{h}$ for μ. In this case, $\mathrm{Fix}(\tilde{h})$ is contained in the interior of a compact disk $D \subset \tilde{N}$ and the *Nielsen index* $I(h,\mu)$ for μ is the usual index obtained as the winding number of the vector field $x - \tilde{h}(x)$ around ∂D (see, e.g. Brown [2]).

If Q is an isolated puncture in N then we say that a Nielsen class μ in $\mathrm{Fix}(h)$ *peripherally contains* Q if for some, and hence every, x in μ there is a properly immersed ray $\alpha \subset N$ based at x that converges to Q such that $h(\alpha)$ is properly homotopic to α relative to x.

Given an isotopy h_t from $h_0 = h$ to some h_1, let $\tilde{h}$ be a lift for μ and let $\tilde{h}_t$ be the lift of h_t that begins with $\tilde{h}_0 = \tilde{h}$. The Nielsen class μ_t for h_t determined by $\tilde{h}_t$ is independent of the choice of $\tilde{h}$. We say that *μ_t is the Nielsen class determined by μ and h_t.*

Lemma 4.1. *Suppose that μ is a non-trivial, compactly covered Nielsen class for $h : N \to N$ and that each fixed end of N is isolated and not peripherally contained in μ. If h_t is an isotopy with $h_0 = h$ and μ_t is determined by h_t and $\mu = \mu_0$, then each μ_t is compactly covered, does not peripherally contain any fixed end and $I(h_t, \mu_t)$ is independent of t.*

Proof. Choose $\tilde{h}_t$ as in the definition of μ_t. To prove that μ_t is compactly covered and that $I(h_t, \mu_t)$ is independent of t, it suffices to show that $\tilde{X} := \cup_t \operatorname{Fix}(\tilde{h}_t)$ is compact in $\tilde{N}$. One can then compute $I(h_t, \mu_t)$ using a disk $D \subset \tilde{N}$ whose interior contains $\operatorname{Fix}(\tilde{h}_t)$ for all t and it is then a standard fact (see Dold [4]) that $I(h_t, \mu_t)$ is independent of t.

Let $Q_1, \dots, Q_k$ be the isolated fixed ends of N. Choose neighborhoods $V_i \subset U_i \subset N$ of Q_i such that $h_t(V_i) \subset U_i$ for all t and such that the one point compactification of U_i obtained by filling in the puncture Q_i is a disk. Then Q_i is peripherally contained in any Nielsen class for h_t that contains an element of $\operatorname{Fix}(h_t) \cap V_i$. Choose a neighborhood W of the remaining ends of N so that $\operatorname{Fix}(h_t) \cap W = \emptyset$ for all t.

Define $H : N \times [0,1] \to N \times [0,1]$ by $H(x,t) = (h_t(x), t)$ and $\tilde{H} : \tilde{N} \times [0,1] \to \tilde{N} \times [0,1]$ by $\tilde{H}(x,t) = (\tilde{h}_t(x), t)$. Let $Y \subset \operatorname{Fix}(H)$ be the projected image of $\tilde{Y} := \operatorname{Fix}(\tilde{H})$. Any two points $(x_s, s), (x_t, t) \in Y$ are connected by an arc $\gamma \subset N \times [0,1]$ such that $H(\gamma)$ is homotopic to γ rel endpoints. Since $\mu_0 = \mu$ is non-trivial and compactly covered, $\tilde{h}_0 = \tilde{h}$ does not commute with any covering translations. It follows that $\tilde{H}$ does not commute with any covering translations and hence that each element of Y has a unique lift to an element of $\tilde{Y}$. We show below that Y is compact. Since points in Y lift uniquely there is a continuous lifting map from Y to $\tilde{Y}$. It follows that $\tilde{Y}$, and hence $\tilde{X}$, is compact.

Denote the set of elements of the equivalence class μ_t by X_t. Then $X := \cup_t X_t \subset N$ is the projected image of $\tilde{X} \subset \tilde{N}$ and of $Y \subset N \times [0,1]$. Obviously $X \cap W = \emptyset$. To complete the proof that Y is compact, it remains to check that $X_t \cap V_i = \emptyset$ for all i and t. We will prove the stronger statement that μ_t does not peripherally contain any puncture, which completes the proof of the lemma. Suppose to the contrary that there is a ray $\alpha_t \subset N$ connecting x_t to Q_i such that $h_t(\alpha_t)$ is properly homotopic to α_t rel x_t. For any $x_0 \in X_0$, there is an arc $\gamma \subset N \times [0,1]$ connecting $(x_0, 0)$ to (x_t, t) such that $H(\gamma)$ is homotopic to γ rel endpoints. Let $\alpha_0 \subset N$ be the concatenation of the projection γ_0 of γ and α_t. Then $h_0(\alpha_0)$ is properly homotopic to α_0 rel x_0. This implies that μ_0 peripherally contains Q_i which is a contradiction. □

Suppose that $\theta : S \to S$ is an orientation preserving homeomorphism of a finitely punctured surface, that C is a compact set in the complement of $\operatorname{Fix}(\theta)$ and that $x \in \operatorname{Fix}(\theta)$. Let N be the component of $S \backslash C$ that contains x and let $h = \theta|_N : N \to N$. Define *the Nielsen class for θ relative to C determined by x* to be the Nielsen class for h determined by x. The remaining 'relative' definitions are made similarly, using h in place of θ.

The following proposition is the main result of this section.

Proposition 4.2. *Suppose that $\theta : S \to S$ is an orientation preserving homeomorphism of a finitely punctured surface, that C is a compact invariant set in the complement of* $\mathrm{Fix}(\theta)$*, that M_i is an essential subsurface of $M := S \setminus C$ with finite negative Euler characteristic and that $\theta|_{M_i}$ is either pseudo-Anosov relative to its puncture set or has finite order greater than one. Suppose further that $z \in \mathrm{Fix}(\theta)$ is contained in the interior of M_i and that $f_t : S \to S$ is an isotopy rel C from $f_0 = \theta$ to some $f = f_1$. Let μ be the Nielsen class for θ relative to C determined by z and let ν be the Nielsen class for f relative to C determined by μ and f_t. Then ν is non-trivial, compactly covered and does not peripherally contain any punctures.*

Proof. There is no loss in assuming that M is connected. Choose lifts $\tilde{z}$ and $\tilde{\theta}$ such that $\tilde{z} \in \mathrm{Fix}(\tilde{\theta})$. Since M_i is essential, the component of the pre-image of M_i in $\tilde{M}$ that contains $\tilde{z}$ is a copy of the universal cover $\tilde{M}_i$ of M_i. The following properties of $\theta|_{M_i}$ and $\tilde{\theta}|_{\tilde{M}_i}$ are well known.

- $\mathrm{Fix}(\tilde{\theta}|_{\tilde{M}_i}) = \{\tilde{z}\}$ and the fixed point index of z with respect to θ is non-zero.
- No component of $\partial\tilde{M}_i$, and hence no component of $\tilde{N} \setminus \tilde{M}_i$, is $\tilde{\theta}$-invariant.
- There does not exist a ray α in M_i connecting z to a puncture of M_i such that $\theta(\alpha)$ is homotopic to α rel endpoints.

The first two items imply that $\mathrm{Fix}(\tilde{\theta}) = \{\tilde{z}\}$ and hence that μ is compactly covered. Suppose that there is a ray α in M connecting z to a puncture of M such that $\theta(\alpha)$ is homotopic to α rel endpoints. Decompose α into an alternating concatenation of maximal subpaths σ_j in M_i and maximal subpaths τ_j in the complement of M_i. With the exception of σ_1 and the final τ_j if it is a ray, all of these subpaths have both endpoints in ∂M_i and we may assume that they cannot be homotoped rel endpoints into ∂M_i. In this case, the component of $\partial\tilde{M}_i$ that contains the terminal endpoint of the lift $\tilde{\sigma}_1$ that begins at $\tilde{z}$ is preserved by θ which contradicts the second item above. We conclude that μ does not peripherally contain any punctures. The proposition now follows from Lemma 4.1 and the first item above. $\square$

5 A Fixed Point Lemma

The Brouwer translation theorem implies that if $f \in \mathrm{Homeo}_+(\mathbb{R}^2)$ has a compact invariant set C then f has a fixed point. We will enhance this by adding some more information about the location of the fixed point relative to C.

Definition 5.1. *Suppose that $f \in \mathrm{Homeo}_+(\mathbb{R}^2)$ and that $C \subset \mathbb{R}^2$ is a compact invariant set with $C \cap \mathrm{Fix}(f) = \emptyset$. Define $P(C, f)$ to be the union of Nielsen classes of* $\mathrm{Fix}(f)$ *rel C which do not peripherally contain ∞.*

We consider first the special case that C is a periodic orbit. The following result is due to Gambaudo [9]. We provide a proof as a convenience to the reader and because it is quite short given our earlier lemmas.

Lemma 5.2 (Gambaudo). *If $f \in \mathrm{Homeo}_+(\mathbb{R}^2)$ has a periodic point p of period greater than one, then $P(\mathrm{Orb}(p), f) \neq \emptyset$.*

Proof. Let $N = \mathbb{R}^2 \setminus \mathrm{Orb}(p)$. By the Thurston classification theorem, there is a homeomorphism $\theta : N \to N$ that is isotopic to f and an essential θ-invariant subsurface N_1 of negative Euler characteristic that contains ∞ as a puncture such that $\theta|_{N_1}$ is either periodic or pseudo-Anosov. No puncture of N other than ∞ is fixed. If ∂N_1 is non-empty then it contains at least two components, each of which bounds a disk in the complement of N_1 that contain punctures corresponding to $\mathrm{Orb}(p)$. Since the action of θ on $\mathrm{Orb}(p)$ is minimal, these disks are not θ-invariant and neither are the components of ∂N_1. The lemma now follows from Lemma 3.6 and Lemma 4.2. $\square$

We can now state and prove our general result on $P(C, f)$.

Proposition 5.3. *Suppose that $f \in \mathrm{Homeo}_+(\mathbb{R}^2)$ and that $C \subset \mathbb{R}^2$ is a non-empty compact f-invariant set with $C \cap \mathrm{Fix}(f) = \emptyset$. Then $P(C, f)$ is non-empty and compact.*

Proof. The proof is by contradiction. Suppose that that each Nielsen class in$\mathrm{Fix}(f)$ peripherally contains ∞. Choose open neighborhoods $V_\infty \subset U_\infty$ of ∞ that are disjoint from C and that satisfy $f(V_\infty) \subset U_\infty$. Similarly, for each $x \in \mathrm{Fix}(f)$ choose open disk neighborhoods $V_x \subset U_x$ whose closures are disjoint from C with the property that $f(V_x) \subset U_x$. Let $V_\infty \cup \{V_{x_i}\}$ be a finite subcover of the cover $V_\infty \cup \{V_x\}$ of $\mathrm{Fix}(f)$. For each x_i there is a path from ∞ to x_i which is homotopic rel endpoints to its f image and the support of the homotopy is disjoint from C. For each $x \in V_{x_i}$ we can form a path from ∞ to x by concatenating the path from ∞ to x_i with a path in V_{x_i} from x_i to x. From this it is clear that for some $\epsilon > 0$ and for any $x \in \mathrm{Fix}(f)$ there is a path α from ∞ to x and a homotopy rel endpoints from $\bar{f}(\alpha)$ to α whose support is disjoint from an ϵ neighborhood of C.

Now let $z \in C$ be a recurrent point. By a small C^0 perturbation of f supported in the ϵ neighborhood of C we can make z be a periodic point. Moreover, we may do this with a perturbation supported on an open set W disjoint from its f image. Hence the perturbed map has precisely the same fixed point set as f and every fixed point still belongs to a Nielsen class that peripherally contains ∞ since f was not modified outside an ϵ neighborhood of C. This contradicts Lemma 5.2.

This proves $P(C, f)$ is not empty. It is immediate that $P(C, f)$ is compact since any point of $\mathrm{Fix}(f) \cap V_\infty$ determines a Nielsen class that peripherally contains ∞. $\square$

Remark 5.4. It is immediate that if $g \in \mathrm{Homeo}_+(\mathbb{R}^2)$ commutes with f and C is g-invariant then $P(C, f)$ is also g-invariant.

6 Proof of Theorem 1.2

We will prove the following strengthened version of Theorem 1.2. The extra condition (2) plays an essential role in our inductive proof.

Theorem 6.1. *Suppose that $\mathcal{F}$ is a finitely generated abelian subgroup of* $\mathrm{Diff}^1_+(\mathbb{R}^2)$ *and that there is a compact $\mathcal{F}$-invariant set $C \subset \mathbb{R}^2$. Then the following hold:*

1. $\mathrm{Fix}(\mathcal{F})$ *is non-empty.*
2. *If $\mathcal{F} = \langle f_1, \dots, f_n \rangle$ and $C \subset \mathrm{Fix}(f_1, \dots, f_{n-1})$ then*

$$(C \cup P(C, f_n)) \cap \mathrm{Fix}(\mathcal{F}) \neq \emptyset.$$

Proof. The proof is by induction on the rank n of $\mathcal{F}$. Our strategy is to use the following intermediate inductive statement.

A_n : (1) holds for $\mathcal{F} = \langle f_1, \dots, f_n \rangle$ under the additional hypothesis that $\mathrm{Fix}(f_n)$ is compact.

We also make use of the following notation.

B_n : Theorem 6.1 holds for $\mathcal{F} = \langle f_1, \dots, f_n \rangle$.

Obviously, the validity of B_n for all $n > 0$ is the result we wish to prove. A_1 follows from the Brouwer translation theorem and B_1 is a consequence of Proposition 5.3 (recall that $\mathrm{Fix}(\emptyset) = \mathbb{R}^2$ so in B_1 the condition that $C \subset \mathrm{Fix}(f_1, \dots f_{n-1})$ is trivially true). Hence by induction the theorem follows from Lemmas 6.2 and 6.3 below.

Lemma 6.2. *$B_{n-1} \Rightarrow A_n$ for all $n \geq 2$.*

Proof. We assume B_{n-1}, the hypothesis of A_n and that $\mathrm{Fix}(\mathcal{F}) = \emptyset$ and show this leads to a contradiction. Denote $\langle f_1, \dots, f_{n-1} \rangle$ by $\mathcal{F}_{n-1}$.

The set $L := \mathrm{Fix}(f_n)$ is non-empty by the Brouwer translation theorem and is compact by the hypothesis of A_n. Define $K = P(L, f_{n-1}) \cap \mathrm{Fix}(\mathcal{F}_{n-1})$. Since K is a subset of $P(L, f_{n-1})$ it is compact. We claim that it is also non-empty. To see this, observe that $J = \mathrm{Fix}(f_1, \dots, f_{n-2}, f_n) \neq \emptyset$ by B_{n-1} and is compact since it is a subset of L. B_{n-1} applied to $\mathcal{F}_{n-1}$ implies

$$(J \cup P(J, f_{n-1})) \cap \mathrm{Fix}(\mathcal{F}_{n-1}) \neq \emptyset.$$

If $J \cap \mathrm{Fix}(\mathcal{F}_{n-1}) \neq \emptyset$ then $\mathrm{Fix}(\mathcal{F}) \neq \emptyset$, a contradiction. It follows, since $J \subset L$, that

$$K = P(L, f_{n-1}) \cap \mathrm{Fix}(\mathcal{F}_{n-1}) \supset P(J, f_{n-1}) \cap \mathrm{Fix}(\mathcal{F}_{n-1}) \neq \emptyset$$

as desired.

Apply Proposition 2.11 to $\mathcal{F}$, K and L and assume the notation of that proposition. Denote $\langle\theta_1,\ldots,\theta_n\rangle$ by Θ and $\langle\theta_1,\ldots,\theta_{n-1}\rangle$ by Θ_{n-1}.

We claim that the subsurface M_1 containing ∞ as a puncture has at least one Θ-invariant boundary component Γ_1. If either $\theta_n|_{M_1}$ is the identity or $\Theta_{n-1}|_{M_1}$ is the identity then this follows from the fact that K and L are disjoint and non-empty. Otherwise, M_1 is finitely punctured and the claim follows from Lemma 3.6, the assumption that $\mathrm{Fix}(\mathcal{F}) = \emptyset$ and the fact that $\mathrm{Fix}(\theta_n)\cap\mathrm{Int}(M_1) = \mathrm{Fix}(f_n)\cap\mathrm{Int}(M_1) = \emptyset$.

Γ_1 is a boundary component of some annulus in $\mathbb{A}$. The other boundary component of this annulus is a boundary component Γ_2 of some subsurface, say S_2, that is Θ-invariant. If ∂M_2 has a second Θ-invariant component Γ_3 then one can repeat this operation to find a third Θ-invariant subsurface and so on. This process must eventually end so there exists a Θ-invariant subsurface M_i with exactly one Θ-invariant boundary component Γ; moreover, the disk D bounded by Γ and containing M_i does not contain ∞.

If both $K\cap D\neq\emptyset$ and $L\cap D\neq\emptyset$ then the argument applied above to M_1 would apply to M_i to produce a Θ-invariant element of R that we have assumed does not exist. We may therefore assume that $K\cap D=\emptyset$ or $L\cap D=\emptyset$. If $L\cap D=\emptyset$ then $K\cap D\neq\emptyset$ and every Nielsen class in L rel $K\cap D$ peripherally contains ∞. This contradicts Proposition 5.3.

It remains to consider the case that $K\cap D=\emptyset$ and $L\cap D\neq\emptyset$. We show below that $C' = L\cap D\cap\mathrm{Fix}(f_1,\ldots,f_{n-2},f_n)$ is non-empty. Assuming this for the moment, we argue to a contradiction thereby finishing the proof of the lemma. Apply B_{n-1} to $\mathcal{F}_{n-1}$ and C' to conclude that $(C'\cup P(C',f_{n-1}))\cap\mathrm{Fix}(\mathcal{F}_{n-1})\neq\emptyset$ and hence, since $\mathrm{Fix}(\mathcal{F})=\emptyset$, that $P(C',f_{n-1})\cap\mathrm{Fix}(\mathcal{F}_{n-1})\neq\emptyset$. This contradicts the fact that $P(C',f_{n-1})\subset D\cap P(L,f_{n-1})$ and the assumption that $K\cap D=\emptyset$.

We now show that $C'\neq\emptyset$ as promised. Let M' be the surface obtained from $M=\mathbb{R}^2\setminus(K\cup L)$ by filling in the punctures corresponding to the compact set $L_D := L\cap D$. Thus $D\subset M'$ and L_D is $\mathcal{F}$-invariant. Let $\tilde{M}'$ be the universal covering space of M', let $\tilde{D}\subset\tilde{M}'$ be a lift of D and let $\tilde{L}_D$ be the lift of L_D that is contained in $\tilde{D}$. Note that $\tilde{L}_D$ is homeomorphic to L_D and is therefore compact. For each $f\in\mathcal{F}$ there is a unique lift $\tilde{f}:\tilde{M}'\to\tilde{M}'$ that setwise fixes $\tilde{L}_D$. The uniqueness of $\tilde{f}$ implies that $\langle\tilde{f}_1,\ldots,\tilde{f}_{n-2},\tilde{f}_n\rangle$ is abelian. Applying A_{n-1} (which follows from B_{n-1}) to this group and $\tilde{L}_D$, we conclude that $\mathrm{Fix}(\tilde{f}_1,\ldots,\tilde{f}_{n-2},\tilde{f}_n)\neq\emptyset$. This set evidently projects into $\mathrm{Fix}(f_1,\ldots,f_{n-2},f_n)$ and it projects into $\mathrm{Fix}(f_n|_{M'})\subset L\cap D$. It therefore projects into C'. $\square$

Lemma 6.3. *A_n together with B_j for $j\leq n-1$ implies B_n for all $n\geq 2$.*

Proof. Let us first reduce to the case that $C\subset\mathrm{Fix}(\mathcal{F}_{n-1})$ by producing a non-trivial compact f_n-invariant subset C_n of $\mathrm{Fix}(\mathcal{F}_{n-1})$. If C is not a subset of $\mathrm{Fix}(\mathcal{F}_{n-1})$ then replace it with C_n.

We prove by induction on $2 \leq k \leq n$ that there is a non-trivial compact $\mathcal{F}_k$-invariant subset C_k of $\mathrm{Fix}(f_1, \ldots, f_{k-1})$. For $k = 2$, this follows from Proposition 5.3 applied to C and Remark 5.4. Assume now that C_{k-1} exists. If $C_{k-1} \cap \mathrm{Fix}(f_k) \neq \emptyset$ define $C_k = C_{k-1} \cap \mathrm{Fix}(f_k)$; otherwise define $C_k = P(C_{k-1}, f_{k-1}) \cap \mathrm{Fix}(f_1, \ldots, f_{k-1})$. Then C_k is non-empty by B_{k-1}, is compact because $P(C_{k-1}, f_{k-1}))$ is compact and is $\mathcal{F}_k$-invariant by Remark 5.4. This completes the induction step and the proof that C_n exists. Hence we may now assume that $C \subset \mathrm{Fix}(\mathcal{F}_{n-1})$.

We will establish B_n if we show $(C \cup P(C, f_n)) \cap \mathrm{Fix}(\mathcal{F}_n) \neq \emptyset$. If $C \cap \mathrm{Fix}(f_n) \neq \emptyset$ we are done so we may assume C is disjoint from $\mathrm{Fix}(f_n)$.

Let K be a subset of C which is f_n minimal. Assume the notation of Proposition 2.11 with $L = \emptyset$ and $M = \mathbb{R}^2 \setminus K$. Denote $\langle \theta_1, \ldots, \theta_n \rangle$ by Θ and let M_1 denote the subsurface which contains ∞ as a puncture. There are two possibilities.

Case 1. *M_1 has finite type:*
If M_1 has punctures corresponding to elements of K then K is finite, $M = M_1$, and the set of reducing curves $R = \emptyset$. Otherwise, ∞ is the only puncture in M_1 and the components of ∂M_1, of which there are at least two, are transitively permuted by θ_n. In either case, θ_n is not the identity and Lemma 3.6 implies that there is a Θ-fixed point x in the interior of M_1. Let $\tilde{M}$ be the universal cover of M, let $\tilde{x} \in \tilde{M}$ be a lift of x, let $\tilde{M}_1$ be the component of the full pre-image of M_1 that contains $\tilde{x}$ and let $\tilde{f}_j$ be the lift of f_j that is equivariantly isotopic to the lift $\tilde{\theta}_j$ of θ_j that fixes $\tilde{x}$. Since $\mathcal{F} = \langle f_j \rangle$ and $\langle \theta_j|_{M_1} \rangle$ are abelian, so is $\tilde{\mathcal{F}} = \langle \tilde{f}_j \rangle$.

The lifts $\tilde{\theta}_j$ of θ_j that fix $\tilde{x}$ generate an abelian subgroup $\tilde{\Theta}$ of $\mathrm{Diff}^1_+(\tilde{M})$. Each f_j has a unique lift $\tilde{f}_j$ that is equivariantly isotopic to $\tilde{\theta}_j$ and $\tilde{\mathcal{F}} := \langle \tilde{f}_1, \ldots, \tilde{f}_n \rangle$ is an abelian subgroup of $\mathrm{Diff}^1_+(\tilde{M})$.

By Proposition 2.11-(3), $\theta_n|_{M_1}$ is either finite order or pseudo-Anosov. Let μ be the f_n-Nielsen class relative to K determined by $\tilde{f}_n$. Proposition 4.2 implies that μ is non-empty, compact and does not peripherally contain ∞. In particular, all elements of μ are contained in $P(K, f_n) \subset P(C, f_n)$. Since $\tilde{\mathcal{F}}_n$ is abelian, $\mathrm{Fix}(\tilde{f}_n)$ is $\tilde{\mathcal{F}}$-invariant. A_n therefore implies that some element of μ is contained in $\mathrm{Fix}(\tilde{\mathcal{F}})$. This verifies B_n in case 1.

Case 2. *M_1 is infinitely punctured:*
n this case, every element of Θ_{n-1} acts as the identity on M_1. Since K is θ_n-minimal, $M = M_1$ and $R = \emptyset$. If there is an essential finite type subsurface $N \subset M_1$ that is preserved up to isotopy by θ_n, then, after modifying θ_n by an isotopy, we can redefine R to be ∂N. This redefines M_1, and as just observed, the new M_1 must be finitely punctured. Case 1 completes the proof in this case so we may assume that no such N exists.

Let $\tilde{M}$ be the universal cover of $M = M_1$, let $\tilde{x} \in \tilde{M}$ be a lift of $x \in P(K, f_n)$ and let $\tilde{f}_n : \tilde{M} \to \tilde{M}$ be the lift of f_n which fixes $\tilde{x}$. Lemma 2.12 implies that f_n does not fix the homotopy class of any essential non-peripheral closed curve. It follows that $\tilde{f}_n$ does not commute with any covering translations. Thus $\mathrm{Fix}(\tilde{f}_n)$ is compact.

Since Θ_{n-1} is the trivial group, each $f \in \mathcal{F}_{n-1}$ is isotopic to the identity rel K.

There is a unique lift $\tilde{f} : \tilde{M} \to \tilde{M}$ that is equivariantly isotopic to the identity. The uniqueness of $\tilde{f}$ implies that $\tilde{F}_n = \langle \tilde{f}_1, \ldots, \tilde{f}_n \rangle$ is abelian. A_n therefore implies that there is a point $\tilde{z}$ in $\mathrm{Fix}(\tilde{\mathcal{F}})$. The projection of this point into M is a point $z \in \mathrm{Fix}(\mathcal{F})$ in the same Nielsen class relative to f_n as x. Thus $z \in P(K, f_n) \subset P(C, f_n)$. This verifies B_n in case 2. $\square$

7 Proof of Theorem 1.1

We begin with a pair of applications of Theorem 1.2, the first of which requires some technical notation.

Suppose that S is a finitely punctured surface, that $\mathcal{F}$ is a finitely generated abelian subgroup of $\mathrm{Homeo}_+(S)$ and that $C \subset S$ is compact and $\mathcal{F}$-invariant. Suppose further that M_1 is an essential subsurface of $M := S \setminus C$ that has finite negative Euler characteristic and that is $\mathcal{F}$-invariant up to isotopy. There is a homomorphism Φ from $\mathcal{F}$ to the mapping class group of M_1 defined by $f \mapsto [\theta|_{M_1}]$ where $\theta \in \mathrm{Homeo}_+(S)$ is isotopic to f and preserves M_1 and where $[h]$ is the isotopy class of h. This is well defined because homeomorphisms of M_1 that are isotopic in M are isotopic in M_1. If each element of the abelian subgroup $\Theta_1^* := \Phi(\mathcal{F})$ is either of pseudo-Anosov type or has finite order then we say that *$\mathcal{F}$ is M_1-irreducible.* For example, if some element of Θ_1^* has pseudo-Anosov type then Lemma 2.10 implies that $\mathcal{F}$ is M_1-irreducible.

If $\mathcal{F}$ is M_1-irreducible then Lemma 2.10 implies that Θ_1^* is either finite or virtually cyclic and lifts to a subgroup $\Theta_1 \subset \mathrm{Homeo}_+(M_1)$. By construction, each $f \in \mathcal{F}$ is isotopic to some $\theta \in \mathrm{Homeo}_+(S)$ where M_1 is θ-invariant and $\theta|_{M_1} \in \Theta_1$.

Proposition 7.1. *Assume notation as above and further that $\mathcal{F}$ is M_1-irreducible, that Θ_1^* is non-trivial and that $\Theta_1 \subset \mathrm{Homeo}_+(M_1)$ is a lift of Θ_1^*. For each $z \in \mathrm{Fix}(\Theta_1)$ that is contained in the interior of M_1, there exists $x \in \mathrm{Fix}(\mathcal{F}) \cap M$ with the following properties: if $\theta_1^* := \Phi(f)$ is non-trivial then the f-Nielsen class rel C that contains x is compactly covered, does not peripherally contain any punctures and corresponds, via the isotopy between f and θ, to the θ-Nielsen class rel C that contains z, where $\theta \in \mathrm{Homeo}_+(S)$ is isotopic to f and $\theta|_{M_1} \in \Theta_1$.*

Proof. We may assume without loss that M is connected. Let $\tilde{M}_1$ be the universal cover of M_1 and let $\tilde{z} \in \tilde{M}_1$ be a lift of z. Each element of Θ_1 has a unique lift to $\tilde{M}_1$ that fixes $\tilde{z}$. This lifts the group Θ_1 to a group $\tilde{\Theta}_1 \subset \mathrm{Homeo}_+(\tilde{M}_1)$.

Lift $\mathcal{F}$ to $\tilde{\mathcal{F}} \subset \mathrm{Homeo}_+(\tilde{M})$ as follows. Given $f \in \mathcal{F}$, let $\tilde{f}_1 : \tilde{M}_1 \to \tilde{M}$ be the unique lift of $f|_{M_1}$ that is equivariantly isotopic to an element of $\tilde{\Theta}_1$ and let $\tilde{f} : \tilde{M} \to \tilde{M}$ be the unique extension of $\tilde{f}_1$ to a lift of f. The uniqueness properties imply that $f \mapsto \tilde{f}$ defines a lift of the group $\mathcal{F}$.

Suppose now that $\theta_1^* := \Phi(f)$ is non-trivial and that ν is the θ-Nielsen class rel C that contains z. Proposition 4.2 implies that the Nielsen class of f rel C that corresponds to ν is non-trivial and compactly covered and hence that $\mathrm{Fix}(\tilde{f})$ is non-trivial, compact and $\tilde{\mathcal{F}}$-invariant. Theorem 1.2 implies that $\mathrm{Fix}(\tilde{\mathcal{F}}) \neq \emptyset$. Choose $\tilde{x} \in \mathrm{Fix}(\tilde{\mathcal{F}})$ and let $x \in \mathrm{Fix}(\mathcal{F})$ be its image in M. □

Lemma 7.2. *Suppose that $\mathcal{F}$ is a finitely generated abelian subgroup of* $\mathrm{Homeo}_+(S^2)$*, that $X \subset S^2$ is a compact $\mathcal{F}$-invariant set and that $\Gamma \subset M := S^2 \setminus X$ is an essential simple closed curve. If every element of $\mathcal{F}$ is isotopic rel X to a homeomorphism θ that preserves Γ and does not interchange the two complementary components of Γ then* $\mathrm{Fix}(\mathcal{F}) \neq \emptyset$.

Proof. Let D be one of the disks bounded by Γ and let M' be the component containing Γ of the surface obtained from M by filling in the punctures corresponding to the compact set $X_D := X \cap D$. Thus $D \subset M'$ and X_D is $\mathcal{F}$-invariant. Let $\tilde{M}'$ be the universal covering space of M', let $\tilde{D} \subset \tilde{M}'$ be a lift of D and let $\tilde{X}_D$ be the lift of X_D that is contained in $\tilde{D}$. Note that $\tilde{X}_D$ is homeomorphic to X_D and is therefore compact. For each $f \in \mathcal{F}$ there is a unique lift $\tilde{f} : \tilde{M}' \to \tilde{M}'$ that setwise fixes $\tilde{X}_D$. If $\mathcal{F} = \langle f_1, \dots, f_n \rangle$ then the uniqueness property implies that $\tilde{\mathcal{F}} = \langle \tilde{f}_1, \dots, \tilde{f}_n \rangle$ is abelian. Theorem 1.2 implies that $\mathrm{Fix}(\tilde{\mathcal{F}})$ and hence $\mathrm{Fix}(\mathcal{F})$ is non-empty. □

The following lemma is an application of the well known fact that any action of a finitely generated abelian group on a tree has a fixed point.

Lemma 7.3. *Suppose that M is a genus zero surface, that $\mathcal{F}$ is a finitely generated abelian subgroup of* $\mathrm{Homeo}_+(M)$ *and that R is a finite collection of simple closed disjoint curves in M that is $\mathcal{F}$-invariant up to isotopy. Let $\mathbb{A}$ be the union of disjoint open annulus neighborhoods of the elements of R. Then there exists either an element Γ of R or a component M_i of $M \setminus \mathbb{A}$ that is $\mathcal{F}$-invariant up to isotopy; in the latter case, we may choose M_i so that there is at most one component of ∂M_i that is $\mathcal{F}$-invariant up to isotopy.*

Proof. Let T be the finite tree that is dual to R. More precisely, T has one vertex for each component M_i of $M \setminus \mathbb{A}$ and an edge connecting the vertices corresponding to M_i and M_j if there exists $\Gamma \in R$ whose annulus neighborhood has one boundary component in M_i and the other in M_j. If each edge of T is assigned length one, then $\mathcal{F}$ induces an action on T by isometries.

Choose generators $\{f_i\}$ of $\mathcal{F}$. The fixed point set for the action of f_1 on T is a non-empty subtree T_1. Since $\mathcal{F}$ is abelian, T_1 is f_2-invariant and the fixed point set for the restricted action of $\mathcal{F}$ on T_1 is a non-empty subtree T_2 of T_1. Repeating this argument shows that the set of points in T that are fixed by $\mathcal{F}$ is a non-empty tree T^*.

If T^* is a single point and is not a vertex then it is the midpoint of some edge of T. In this case there is an $\mathcal{F}$-invariant $\Gamma \in R$. Otherwise, T^* is either a single vertex or is a union of edges of T. Let v be a vertex of T^* whose valence in T^* is zero or one. The subsurface M_i corresponding to v is $\mathcal{F}$-invariant up to isotopy and components of

∂M_i that are $\mathcal{F}$-invariant up to isotopy are in one-to-one correspondence with edges of T^* that are incident to v. $\square$

The index two part of Theorem 1.1 is handled by our next lemma.

Lemma 7.4. *If a finitely generated abelian subgroup $\mathcal{F}$ of $\mathrm{Homeo}_+(S^2)$ has a finite index subgroup F' such that $\mathrm{Fix}(\mathcal{F}') \neq \emptyset$ then it has such a subgroup with index at most two.*

Proof. The $\mathcal{F}$-orbit P of a point in $\mathrm{Fix}(\mathcal{F}')$ is finite. If P has cardinality one or two then its stabilizer has index at most two and we are done. We may therefore assume that $M := S^2 \setminus P$ has negative Euler characteristic. Assume the notation of Proposition 2.11 with $K = L = \emptyset$. (The existence of normal forms in this case follows directly from the Thurston classification theorem without reference to Proposition 2.11 but it is convenient to use our usual notation.) Denote $\langle \theta_1, \dots, \theta_n \rangle$ by Θ.

If there is a Θ-invariant element $\Gamma \in R$ then Lemma 7.2 allows us to choose $\mathcal{F}'$ with index at most two. We are therefore reduced, by Lemma 7.3, to the case that there is a Θ-invariant subsurface M_i whose boundary does not have any Θ-invariant components. By Lemma 3.6 there is a subgroup Θ' of Θ with index at most two that fixes either a puncture, a component of ∂M or a point in the interior of M_i. It therefore suffices to show that the corresponding subgroup $\mathcal{F}'$ of $\mathcal{F}$ has a global fixed point. This is obvious if Θ' fixes a puncture and follows from Lemma 7.2 if Θ' fixes a component of ∂M_i. In the remaining case, Θ' fixes a point x in the interior of M_i and Proposition 7.1 completes the proof. $\square$

Proof of Theorem 1.1 We assume at first that $\mathcal{F}$ is finitely generated and argue by induction on the rank n of $\mathcal{F}$. The $n = 1$ case is obvious so we assume inductively that the statement holds for rank $n - 1$. Choose generators $f_1, \dots, f_n$ for $\mathcal{F}$ and denote $\langle f_1, \dots, f_{n-1} \rangle$ by $\mathcal{F}_{n-1}$. By the inductive hypothesis there is a subgroup $\mathcal{F}'_{n-1}$ of $\mathcal{F}_{n-1}$ with index at most two such that $K = \mathrm{Fix}(\mathcal{F}'_{n-1})$ is non-empty. If $w(f, g) = 0$ for all $f, g \in \mathcal{F}$ then we may assume that $\mathcal{F}'_{n-1} = \mathcal{F}_{n-1}$. Let $L = \mathrm{Fix}(f_n)$ and let $\mathcal{F}^*$ be the subgroup of $\mathcal{F}$ generated by $\mathcal{F}'_{n-1}$ and f_n. Thus $\mathcal{F}^*$ has index at most two and equals $\mathcal{F}$ if $w(f, g) = 0$ for all $f, g \in \mathcal{F}$.

If $K \cap L \neq \emptyset$ then $\mathcal{F}' = \mathcal{F}^*$ satisfies the conclusions of the theorem and we are done. It therefore suffices to assume that $K \cap L = \emptyset$ and argue to a contradiction. Assume the notation of Proposition 2.11 applied to K, L and $\mathcal{F}^*$. Let Θ'_{n-1} be the subgroup of normal forms corresponding to $\mathcal{F}'_{n-1}$, let θ_n correspond to f_n and let Θ^* be the subgroup generated by Θ'_{n-1} and θ_n.

If $R = \emptyset$ then $K \cup L$ is finite and its stabilizer has finite index in $\mathcal{F}^*$. If $R \neq \emptyset$ then there is a finite index subgroup of Θ^* that setwise fixes each M_i. Lemma 7.2 implies that the corresponding subgroup of $\mathcal{F}^*$ has a global fixed point. In either case, Lemma 7.4 implies that some subgroup of $\mathcal{F}$ with index at most two has a global fixed point.

To complete the proof in the case that $\mathcal{F}$ is finitely generated, we must show that $\mathrm{Fix}(\mathcal{F}) \neq \emptyset$ if and only if $w(f,g) = 0$ for all $f, g \in \mathcal{F}$. The "only if" part is a consequence of Corollary 3.9. Thus we assume that $w(f,g) = 0$ for all $f, g \in \mathcal{F}$ and prove that $\mathcal{F}^* = \mathcal{F}$ has a global fixed point. Suppose at first that $\Gamma \in R$ is Θ^*-invariant. Since K is non-empty, elements of Θ'_{n-1} do not interchange the two complementary components of Γ. Similarly, θ_n does not interchange the two complementary components of Γ because L is non-empty. It follows that no element of Θ^* interchanges the two complementary components of Γ. Lemma 7.2 then completes the proof.

We are therefore reduced, by Lemma 7.3, to the case that there is a Θ^*-invariant subsurface M_i whose boundary does not have any Θ^*-invariant components. Since θ_n fixes either a puncture in M_i or a component of ∂M_i, it cannot be that Θ^*_{n-1} is the identity. Similarly θ_n can not be the identity. Thus M_i has finite type. Lemma 3.6 implies that Θ^* fixes either a puncture, a boundary component or a point in the interior of M_i. The first contradicts the assumption that $K \cap L = \emptyset$, the second contradicts the choice of M_i and the third contradicts the fact that $L = \mathrm{Fix}(\theta_n|_{M_i})$. This completes the proof when $\mathcal{F}$ is finitely generated.

We complete the proof by showing the result for finitely generated $\mathcal{F}$ implies the general result. This follows from Proposition 7.5 below applied with $n = 1$ and $n = 2$. □

Proposition 7.5. *Suppose Λ is a compact metric space and $\mathcal{F}$ is a subgroup of* $\mathrm{Homeo}(\Lambda)$ *with the property that every finitely generated subgroup $\mathcal{F}_0$ has a subgroup of index $\leq n$ with a global fixed point. Then $\mathcal{F}$ has a subgroup of index $\leq n$ with a global fixed point.*

Proof. We give a proof by contradiction. A point $x \in \Lambda$ is a fixed point of a subgroup of $\mathcal{F}$ of index $\leq n$ if and only if the $\mathcal{F}$ orbit of x contains at most n points. Thus if x is not a fixed point for a subgroup of $\mathcal{F}$ of index $\leq n$ there are elements $f_{0,x}, f_{1,x}, \dots f_{n,x} \in \mathcal{F}$ such that $f_{0,x}(x), f_{1,x}(x), \dots f_{n,x}(x)$ are all distinct. This is an open condition, i.e. there is a neighborhood U_x of x such that for all $y \in U_x$ the points $f_{0,x}(y), f_{1,x}(y), \dots f_{n,x}(y)$ are all distinct. Assume that the result we want to prove is false, i.e. there is no point of Λ which is fixed by an index n subgroup of $\mathcal{F}$. Then the collection $\{U_x\}$ is an open cover of Λ which has a finite subcover $\{U_j\}_{j=1}^m$. We will denote by $f_{0,j}, f_{1,j}, \dots f_{n,j}$ the corresponding elements of $\mathcal{F}$. Then for any $x \in \Lambda$ there is a j such that $f_{0,j}(x), f_{1,j}(x), \dots f_{n,j}(x)$ are all distinct. But this is impossible since it would imply that the group $\mathcal{F}_0$ generated by $\{f_{i,j}\}$ has no point fixed by an subgroup of index $\leq n$ which contradicts the hypothesis. □

Proof of Corollary 1.3 By Proposition 7.5 it suffices to prove the result when $\mathcal{F}$ is finitely generated. Let $\mathcal{G} \subset \mathrm{Homeo}_+(S^1)$ be the group of restrictions to $S^1 = \partial \mathbb{D}^2$ of elements of $\mathcal{F}$. If every element of $\mathcal{G}$ has a fixed point then $\mathrm{Fix}(\mathcal{G}) \neq \emptyset$. This is because the group $\mathcal{G}$ is abelian and hence there is a $\mathcal{G}-$invariant measure μ on S^1. But for each $g \in \mathcal{G}$ $\mathrm{Fix}(g) \neq \emptyset$ which implies $supp(\mu) \subset \mathrm{Fix}(g)$. Hence $supp(\mu) \subset \mathrm{Fix}(\mathcal{G})$.

We are left with the case that there exists $f_0 \in \mathcal{F}$ such that $\mathrm{Fix}(f_0) \cap \partial\mathbb{D}^2 = \emptyset$. In that case we apply Theorem 1.2 to $\mathbb{D}^2 \setminus \partial\mathbb{D}^2$ observing that $\mathrm{Fix}(f_0)$ is a compact $\mathcal{F}$ invariant set. $\square$

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