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Kaon decays and predictions of chiral symmetry

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Kaon transitions and predictions of chiral symmetry

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We describe a set of kaonic electromagnetic and semileptonic weak decay processes which are completely predicted within the framework of chiral symmetry (and, therefore, of low-energy QCD), emphasizing where present problems exist and suggesting future experiments.

I. INTRODUCTION

The study of quantum chromodynamics (QCD) has occupied a central role in the program of modern particle physics.1 Traditional experiments have involved the very-high-energy machines, using the asymptotic-freedom property of QCD (Ref. 2) in order to confront QCD perturbation theory,3 which is an expansion in $1/\ln E$. It is also possible to pose significant tests of QCD at low $[s<500 \text{ MeV}^2]$ energies using the feature of chiral symmetry,4 i.e., invariance under separate global rotations among left- and right-handed $u$, $d$, and $s$ quarks. This SU(3)$_L \otimes$ SU(3)$_R$ symmetry is broken, of course, by the existence of a nonzero quark mass, but such effects can be included perturbatively and it is possible to construct an effective Lagrangian which describes the low-energy interactions of $0^-$ Goldstone bosons of the theory in a rigorous fashion provided that the underlying quark-gluon interactions are indeed chiral symmetric.5 Here predictions are given in an expansion in powers of the energy and/or quark masses. As in the case of many high-energy tests there exist drawbacks within such an approach—specifically that absolute predictions are in general not possible, only relations between empirical observables. Also the restriction to low-energy processes poses a serious limitation. Nevertheless this technique, called chiral perturbation theory, offers an attractive avenue by which to probe the underlying structure of strong interactions in a regime wherein relatively precise measurements are presently possible.

In an earlier paper we studied one aspect of such a program—weak and electromagnetic interactions of the charged pion—and identified various relations between pionic interaction parameters required by chiral symmetry.6 Confrontation with experimental results revealed good agreement except in the case of the pion polarizability,7 which strongly suggests remeasurement of this parameter. In this paper we extend our discussion of low-energy tests of chiral symmetry to include the kaonic sector, which, because of the greater kaon mass and plethora of semileptonic decay modes, allows for a much richer program of experimental probes. Indeed, as we shall show, there exist kaonic analogs of each of the pionic tests discussed in Ref. 6 as well as many additional experimental possibilities allowed by the existence of $K_{13}$, $K_{137}$, $K_{14}$, etc., modes. We shall consider only semileptonic processes here, as nonleptonic and nonleptonic-radiative kaon transitions have been treated elsewhere. The greater kaon mass also presents a difficulty for chiral perturbation theory in that the corrections to the lowest-order predictions will be larger than they are in the case of pionic transitions. In some situations, chiral perturbation theory will use SU(3) breaking in order to predict these modifications. In known examples, the size of the corrections is about 25–30%. It is important to note that these modifications do not invalidate chiral perturbation theory, but fit naturally into the framework using the energy expansion.

In the next section then we present for completeness a brief outline of the effective Lagrangian methods used in our analysis. In Sec. III we examine the kaonic analogs of the pionic processes of Ref. 6 while in Sec. IV, we study the richer vein offered by $K_{13}$, $K_{137}$ studies. Finally, we summarize our findings in a concluding Sec. V.

II. FORMALISM

Since the chiral Lagrangian technique has been carefully explained in other works, there is no need to belabor this formalism here. Nevertheless for completeness we outline the Gasser-Leutwyler procedure which will be employed in subsequent sections.5

In the limit that the $u$-, $d$-, and $s$-quark masses vanish, QCD possesses an exact global SU(3)$_L \otimes$ SU(3)$_R$ chiral symmetry:

$$ q_L \rightarrow \exp \left[ i \sum_{j=1}^{q} \lambda_j \alpha_j \right] q_L \equiv Lq_L,$$

$$ q_R \rightarrow \exp \left[ i \sum_{j=1}^{q} \lambda_j \beta_j \right] q_R \equiv Lq_R,$$

where

$$ q = \begin{bmatrix} u \\ d \\ s \end{bmatrix} \tag{2}$$

denotes a three-component column vector and $\lambda_i$ are the Gell-Mann matrices. This chiral SU(3)$_L \otimes$ SU(3)$_R$ invariance is dynamically broken to SU(3)$_V$ and Goldstone’s theorem implies the existence of eight “massless” $J^P=0^-$ Goldstone bosons, which are identified with $\pi$, $K$, and $\eta$ (Ref. 8). The axial $U(1)_A$ transformation
\[ q \rightarrow e^{i\omega \gamma_5} q \] (3)

is, however, not a quantum-mechanical invariant and is associated with the well-known QCD anomaly.\(^9\) In addition the quarks are not massless and the inclusion of appropriate mass terms introduces a small explicit breaking of the chiral symmetry, which can be accounted for via a perturbative expansion in the energy.\(^10\)

The manifestation of chiral symmetry within the interactions of the (pseudo-)Goldstone bosons is most succinctly realized in terms of the nonlinear order parameter

\[ U = \exp \left[ \frac{i}{F} \sum_{j=1}^{8} \lambda_j \phi_j \right] , \]

(4)

where \( \phi_j \) are the pseudoscalar fields and \( F = 94 \text{ MeV} \) is the pion decay constant. Under SU(3)\(_L\) \( \otimes \) SU(3)\(_R\) the matrix \( U \) is defined to transform as

\[ U \rightarrow L U R^\dagger . \] (5)

Including the effects of quark mass, the simplest Lagrangian consistent with chiral and Lorentz invariance is then

\[ \mathcal{L}^{(2)} = \frac{F^2}{4} \text{Tr} D_{\mu} U D^{\mu} U^\dagger + \frac{F^2}{4} \text{Tr}m (U + U^\dagger) . \] (6)

Here \( F^{LR}_{\mu\nu} \) are external field-strength tensors defined via

\[ F^{LR}_{\mu\nu} = \partial_\mu F^{L\nu}_\rho - \partial_\nu F^{L\mu}_\rho - i[F^{LR}_{\mu\rho}, F^{L\nu}_{\rho}] , \]

(7)

and \( D_\mu \) is the covariant derivative, defined by

\[ D_\mu = \partial_\mu - i[V_\mu, ] + i[A_\mu, ] . \]

(8)

The mass matrix \( m \) characterizes chiral-symmetry breaking and is given by

\[ m = 2 B_0 \begin{bmatrix} m_u & 0 \\ 0 & m_d \end{bmatrix} . \] (9)

Obviously the first piece of \( \mathcal{L}^{(2)} \) contains the meson kinetic energy and is chiral invariant, while the second term includes the pseudoscalars and transforms as \((3_L, \bar{3}_R) \otimes (\bar{3}_L, 3_R)\) under chiral rotations. Comparison with experimental values of the 0\(^-\) masses yields the normalization\(^11\)

\[ 2B_0 = \frac{2m^2}{m_s + m_u} = \frac{2m^2}{m_d + m_u} = \frac{6m^2}{m_s} . \] (10)

Such effective Lagrangians have been known for over two decades and the tree-level evaluation of \( \mathcal{L}^{(2)} \) yields [to \( O(p^2, m^2) \)] the familiar Weinberg \( \pi \pi \) scattering amplitude\(^12\)

\[ T^{\pi\pi}(s,t,u) = \frac{1}{F^2} \left[ \delta^{0d}\delta^{0d}(s - m^2) + \delta^{ad}\delta^{bd}(t - m^2) \right. \]

\[ \left. + \delta^{ad}\delta^{bd}(u - m^2) \right] \] (11)

which is roughly borne out experimentally, as well as other successful predictions. Of course, loop diagrams associated with \( \mathcal{L}^{(2)} \) are required for unitarity and produce effects of higher order \([ O(p^4, p^2 m^2, m^4) \)] along with divergences. The infinities can be eliminated by renormalization of phenomenological chiral couplings of order 4. Gasser and Leutwyler have given the most general such Lagrangian:\(^13\)

\[ \mathcal{L}^{(4)} = L_1 (\text{Tr} D_{\mu} U D^{\mu} U^\dagger)^2 + L_2 (\text{Tr} D_{\mu} U D^{\mu} U^\dagger)^2 + L_3 (\text{Tr} D_{\mu} U D^{\mu} U^\dagger)^2 + L_4 (\text{Tr} D_{\mu} U D^{\mu} U^\dagger) \text{Tr}m (U + U^\dagger) 
\]

\[ + L_5 (\text{Tr} D_{\mu} U D^{\mu} U^\dagger) mU + \text{Tr}m (U + U^\dagger) + L_6 (\text{Tr} D_{\mu} U D^{\mu} U^\dagger) + L_7 \text{Tr}m^2 \] (12)

Here the coefficients \( L_1, \ldots, L_{12} \) are arbitrary and unphysical (bare) since they can be used in order to absorb divergent loop contributions from \( \mathcal{L}^{(2)} \). The physical (renormalized) couplings are found to be

\[ \mathcal{L}'(\mu) = \mathcal{L}_1 + \frac{\Gamma_1}{32\pi^2} \left[ \frac{2}{\epsilon} + \ln 4\pi + 1 - \gamma \right] \mu^\epsilon , \] (13)

where \( \gamma \) is Euler's constant, \( \epsilon = d - 4 \) represents the dimensionality, and

\[ \Gamma_1 = \frac{1}{3}, \quad \Gamma_2 = \frac{3}{5}, \quad \Gamma_3 = 0, \quad \Gamma_4 = \frac{1}{2}, \quad \Gamma_5 = \frac{1}{2}, \]

(14)

\[ \Gamma_6 = \frac{11}{54}, \quad \Gamma_7 = 0, \quad \Gamma_8 = \frac{1}{4}, \quad \Gamma_9 = \frac{1}{4}, \quad \Gamma_{10} = -\frac{1}{4}, \quad \Gamma_{11} = -\frac{1}{8}, \quad \Gamma_{12} = \frac{3}{24} \]

are constants chosen in order to cancel the divergences.

Finally, we must include also the effects associated with the anomaly, which is also of order 4. Including the gauge dependence only for the photon field \( A_\mu \), we find\(^14\)

\[ S_{\text{anom}} = \int d^4 x \epsilon^{\mu\nu\lambda\sigma} \text{Tr} L_{\gamma L_{\mu} L_{\nu} L_{\lambda} L_{\sigma}}, \]

(15)

with

\[ L_{\mu} = \partial_\mu U U^\dagger, \quad R_\mu = U^\dagger \partial_\mu U , \]

\[ j^\mu = \epsilon^{\mu\nu\lambda\sigma} \text{Tr} (Q_L L_{\nu} L_{\lambda} + Q_R R_{\nu} R_{\lambda}) , \]

(16)

\[ T_\beta = \text{Tr} (Q^2 L_\beta + Q^2 R_\beta + \frac{1}{2} Q U U^\dagger L_\beta + \frac{1}{2} Q U U^\dagger R_\beta) , \]
and

$$Q = \begin{pmatrix} \frac{2}{5} & 0 & 0 \\ 0 & -\frac{1}{5} & 0 \\ 0 & 0 & -\frac{1}{5} \end{pmatrix}$$  \hspace{1cm} (17)$$

being the quark charge matrix. The full non-Abelian anomaly has also been given but will not be needed here.  \(^{15}\)

The use of such an effective chiral Lagrangian in order to make contact with experimental quantities has been described elsewhere.  \(^{5,6}\) Thus, we quote henceforth only the results obtained from such evaluations. For simplicity of exposition we shall omit \(\mathcal{L}^{(2)}\) loop contributions in the main body of the paper. Indeed such terms are \(O(1/N)\) with respect to the leading tree effects and explicit calculation confirms the dominance of the tree-level \(\mathcal{L}^{(4)}\) coefficients over their \(\mathcal{L}^{(2)}\) loop counterparts.  \(^{16}\) For completeness, however, we include the effect of loop terms in a brief appendix.

### III. KAONIC ANALOGS OF PIONIC PROCESSES

As mentioned above, the kaon, because of its greater mass, offers a much expanded laboratory for theoretical and experimental analysis than does its pionic counterpart. Before exploring some of the additional processes which the kaonic system offers, however, it is useful to emphasize that each of the pionic reactions discussed in Ref. \(6\) has a direct kaonic counterpart. Thus we define the following: (i) Electromagnetic form factor:

$$\langle K^+(p_2) | V_{\mu}^{\text{em}} | K^+(p_1) \rangle = f_{K^+}^{\mu}(q^2)(p_1 + p_2)_\mu$$  \hspace{1cm} (18)$$

with

$$f_{K^+}^{\mu}(q^2) \equiv 1 + \frac{1}{6} \langle r^2_{K^+} \rangle q^2 \cdots .$$  \hspace{1cm} (19)$$

(ii) Radiative kaon decay \(-K^+ \rightarrow e^+ \nu e^+ \nu e^{-}\):

$$K^+ \rightarrow e^+ \nu e^+ \nu e^{-}.$$  \hspace{1cm} (20)$$

Here \(h_A, r_A\) represent axial structure constants while \(h_V\) is a form factor which arises from the polar vector current. Of course, the axial structure constant \(r_A\) is relevant only for the \(e^+ \nu e^+ \nu e^{-}\) mode. (iii) Kaonic Compton scattering:

$$A_{\mu\nu}(p_1, q_1, q_2) \equiv \int d^4x \, e^{iq_2 \cdot x} \langle K^+(p_2) | T[V_{\mu}^{\text{em}}(x)V_{\nu}^{\text{em}}(0)] | K^+(p_1) \rangle$$

$$\equiv \frac{\sqrt{2} F_K}{m_K} r_A[(p - q)_\mu q_\nu - g_{\mu\nu}(p - q)_\mu q_\nu] + h_V [\epsilon_{\mu\nu\rho\sigma}q^\rho r_A(g_{\sigma\nu} q^\mu - q^\mu g_{\sigma\nu})].$$  \hspace{1cm} (21)$$

where

$$T_\mu(p_2, p_1) = (p_1 + p_2)_\mu(1 + \frac{1}{6} \langle r^2_{K^+} \rangle q^2) - q_\mu \frac{1}{6} \langle r^2_{K^+} \rangle (p_1^2 - p_2^2).$$  \hspace{1cm} (22)$$

is the (off-shell) kaon electromagnetic vertex. As discussed in Ref. \(6\), \(\gamma_K\) represents the kaonic polarizability with

$$\alpha_E(K^+) = -\beta_M(K^+) = \gamma_K.$$  \hspace{1cm} (23)$$

That the electric (∂E) and magnetic (∂M) polarizabilities must be negatives of one another is associated with the chiral-symmetry requirement, as described in Ref. \(6\).

In terms of the effective chiral Lagrangian outlined above we can represent five of these experimental quantities in terms of just two of the chiral parameters—\(L_9\) and \(L_{10}\):

$$\langle r^2_{K^+} \rangle \, \text{theor} = \frac{12L_9}{F_{\pi}^2},$$

$$\frac{h_A}{h_V} \, \text{theor} = 32\pi^2(L_9 + L_{10}) \,$$

$$\frac{r_A}{h_V} \, \text{theor} = 32\pi^2L_9 \, ,$$

$$\alpha_E(K) + \beta_M(K) \, \text{theor} = 0 \, ,$$

$$\alpha_E(K) \, \text{theor} = \frac{4\alpha}{m_K^2 F_{\pi}^2}(L_9 + L_{10}) \, .$$

Staying strictly within the kaonic system we observe that chiral symmetry requires three relations to be valid:
and verifying these predictions constitutes an important test of the underlying theory—i.e., QCD. While the absolute values of \( r_A/h_V \) or of \( \alpha_E(K) \) requires the use of a specific model, the relations between these parameters must obtain in any chiral model. However, we can go even further. Since \( L_9, L_{10} \) were already determined in the pionic system, we can make here an absolute prediction for each of these experimental quantities in terms of the corresponding pion values. Unfortunately the present kaonic data base is quite limited. While the \( K^+ \) charge radius has been measured,

\[
\frac{r_A}{h_V} = \frac{8\pi^2}{3} F_\pi^2 \langle r_K^2 \rangle,
\]

(25)

and

\[
\alpha_E(K) = -\beta_M(K) = \frac{\alpha}{8\pi^2 m_K^2 F_K^2} \frac{h_A}{h_V},
\]

(26)

Finally, in the case of the polarizability only a crude upper limit is available from the kaonic atom measurements:

\[
|\alpha_E(K)|^{\text{exp}} < 2 \times 10^{-2} \text{ fm}^3 \quad \text{vs} \quad |\alpha_E(K)|^{\text{theor}} = 5.8 \times 10^{-5} \text{ fm}^3.
\]

(30)

In view of this situation it is clearly out of the question to effect a meaningful test of chiral symmetry at the present time. On the other hand, the possibility of producing such a chiral comparison should serve as a challenge to experimentalists. The gathering of such data ought certainly to be pursued as part of the experimental program of the new kaonic facilities which may be coming on line in the future. (One difficulty in this regard is that the expected polarizability is smaller than that of the pion by the factor \( F_K^2/m_K^2 - m_\pi^2 \approx 4.8 \). Hopefully this can be overcome.)

IV. \( K_{13s}, K_{13y} \) DECAY PROCESSES

We have observed in the previous section a series of experiments involving the charged kaon which can be expressed in terms of only two chiral parameter, \( L_9 \) and \( L_{10} \), which have already been determined in the pion sector. A second class of kaonic reactions

\[
K_{13s}^+ \to \pi^0 l^+ v_l, \quad K_{13y}^+ \to \pi^0 l^+ \gamma,
\]

can be described in terms of \( L_9, L_{10} \) and a third parameter \( L_7 \). Since the latter is well determined by the measured value of kaonic-to-pionic decay constants\(^5\)

\[
\frac{F_K}{F_\pi} = 1 + \frac{4}{F_\pi^2} (m_K^2 - m_\pi^2) L_5 = 1.22 \pm 0.02
\]

(31)

chiral symmetry also makes unambiguous predictions for these \( K_{13s}, K_{13y} \) processes.

In the case of \( K_{13s} \) decays, \( K^+ \to \pi^0 l^+ v_l, K_L^0 \to \pi^0 l^- v_l, \)

it is traditional to parametrize the hadronic matrix elements in terms of form factors \( f_+(q^2), f_-(q^2) \):

\[
\langle \pi(p_2) | V_{\mu}^{K^+} | K(p_1) \rangle = \sqrt{1/2} f_+(q^2)(p_1+p_2)_\mu + f_-(q^2)(p_1-p_2)_\mu.
\]

(32)

Alternatively, one often finds results expressed in terms of \( s \)- and \( p \)-wave projections in the cross channel:

\[
s \text{ wave } f_0(q^2) = f_+(q^2) + \frac{q^2}{m_K^2 - m_\pi^2} f_-(q^2),
\]

\[
p \text{ wave } f_+(q^2).
\]

In either case it is common to employ a linear extrapolation in order to characterize the momentum-transfer dependence:

\[
f(q^2) = f(0)(1 + \frac{1}{2} \langle r^2 \rangle q^2 + \cdots )
\]

(34)

The chiral-symmetry prediction can be obtained from the effective Lagrangian of Eqs. (6) and (12) yielding\(^6\)

\[
f_+(p_1^2, p_2^2, q^2) = 1 + \frac{2L_9}{F_\pi^2} q^2,
\]

(35)

\[
f_-(p_1^2, p_2^2, q^2) = \frac{4}{F_\pi^2} (m_K^2 - m_\pi^2) L_5 - \frac{4L_9}{F_\pi^2} (p_1^2 - p_2^2)
\]

\[
= \frac{F_K}{F_\pi} - 1 - \frac{2L_9}{F_\pi^2} (p_1^2 - p_2^2).
\]
This form is consistent with (and required by) the Callan-Treiman condition 21
\begin{align*}
\langle \pi^0(p_2) | V_{\mu}^- K^- | K^+(p_1) \rangle & \sim \frac{i}{F_{\pi}} \langle 0 | [Q_{2}^{0}, V_{\mu}^-] | K^+(p_1) \rangle \\
& = -\frac{i}{2F_{\pi}} \langle 0 | A_{\mu}^- | K^+(p_1) \rangle = -\frac{1}{\sqrt{2}} \frac{F_{K}}{F_{\pi}} P_{\mu} \tag{36}
\end{align*}

first derived by current-algebra–PCAC (partial conservation of axial-vector current) techniques. We note that
\begin{equation}
\frac{F_{K}}{F_{\pi}} \frac{f_{+}(p_{1}^{2}, 0, p_{1}^{2}) + f_{-}(p_{1}^{2}, 0, p_{1}^{2})}{f_{+}(0, p_{1}^{2}, p_{1}^{2})} = \frac{F_{K}}{F_{\pi}} \tag{37}
\end{equation}

Similarly the soft-kaon condition
\begin{equation}
\frac{f_{+}(0, p_{2}^{2}, p_{2}^{2}) + f_{-}(0, p_{2}^{2}, p_{2}^{2})}{2} = 1 - \frac{2L_{9}}{F_{\pi}} (m_{K} - m_{\pi}) \tag{38}
\end{equation}
is obtained.

In comparing with experiment we use mass-shell conditions, yielding
\begin{align*}
f_{+}(m_{K}^{2}, m_{\pi}^{2}, 0) & = 1 \tag{39} \\
f_{-}(m_{K}^{2}, m_{\pi}^{2}, 0) & = -0.13 \tag{40}
\end{align*}

We observe that \( f_{-}(m_{K}^{2}, m_{\pi}^{2}, 0) \) vanishes in the SU(3)-symmetry limit as expected and that the \( L_{9} \) contribution is required in order to yield the negative sign indicated by present \( K_{\mu 3} \) experiments.\(^{22}\)
\begin{equation}
\xi(0) = \frac{f_{-}(m_{K}^{2}, m_{\pi}^{2}, 0)}{f_{+}(m_{K}^{2}, m_{\pi}^{2}, 0)} = \begin{cases} 
-0.11 \pm 0.09 \text{ from } K_{\mu 3}^{0} \\
-0.35 \pm 0.15 \text{ from } K_{\mu 3}^{+}.
\end{cases} \tag{40}
\end{equation}

Likewise the predicted slopes
\begin{align*}
\lambda_{+} & = \frac{2L_{9}}{F_{\pi}^{2}} = 0.067 \text{ fm}^{2} \tag{41} \\
\lambda_{0} & = \frac{1}{m_{K}^{2} - m_{\pi}^{2}} \left[ \frac{F_{K}}{F_{\pi}} - 1 \right] = 0.040 \text{ fm}^{2}
\end{align*}
are quite consistent with the values determined experimentally:\(^{23}\)
\begin{align*}
\lambda_{+} & = \begin{cases} 
0.060 \pm 0.003 \text{ fm}^{2} \text{ from } K_{\epsilon 3}^{0} \\
0.056 \pm 0.008 \text{ fm}^{2} \text{ from } K_{\epsilon 3}^{+} \\
0.068 \pm 0.010 \text{ fm}^{2} \text{ from } K_{\mu 3}^{0} \\
0.067 \pm 0.017 \text{ fm}^{2} \text{ from } K_{\mu 3}^{+} \tag{42}
\end{cases} \\
\lambda_{0} & = \begin{cases} 
0.050 \pm 0.012 \text{ fm}^{2} \text{ from } K_{\mu 3}^{0} \\
0.008 \pm 0.014 \text{ fm}^{2} \text{ from } K_{\mu 3}^{+}
\end{cases}
\end{align*}
extcept for the case of \( \lambda_{0}(K_{\mu 3}^{+}) \) for which a significant discrepancy exists. Additional experimental work is strongly suggested in order to resolve this problem.

Two other points should be noted here. First, one might at first be surprised that the predicted \( K_{\mu 3}^{+} \) charge radius
\begin{equation}
\langle r^{2} \rangle = \frac{12L_{9}}{F_{\pi}^{2}} \tag{43}
\end{equation}
is identical to that for the electromagnetic form factor, since from vector dominance one would expect differing values
\begin{equation}
\langle r^{2}_{K^{0}} \rangle = \frac{1}{m_{K}^{*}}, \quad \langle r^{2}_{K^{+}} \rangle = \frac{1}{m_{K}^{+}}. \tag{44}
\end{equation}
The difference, however, between these two charge radii would arise only at order \( \mathcal{O}(\alpha) \) and hence is outside the scope of the present investigation. A second point to be noted is that when the \( u, d \) mass difference is included, a slight difference in the expected rates for charged and neutral \( K_{\mu 3} \) rates is introduced due to \( m_{d} - m_{u} \) mixing. Thus,
\begin{equation}
|\pi^{0}(p_{1})| = \cos \phi^{3} + \sin \phi^{3} \tag{45}
\end{equation}
with \( \epsilon = \sqrt{3} \frac{m_{d} - m_{u}}{4 m_{s} - \frac{3}{4}(m_{u} + m_{d})} \)
leading to
\begin{equation}
\frac{f_{+}^{K_{\mu 3}^{+} \pi^{0}}(0)}{f_{+}^{K_{\mu 3}^{0} \pi^{0}}(0)} \bigg|_{\text{theor}} = \cos \phi + \sqrt{3} \sin \phi \tag{46}
\end{equation}
which is in good agreement with the measured isospin violation\(^{20}\)
\begin{equation}
\left( \frac{f_{+}^{K_{\mu 3}^{+} \pi^{0}}(0)}{f_{+}^{K_{\mu 3}^{0} \pi^{0}}(0)} \right)_{\text{exp}} = 1.029 \pm 0.010. \tag{47}
\end{equation}

We have overall then an excellent picture of the \( K_{\mu 3} \) system within the chiral framework except for \( \lambda_{0}(K_{\mu 3}^{+}) \).

A second process which can be understood simply in terms of \( L_{9}, L_{10}, L_{5} \) is that of radiative \( K_{\mu 3} \) decay for which we find (we quote only the \( K^{+} \rightarrow \pi^{0} \) result—the \( K_{L} \rightarrow \pi^{0} \) result is similar)
\[ A_{\mu}((p_1, q_1, q_2) = \int d^4x \ e^{iq_1 \cdot x}[\pi_0(p_2)] \left[ T[V^{em}_\mu(x) V^{wk}_\nu(0)]K^+(p_1) \right] \]

\[ = \frac{1}{(p_1 - q_1)^2 - m_k^2} T_\mu(p_1 - q_1, p_1) (\pi_0(p_2)] V^{wk}_\nu K^+(p_1 - p_2) \]

\[ + \sqrt{1/2} \left[ \frac{F_K}{F_\pi} g_{\mu\nu} + \frac{L_9 + L_{10}}{F_\pi^2} (q_{1 \mu} - q_{1 \nu} q_{2 \mu} - q_{1 \nu} q_{2 \nu}) + \frac{L_9}{F_\pi^2} (q_{1 \mu} q_{1 \nu} - q_{1 \mu} q_{2 \nu}) \right] \]

\[ + \frac{L_9}{F_\pi^2} (q_{1 \nu} (p_2 \cdot q_2 - p_1 \cdot q_1) - q_{2 \mu} p_{2 \nu} + q_{1 \nu} p_{1 \mu} - p_{1 \mu} p_{2 \nu} + p_{1 \nu} p_{2 \mu}) \]  

It is straightforward to verify that this form satisfies both the gauge invariance

\[ q_i A_{\mu}((p_1, q_1, q_2) = (\pi_0(p_2)] V^{wk}_\nu K^+(p_1 - q_1) \]

\[ - (\pi_0(p_2)] V^{wk}_\nu K^+(p_1) \]  

(49)

and soft-pion requirements

\[ A_{\mu}((p_1, q_1, q_2) \sim \frac{1}{q_2 \to 0} M_{\mu\nu}(q_1, p_1) \].  

(50)

We observe that in addition to the pole term associated with inner bremsstrahlung, there exist substantial and distinctive structure-dependent contributions required by chiral symmetry. The size of this structure dependence is completely determined in terms of known properties of the charged pion and verification of such structure dependence would constitute an additional and a significant test of low-energy chiral symmetry. As with all such radiative processes, experimentally isolating the structure dependence from the large and generally dominant inner bremsstrahlung component is none too simple but could be within the reach of a dedicated experiment. In fact the best present limit\(^{23}\)

\[ \frac{\Gamma_{SD}(K^+ \to \pi^0 e^+ e^-; E > 10 \text{ MeV})}{\Gamma(K^+ \to \text{all})} < 5.3 \times 10^{-5} \]  

(51)

is tantalizingly close to the level at which such structure dependence should begin to be observed.\(^{24}\)

V. CONCLUSIONS

We have used the fact that higher-order effects in models which possess chiral symmetry (such as QCD) can be expressed in terms of a small number of phenomenological parameters \(L_1, \ldots, L_{10}\). In a previous paper (Ref. 6) we discussed how experiments involving the chiral pion could be used in order to test chiral symmetry and to measure two of these parameters — \(L_9\) and \(L_{10}\). In this work we have demonstrated how these values of \(L_9, L_{10}\) plus one additional quantity — \(L_5\), which is given in terms of \(F_K/F_\pi\) — can be used in order to determine an entire range of electromagnetic and semileptonic weak kaon transitions. These are explicit predictions of chiral symmetry and large violation of any such prediction would be difficult to understand within the framework of QCD. Certainly in this regard experimental work at present and/or future kaon facilities is strongly suggested.

Careful readers will note that we have not included the \(K\to\pi\) semileptonic process in our discussion, for which some considerable and precise experimental data are available. This omission was purposeful, since analysis of \(K\to\pi\) decay involves four additional chiral parameters, \(L_1, \ldots, L_4\), and it is not simply predicted in terms of available data. Rather it is the other way around. The very careful \(K\to\pi^+\pi^-\) data\(^{25}\) can be used in order to place restrictions on the size of the chiral parameters which are simply unavailable by other means.\(^{26}\) This is a very important program and will be described in a separate communication.\(^{27}\)

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APPENDIX

As stated in Sec. I, at low energies the loop corrections arising from the lowest-order Lagrangian \(\mathcal{L}^{(2)}\) are in general small and make only a small modification to the tree-level order \(\mathcal{L}^{(2)} + \mathcal{L}^{(4)}\) predictions. For completeness we list here the \(\mathcal{L}^{(2)}\) loop contributions to various quantities.

(i) Charged-kaon form factor:

\[ f_{K^+}(q^2) = 1 + q^2 \left[ \frac{2L_9}{F_\pi} + \frac{1}{32\pi^2 F_\pi^2} \left[ 2 \frac{m_k^2}{q^2} H \left( \frac{q^2}{m_k^2} \right) + \frac{m^2}{q^2} H \left( \frac{q^2}{m^2} \right) - \frac{1}{3} \ln \frac{m_k^2}{\mu^2} - \frac{1}{6} \ln \frac{m^2}{\mu^2} \right] \right] \]

(A1)

where

\[ H(x) = -\frac{4}{3} + \frac{5}{13} x + (\frac{3}{5} - \frac{1}{5}) x^{1/2} \ln Q(x) \]

(A2)

and
\[ Q(x) = \frac{\sqrt{x - 4} + \sqrt{x}}{\sqrt{x - 4} - \sqrt{x}}. \]  

(A3)

For \( q^2 < m_\pi^2 \) we then find

\[ \langle r_{K^+}^2 \rangle = 12 \frac{L_5^*}{F_\pi^2} - \frac{1}{16\pi^2 F_\pi^2} \left[ \frac{3}{2} + \ln \frac{m_K^2}{\mu^2} + \frac{1}{2} \ln \frac{m_\pi^2}{\mu^2} \right]. \]  

(A4)

Note that \( 12L_5^* \approx 0.084 > \frac{3}{32\pi^2} \approx 0.01. \)

(ii) Kaonic Compton scattering:

\[ \alpha_{K}(K) = \frac{-\alpha}{m_K F_K} \left[ 4(L_5^* + L_1^*) - \frac{1}{16\pi^2} \left[ \frac{3}{2} + \frac{m_K^2}{t} \ln^2 Q \left( \frac{t}{m_K^2} \right) \left( \ln \frac{Q}{m_K^2} \right) \ln \frac{Q}{m_\pi^2} \right] \right], \]  

(A5)

where \( t = (q_1 - q_2)^2 = -2q_1 \cdot q_2 \) is the momentum transfer. Since for small arguments we have

\[ \frac{1}{x} \ln^2 Q(x) \approx -1 + \frac{x}{12}. \]  

(A6)

we see that

\[ \alpha_{K}(K) = \frac{-\alpha}{m_K F_K} \left[ 4(L_5^* + L_1^*) - \frac{1}{384\pi^2} \frac{t}{m_\pi^2} \ln^2 Q \right], \]  

(A7)

so that the tree-order amplitude obtains at low momentum transfer.

(iii) \( K_{13} \) (Ref. 20):

\[ f_+(q^2) = 1 + \frac{1}{8F_\pi^2} \left[ 5q_2^2 - 2(m_K^2 + m_\pi^2) - 3 \left( \frac{m_K^2 - m_\pi^2}{q^2} \right)^2 (q_2^2) \right] + \frac{1}{24F_\pi^2} \left[ 3q_2^2 - 2(m_K^2 + m_\pi^2) - \left( \frac{m_K^2 - m_\pi^2}{q^2} \right)^2 (q_2^2) \right] \]  

\[ + \frac{4L_5^*}{F_\pi^2} q^2, \]  

(A8)

where

\[ F_{ij}(q^2) = \frac{q^2}{F_\pi^2} \left[ \frac{2}{3} L_{ij}^* + M_{ij}^*(q^2) - \frac{1}{q^2} L_{ij}(q^2) \right], \]  

(A9)

\[ \mu_i = \frac{m_i^2}{32\pi^2 F_\pi^2} \ln \frac{m_i^2}{\mu^2}, \]

and the functions \( \tilde{J}_{ij}, M_{ij}, L_{ij} \) have been defined in Ref. 5(b).

These forms are somewhat complex. However, a simple result is obtained for \( f_+(0) \):

\[ f_+(0) = 1 + \frac{1}{2} F_{K+,0} + \frac{1}{2} F_{K,0}, \]  

(A10)

where

\[ F_{ij}(0) = -\frac{1}{128\pi^4 F_\pi^2} \left( m_i^2 + m_j^2 \right) \left[ \frac{2}{3} \frac{m_i^2}{m_j^2} \ln \frac{m_i^2}{m_j^2} + \frac{2}{3} \frac{m_j^2}{m_i^2} \ln \frac{m_j^2}{m_i^2} \right]. \]  

(A11)

Note that

\[ \lim_{m_i \rightarrow m_j} F_{ij}(0) = 1 \]  

(A12)

so that \( f_+(0) = 1 \) in the SU(3)-symmetry limit. However, using physical values for the masses we find

\[ f_+(0) = 0.978. \]  

(A13)

Similarly we can address the \( q^2 \) dependence of the vector form factor:

\[ \lambda_{K} = \frac{1}{8\pi^2} \left( r_{K}^2 \right) - \frac{1}{128\pi^2 F_\pi^2} \left[ w_1 \left( \frac{m_\pi^2}{m_K^2} \right) + w_2 \left( \frac{m_\pi^2}{m_K^2} \right) \right] + \frac{5}{6} \ln \frac{m_K^2}{m_\pi^2} + \frac{1}{2} \ln \frac{m_\pi^2}{m_K^2} - 2, \]  

(A14)

where

\[ w_1(x) = \frac{1}{2} \frac{x^2 - 3x^2 - 3x + 1}{(x - 1)^2} \ln x + \frac{1}{2} \frac{x + 1}{x - 1} \left( \frac{x + 1}{x - 1} \right)^2 - \frac{1}{3}. \]  

(A15)

We then predict a small reduction from the tree-order prediction:
\[ \lambda^\text{tree}_+ = 0.067 \text{ fm}^2 \rightarrow \lambda^\text{loop}_+ = 0.060 \text{ fm}^2. \]  
\[ w_2(x) = \frac{3}{2} \left[ \frac{1+x}{1-x} \right]^2 + 3x(1-x) \ln x. \]

For the scalar form factor we find

\[ \lambda_0 = \frac{1}{m_K^2 - m_\pi^2} \left( \frac{F_K}{F_\pi} - 1 \right) + \frac{1}{192\pi^2 F_\pi^2} \left( \frac{5}{2} w_2 \left( \frac{m_K^2}{m_\pi^2} \right) 
+ \frac{19m_K^2 + 3m_\pi^2}{6(m_K^2 + m_\eta^2)} w_2 \left( \frac{m_\pi^2}{m_\eta^2} \right) - 3 \right), \]

where

\[ \lambda^\text{tree}_0 = 0.040 \text{ fm}^2 \rightarrow \lambda^\text{loop}_0 = 0.034 \text{ fm}^2. \]

(iv) \( K_{13y} \): Just as the \( K_{13} \) loop amplitude is much more complex than that for \( \pi_{13} \), the \( K_{13y} \) loop correction is considerably lengthier than its Compton-scattering counterpart. We shall thus not present it here but will instead give the complete form in a separate communication, wherein the \( K_{13y} \) process will be more carefully explored.

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1See, e.g., K. Gottfried and V. F. Weisskopf, Concepts of Particle Physics (Oxford University Press, New York, 1986), Vol. II.
13Gasser and Leutwyler [Ref. 5(b)].
16For example, at a renormalization point \( \mu = m_\eta \) one finds

\[ \langle r_\sigma^2 \rangle = \langle r_\sigma^2 \rangle_{L_{(2)}^\text{loop}} + \langle r_\sigma^2 \rangle_{L_{(4)}}, \]

with

\[ \langle r_\sigma^2 \rangle_{L_{(4)}^\text{loop}} = -16.5 \langle r_\eta^2 \rangle_{L_{(2)}^\text{loop}}. \]
20Gasser and Leutwyler [Ref. 5(c)].
26In Ref. 5(a) Gasser and Leutwyler use a Zweig-rule argument in order to deduce in \( L_1, \ldots, L_4 \) from \( \pi \pi \) scattering measurements.