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1989

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ONE LOOP RENORMALIZATION OF THE WESS–ZUMINO–WITTEN ANOMALY LAGRANGIAN*

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Received 29 August 1988

We study the effect of meson loops on the chiral anomaly action given by Witten as an integral over a five dimensional space. The coefficient of the anomaly, quantized in terms of the number of colors, is not modified by loops. All of the loop corrections are given by local four-dimensional chiral Lagrangians which contribute, in higher order in the momenta, to the same physical processes as the anomaly. We discuss the modifications of the decays $\pi^0 \rightarrow \gamma\gamma$, $\eta \rightarrow \gamma\gamma$, $\pi^0 \rightarrow \gamma\ell^+\ell^-$, $\eta \rightarrow \gamma\ell^+\ell^-$.

1. Introduction

The structure of Quantum Chromodynamics (QCD) at low energy is best described in terms of the nonlinear effective chiral Lagrangians [1–4]. These involve the low energy fields, pions, kaons, etc., and photons, and incorporate all of the interactions between them in a relatively simple dynamical framework. The effective Lagrangians are written as an expansion in terms of the energy and mass of the particles. In coordinate space this involves an expansion in the number of derivatives, where the lowest order term with two derivatives yields the dominant effects at low energy. Lagrangians with four derivatives provide corrections to these predictions. However, the main theoretical interest actually focuses on these next to leading (four derivative) terms. This is because the lowest order predictions are fixed by chiral symmetry alone (in terms of F_π and m_π), while the higher order Lagrangians (whose effects are clearly visible in the data) vary according to the underlying theory. They are in principle uniquely predicted in QCD and can form tests of our calculational methods or our understanding of QCD. There are many

* This paper is dedicated to the memory of H. Pagels.

attempts directed at characterizing the structure of the low energy world in terms of chiral Lagrangians, and the study of the connection between the phenomenological results and QCD.

In this paper we propose to begin the extension of this program to the anomalous sector of QCD [5–8]. Besides the obvious interest in the structure of QCD, this effort has significant extra theoretical interest. The effect of the anomaly cannot be given in terms of a standard local effective Lagrangian. Remarkably, however, it can be written as an integral of a local density over a five dimensional space whose boundary is on four dimensional space–time, the Wess–Zumino–Witten (WZW) anomaly construction [7, 8]. Even more remarkably, the coefficient of the Lagrangian is quantized and is given by the number of colors in QCD. This is a fascinating object whose structure has an intrinsic interest. In the effective Lagrangian program the meson loops must be considered, and it is of interest to see whether the quantization of the Lagrangian is modified by renormalizations. Also loops will always generate new forms of Lagrangians with more derivatives. Will these new Lagrangians also be defined in five dimensional space or are they local in four dimensions? On a more phenomenological level, are the well-known predictions of $\pi^0 \rightarrow \gamma\gamma$, $\eta \rightarrow \gamma\gamma$ modified, and can we understand the off-shell behavior contained in $\pi^0 \rightarrow \gamma\gamma^* \rightarrow \gamma e^+ e^-$? In this paper we perform the one loop renormalization of the anomaly and we will answer these questions. We will address both formal and phenomenological issues.

The treatment of loops in a technically nonrenormalizable theory is slightly more subtle than usual, but in fact is well defined in the context of a low energy expansion. Loops are required in order to include low energy effects dictated by unitarity, i.e. the inclusion of physical intermediate states in various transitions. However, the high momentum portions of loop calculations will produce divergences. These effects can all be absorbed into the renormalization of the physical parameters of both the lower order and higher order Lagrangians. The need to include the higher order Lagrangians is related to the sense in which the theory is nonrenormalizable. The inclusion of more loops requires the introduction of higher and higher order terms in the Lagrangian, until ultimately an infinite number of parameter need to be introduced. However, in a low energy expansion, we can work to any fixed number of derivatives and there is only a small finite number of parameters which need to be included. One can always choose an energy low enough so that the contribution of the remaining terms (infinite in number) is negligible. In QCD, one does very well at low energies including up to four derivatives in the Lagrangian.

In sect. 2, we begin by reviewing the structure of chiral Lagrangians, with particular emphasis on the WZW anomaly construction. Then in sect. 3 we calculate the one loop effects using heat kernel techniques [4] along with dimensional regularization. Sect. 4 explores the theoretical and phenomenological consequences of our results, while sect. 5 contains a summary and conclusions.

2. Chiral Lagrangians

The fundamental object in chiral SU(3) theories is the 3×3 chiral matrix U , which transforms as

$$U \rightarrow R^+ U L \quad (1)$$

under chiral rotation, where $L(R)$ are the SU(3) matrices describing the left handed (right handed) transformation. The small fluctuations of U around the identity matrix are the pseudoscalar mesons, and we parametrize U by

$$U = \exp(i\lambda^a \phi^a / F), \quad (2)$$

where ϕ^A , $A = 1, \dots, 8$ is the meson octet and F can be identified, to first order, with the pion decay constant, $F_\pi = 0.094$ GeV.

The lowest order Lagrangian containing U must be of the form

$$\mathcal{L}_0 = \frac{1}{4} F^2 \text{Tr}(\partial_\mu U \partial^\mu U^\dagger) + \frac{1}{4} F^2 \text{Tr}(m^2 U + m^2 U^\dagger), \quad (3)$$

where

$$m^2 = \begin{pmatrix} m_\pi^2 & 0 & 0 \\ 0 & m_\pi^2 & 0 \\ 0 & 0 & 2m_K^2 - m_\pi^2 \end{pmatrix}.$$

The first term is invariant under chiral rotation. The second is not, reflecting the small breaking of chiral symmetry due to quark masses. Both terms are of the same order in the energy expansion because a factor of m^2 is equivalent in the counting of energy to two derivatives, i.e. $\partial_\mu \partial^\mu \sim q^2 \sim m^2$.

In order to add photons to the theory at this order, one need only to change the usual derivatives into covariant derivatives, with

$$\partial_\mu U \rightarrow D_\mu U = \partial_\mu U + ie[Q, U] A_\mu, \quad (4)$$

where

$$Q = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \quad (5)$$

is the charge matrix, leading to the usual charge assignment for the mesons.

The Lagrangians to next order (four derivatives) have been catalogued by Gasser and Leutwyler [4]. These authors have also performed the one loop renormalizations of the non-anomalous sector of the theory, and our method will be similar to theirs.

In order to write out the Wess–Zumino–Witten (WZW) anomaly Lagrangian in an economical fashion, it is advantageous to introduce left and right handed currents

$$L_\mu = U^\dagger \partial_\mu U, \quad R_\mu = (\partial_\mu U) U^\dagger. \quad (6)$$

The fields can be extended into a hypothetical fifth dimension. The WZW action including photons is then [7–9]

$$S_{\text{WZW}} = \frac{iN_c}{240\pi^2} \int_{M_4} d^5x \epsilon^{ijklm} \text{Tr}(L_i L_j L_k L_l L_m) \\ - \frac{N_c e}{48\pi^2} \int d^4x A_\mu J_\mu + \frac{iN_c e^2}{24\pi^2} \int d^4x \epsilon^{\mu\nu\alpha\beta} \partial_\mu A_\nu A_\alpha T_\beta, \quad (7)$$

where

$$J^\mu = \epsilon^{\mu\nu\alpha\beta} \text{Tr}(Q L_\nu L_\alpha L_\beta + Q R_\nu R_\alpha R_\beta), \quad (8)$$

$$T_\beta = \text{Tr}(Q^2 L_\beta + Q^2 R_\beta + \tfrac{1}{2} U^\dagger Q U Q L_\beta + \tfrac{1}{2} U Q U^\dagger Q R_\beta). \quad (9)$$

The five dimensional integral has as its boundary the normal four dimensional space–time, and for this to make sense each term in the expansion of the integrand must be expressible in terms of a total derivative such that only the surface term is relevant to the integration, i.e. it must ultimately only depend on the field values in normal space–time. Explicit calculation can be used to verify this term by term. Note that this form has been corrected from the result given in ref. [8] in order to make it parity conserving [9]. Physical processes which are contained in this Lagrangian include $K^+ K^- \rightarrow \pi^+ \pi^- \pi^0$, $e^+ e^- \rightarrow \gamma^* \rightarrow 3\pi$, and $\pi^0 \rightarrow \gamma\gamma$, one from each of the three terms in ref. [7].

3. One loop expansion of the anomaly

In order to compute the first quantum corrections to the effective action, we expand the chiral matrix U about a unitary background value $\bar{U}(x)$,

$$U = \bar{U} + \delta U = \bar{U}(1 + \Delta), \quad (10)$$

with

$$\Delta = \bar{U}^{-1} \delta U. \quad (11)$$

In order to maintain unitarity, we use

$$U^\dagger = \bar{U}^\dagger - \Delta \bar{U}^\dagger + \Delta \Delta U^\dagger + \dots \quad (12)$$

More importantly, the currents have the expansion

$$L_\mu = \bar{L}_\mu + D_\mu^0 \Delta - \Delta D_\mu^0 \Delta, \\ R_\mu = \bar{R}_\mu + \bar{U} \partial_\mu \Delta \bar{U}^\dagger - \bar{U} (\partial_\mu \Delta) \Delta U^\dagger = \bar{U} \{ \bar{L}_\mu + \partial_\mu \Delta - (\partial_\mu \Delta) \Delta \} \bar{U}^\dagger, \quad (13)$$

where

$$D_\mu^0 = \partial_\mu + [\bar{L}_\mu, \quad]. \quad (14)$$

There will be a corresponding expansion of the Lagrangian in powers of Δ ,

$$\mathcal{L} = \mathcal{L}_0(\bar{U}) + \mathcal{L}_1(\bar{U}, \Delta) + \mathcal{L}_2(\bar{U}, \Delta), \quad (15)$$

where $\mathcal{L}_2$ is quadratic in Δ . The linear term, $\mathcal{L}_1$, will vanish for all matrix fields U satisfying the equations of motion. In order to obtain the one loop effective action for $\bar{U}$, one integrates $\mathcal{L}_2$ over all of the fluctuation Δ . In what follows all quantities appearing will be $\bar{U}$, $\bar{L}_\mu$ or $\bar{R}_\mu$, and we will drop the bars.

The expansion of the four dimensional terms in the action is straightforward. The five dimensional ones require some extra effort to remove the dependence on the fifth coordinate. In order to see how this may be accomplished consider a particular term in the expansion of $\mathcal{L}_{\text{WZW}}$,

$$\begin{aligned} I &= \int d^5x \epsilon^{ijklm} \text{Tr}(\Delta L_i L_j \partial_k \Delta L_l L_m) \\ &= \frac{1}{2} \int d^5x \epsilon^{ijklm} \{ \text{Tr}(\partial_i \Delta L_j L_k \Delta L_l L_m) + \text{Tr}(\Delta L_j L_k \partial_i \Delta L_l L_m) \}. \end{aligned} \quad (16)$$

We may now use the facts that

$$\partial_i L_j = -L_i L_j + U \cdot \partial_i \partial_j U, \quad (17)$$

$$\epsilon^{ijklm} \partial_i (L_j L_k) = \epsilon^{ijklm} (-L_i L_j L_k - L_j L_i L_k) = 0 \quad (18)$$

in order to write the integrand as a total derivative

$$I = \frac{1}{2} \int d^5x \epsilon^{ijklm} \partial_i \text{Tr}(\Delta L_j L_k \Delta L_l L_m), \quad (19)$$

which can be turned into a surface (i.e. four dimensional) integral

$$I = \frac{1}{2} \int d^4x \epsilon^{\mu\nu\alpha\beta} \text{Tr}(\Delta L_\mu L_\nu \Delta L_\alpha L_\beta). \quad (20)$$

With some effort, all of the terms in the expansion can be integrated similarly and turned into four-dimensional integrals. We find a second variation of the form

$$\begin{aligned} \delta^2 S_{\text{WZW}} &= \delta^2 \left(\frac{-iN_c}{240\pi^2} \int d^5x \epsilon^{ijklm} \text{Tr}(L_i L_j L_k L_l L_m) \right) \\ &= \frac{-iN_c}{48\pi^2} \int d^4x \epsilon^{\mu\nu\alpha\beta} \{ \text{Tr}(\Delta D_\mu^0 \Delta L_\nu L_\alpha L_\beta) \\ &\quad + \text{Tr}(\Delta L_\mu D_\nu^0 \Delta L_\alpha L_\beta) - \frac{1}{2} \text{Tr}(\Delta L_\mu L_\nu \Delta L_\alpha L_\beta) \} \\ &\quad + \text{variation of 4-dim. terms.} \end{aligned} \quad (21)$$

Already at this stage, one of our theoretical questions is answered. It is clear that all of the one loop counterterms will be four dimensional rather than a five dimensional construction. This in turn implies that the quantized coefficient of the WZW term cannot be modified. We will return to these issues later. Overall the variation of $\mathcal{L}_{\text{WZW}}$, including photons, can be written most compactly using the notation

$$\begin{aligned} d_\mu \Delta &= \partial_\mu \Delta + ie[Q, \Delta] A_\mu, \\ \mathcal{L}_\mu &= L_\mu + ie(\tilde{Q} - Q) A_\mu, \quad \tilde{Q} = U^\dagger Q U, \end{aligned} \quad (22)$$

as

$$\begin{aligned} \delta^2 \mathcal{L}_{\text{WZW}} &= \frac{-iN_c}{96\pi^2} \epsilon^{\mu\nu\alpha\beta} \left\{ \text{Tr} \left([\Delta d_\mu \Delta - d_\mu \Delta \Delta] \mathcal{L}_\nu \mathcal{L}_\alpha \mathcal{L}_\beta \right) \right. \\ &\quad - \text{Tr} \left(\Delta \mathcal{L}_\nu d_\mu \Delta \mathcal{L}_\alpha \mathcal{L}_\beta - d_\mu \Delta \mathcal{L}_\nu \Delta \mathcal{L}_\alpha \mathcal{L}_\beta + \Delta^2 \mathcal{L}_\mu \mathcal{L}_\nu \mathcal{L}_\alpha \mathcal{L}_\beta \right) \\ &\quad - \frac{1}{2} i F_{\alpha\beta} \text{Tr} \left([\Delta d_\mu \Delta - d_\mu \Delta \Delta] \{ (Q + \tilde{Q}), \mathcal{L}_\nu \} \right. \\ &\quad \quad \left. - (Q + \tilde{Q})(\Delta \mathcal{L}_\nu d_\mu \Delta - d_\mu \Delta \mathcal{L}_\nu \Delta) \right) \\ &\quad + \frac{1}{2} i F_{\alpha\beta} \text{Tr} \left((Q + \tilde{Q}) [\Delta^2 \mathcal{L}_\mu \mathcal{L}_\nu - \mathcal{L}_\mu \Delta^2 \mathcal{L}_\nu + \mathcal{L}_\mu \mathcal{L}_\nu \Delta^2] \right. \\ &\quad \quad \left. + Q(\Delta^2 \mathcal{L}_\mu \mathcal{L}_\nu - \Delta \mathcal{L}_\mu \Delta \mathcal{L}_\nu + \mathcal{L}_\mu \Delta \mathcal{L}_\nu \Delta - \mathcal{L}_\mu \mathcal{L}_\nu \Delta^2) \right) \\ &\quad \left. + F_{\mu\nu} F_{\alpha\beta} \text{Tr}(\Delta^2(Q^2 + \tilde{Q}^2 + \tilde{Q}Q)) \right\}. \end{aligned} \quad (23)$$

In order to evaluate loop effects, we need to add this to the variation of the lowest order Lagrangian. Using

$$\delta^2 S_{\text{TOT}} = \delta^2 \left(\int d^4x \mathcal{L}_0 + S_{\text{WZW}} \right) \equiv \int d^4x \delta^2 \mathcal{L}_{\text{TOT}},$$

we find that

$$\delta^2 \mathcal{L}_{\text{TOT}} = \frac{1}{2} F^2 \Delta^A (D^2 + \sigma)^{AB} \Delta^B, \quad (24)$$

where

$$\begin{aligned} \Delta^A &= \frac{1}{2} \text{Tr}(\lambda^A \Delta), \\ D_\mu^{AB} &= \delta^{AB} \partial_\mu + \Gamma_\mu^{AB} + S_\mu^{AB}, \\ \sigma^{AB} &= \sigma_0^{AB} + \sigma_{\text{WZW}}^{AB}. \end{aligned} \quad (25)$$

Here Γ_μ^{AB} and σ_0^{AB} follows from $\mathcal{L}_0$

$$\Gamma_\mu^{AB} = -\frac{1}{4} \text{Tr}([\lambda^A, \lambda^B](\mathcal{L}_\mu + 2iQA_\mu)), \quad (26a)$$

$$\sigma_0^{AB} = \frac{1}{8} \text{Tr}([\lambda^A, \mathcal{L}_\mu][\lambda^B, \mathcal{L}^\mu]) + \frac{1}{4} \text{Tr}(\{\lambda^A, \lambda^B\} \frac{1}{2} m^2 (U + U^-)), \quad (26b)$$

and are equivalent to the results of Gasser and Leutwyler, although written in a different basis. The cross terms involving S_μ^{AB} in D^2 , as well as σ_{WZW}^{AB} , contain the anomaly terms

$$S_\mu^{AB} = \frac{-iN_c}{96\pi^2 F^2} \epsilon^{\mu\nu\alpha\beta} \left\{ \text{Tr}([\lambda^A, \lambda^B](\mathcal{L}_\nu \mathcal{L}_\alpha \mathcal{L}_\beta - \frac{1}{2} i F_{\alpha\beta} \{Q + \tilde{Q}, \mathcal{L}_\nu\})) \right. \\ \left. - \text{Tr}([\lambda^A \mathcal{L}_\nu \lambda^B - \lambda^B \mathcal{L}_\nu \lambda^A](\mathcal{L}_\alpha \mathcal{L}_\beta - \frac{1}{2} i (Q + \tilde{Q}) F_{\alpha\beta})) \right\}. \quad (27)$$

$$\sigma_{\text{WZW}}^{AB} = \frac{-iN_c}{96\pi^2 F^2} \epsilon^{\mu\nu\alpha\beta} \\ \times \left\{ \frac{1}{4} F_{\mu\nu} F_{\alpha\beta} \text{Tr}(\{\lambda^A, \lambda^B\}(\tilde{Q}Q - Q\tilde{Q})) \right. \\ \left. + \frac{1}{4} i F_{\alpha\beta} \text{Tr}((Q - \tilde{Q})([\{\lambda^A, \lambda^B\}, \mathcal{L}_\mu \mathcal{L}_\nu] - \lambda^A \mathcal{L}_\mu \lambda^B \mathcal{L}_\nu \right. \\ \left. - \lambda^B \mathcal{L}_\mu \lambda^A \mathcal{L}_\nu + \mathcal{L}_\mu \lambda^A \mathcal{L}_\nu \lambda^B + \mathcal{L}_\mu \lambda^B \mathcal{L}_\nu \lambda^A)) \right\}. \quad (28)$$

In writing eq. (28) we have left out a piece which is proportional to the equation of motion (that is the coefficient of the first variation of the action in Δ); see appendix B. Terms of order $S_\mu S^\mu$ are higher order $O(p^8)$ and they are irrelevant for our work.

The integration over the quantum variables may now be performed easily

$$e^{iZ(U)} = \int [d\Delta^A] \exp -i \int d^4x \frac{F^2}{2} \Delta^A (D^2 + \sigma)^{AB} \Delta^B \\ = e^{(i/2) \ln \det(D^2 + \sigma)}. \quad (29)$$

In order to evaluate this we can use heat kernel techniques in order to write the determinant as

$$\ln \det D = \text{Tr}' \ln D = - \int_0^\infty \frac{dz}{z} \text{Tr} e^{-Z(D^2 + \sigma)} \\ = \frac{-i}{(4\pi)^{d/2}} \int \frac{dz}{z^{1+d/2}} \int d^4x \text{Tr} \langle x | H(z) | x \rangle, \quad (30)$$

where Tr' includes a trace over the space-time indices, while Tr (without the prime) does not. We have gone to a dimensional regularization scheme (primarily in order to be compatible with earlier work [4]) although other regularization schemes are also possible. The divergences in the heat kernel are uncovered by a power series expansion

$$\langle x|H(z)|y\rangle = \langle x|H_0|y\rangle + z\langle x|H_1|y\rangle + z^2\langle x|H_2|y\rangle + \dots \quad (31)$$

which are related by the heat equation

$$\left((x-y)^\mu D_\mu + n + 1\right)\langle x|H_{n-1}|y\rangle + D^\mu D_\mu \langle x|H_n|y\rangle + \sigma \langle x|H_n|y\rangle = 0, \quad (32)$$

and which are solved by

$$\langle x|H_0|x\rangle = 1, \quad \langle x|H_1|x\rangle = -(\sigma_0 + \sigma_{\text{WZW}}),$$

$$\langle x|H_2|x\rangle = -\frac{1}{2}I_{\mu\nu}I^{\mu\nu} + \frac{1}{2}(\sigma_0^2 + 2\sigma_0\sigma_{\text{WZW}}) - \frac{1}{6}[D^\mu, [D_\mu, \sigma_0 + \sigma_{\text{WZW}}]] - \frac{1}{6}I_{\mu\nu}S^{\mu\nu}, \quad (33)$$

$$I_{\mu\nu} = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu + [\Gamma_\mu, \Gamma_\nu], \quad (34)$$

$$S_{\mu\nu} = \partial_\mu S_\nu - \partial_\nu S_\mu + [\Gamma_\mu, S_\nu] + [S_\mu, \Gamma_\nu], \quad (35)$$

where we have dropped terms of order S^2 and σ_{WZW}^2 . The heat parameter integration can be done in order to obtain an effective potential

$$Z = \int d^4x \left\{ \frac{1}{4\pi(d-3)} \text{Tr}(\sigma_0 + \sigma_{\text{WZW}}) + \frac{1}{(4\pi)^2(d-4)} \text{Tr}\left(\frac{1}{12}I_{\mu\nu}I^{\mu\nu} + \frac{1}{6}I_{\mu\nu}S^{\mu\nu} + \frac{1}{2}\sigma_0^2 + \sigma_0\sigma_{\text{WZW}} - \frac{1}{6}[D^\mu, [D_\mu, \sigma_0 + \sigma_{\text{WZW}}]]\right) \right\}. \quad (36)$$

4. The effective action

We have verified that the nonanomalous terms in eq. (36) reproduce the results of Gasser and Leutwyler. The new physics lies in anomalous terms. The first step is verifying that

$$\text{Tr}(\sigma_{\text{WZW}}) = 0. \quad (37)$$

The explicit verification is cumbersome; this equation, however, is required by general consideration. σ_{WZW} is of order p^4 , and it is well known that there is no

four dimensional local Lagrangian involving an $\varepsilon^{\mu\nu\alpha\beta}$ tensor at order p^4 (hence the need for Witten's five dimensional construction).

The remaining anomalous effective action is obtained through a somewhat tedious calculation. For compactness of presentation let us define

$$\begin{aligned}\mathcal{H}_{\mu\nu} &= U^\dagger D_\mu D_\nu U - \frac{1}{2} \{ \mathcal{L}_\mu, \mathcal{L}_\nu \} + \frac{1}{2} i (\tilde{Q} - Q) F_{\mu\nu}, \\ \bar{\mathcal{H}}_{\mu\nu} &= \mathcal{H}_{\mu\nu} - g_{\mu\nu} \mathcal{H}^\lambda{}_\lambda, \\ A_0 &= \mathcal{L}_\lambda \mathcal{L}^\lambda - m^2 (U + U^\dagger).\end{aligned}\tag{38}$$

Note that

$$\mathcal{L}_0 = -\frac{1}{4} F^2 \text{Tr} A_0\tag{39}$$

The divergent terms in the effective anomalous Lagrangian are now given by

$$\begin{aligned}Z_A &= \frac{-iN_c}{96\pi^2 F^2} \frac{1}{16\pi^2(d-4)} \cdot \int d^4x \varepsilon^{\mu\nu\alpha\beta} \\ &\times \left\{ 2 \text{Tr} \left(\mathcal{L}_\nu \mathcal{L}_\alpha \mathcal{L}_\beta \left[\bar{\mathcal{H}}_{\lambda\mu}, \mathcal{L}^\lambda \right] \right) - 2i\delta^\lambda F_{\lambda\mu} \text{Tr} \left((Q + \tilde{Q}) \mathcal{L}_\nu \mathcal{L}_\alpha \mathcal{L}_\beta \right) \right. \\ &+ iF_{\alpha\beta} \text{Tr} \left(\bar{\mathcal{H}}_{\lambda\mu} \left[\{ Q + \tilde{Q}, \mathcal{L}_\nu \}, \mathcal{L}^\lambda \right] \right) - \frac{1}{9} iF_{\alpha\beta} \text{Tr} \left(\bar{\mathcal{H}}_{\lambda\mu} \left[\mathcal{L}^\lambda, \left[\mathcal{L}_\nu, (Q + \tilde{Q}) \right] \right] \right) \\ &- 2 \partial^\lambda F_{\lambda\mu} F_{\alpha\beta} \text{Tr} \left((Q + \tilde{Q})^2 \mathcal{L}_\nu \right) - \frac{3}{2} F_{\mu\nu} F_{\alpha\beta} \text{Tr} \left((A_0 + \frac{8}{9} m^2 (U + U^\dagger)) (\tilde{Q}Q - Q\tilde{Q}) \right) \\ &- iF_{\alpha\beta} \left[\text{Tr} \left(\mathcal{L}_\mu \mathcal{L}_\nu (Q + \tilde{Q}) \right) \text{Tr} \left(A_0 - \frac{4}{9} m^2 (U + U^\dagger) \right) \right. \\ &+ \frac{11}{6} \text{Tr} \left(\left[(Q - \tilde{Q}), (A_0 + \frac{8}{11} m^2 (U + U^\dagger)) \right] \mathcal{L}_\mu \mathcal{L}_\nu \right) - \text{Tr} \left(\mathcal{L}_\lambda \mathcal{L}_\mu \right) \\ &\left. \times \text{Tr} \left(\{ \mathcal{L}^\lambda, (Q - \tilde{Q}) \} \mathcal{L}_\nu \right) + 2 \text{Tr} \left(m^2 (U + U^\dagger) \mathcal{L}_\mu \right) \text{Tr} \left((Q - \tilde{Q}) \mathcal{L}_\nu \right) \right\}.\end{aligned}\tag{40}$$

The most important phenomenological application of the anomaly is in the decays

$\pi^0 \rightarrow \gamma\gamma$ and $\eta \rightarrow \gamma\gamma$. The corresponding amplitudes which follow from eq. (7) are

$$A(\pi^0 \rightarrow \gamma\gamma) = \frac{iN_c\alpha}{3\pi F} \epsilon^{\mu\nu\alpha\beta} \epsilon_\mu^* q_\nu \epsilon_\alpha^* q'_\beta, \quad (41a)$$

$$A(\eta^0 \rightarrow \gamma\gamma) = \frac{iN_c\alpha}{3\sqrt{3}\pi F} \epsilon^{\mu\nu\alpha\beta} \epsilon_\mu^* q_\nu \epsilon_\alpha^* q'_\beta, \quad (41b)$$

where the polarization vectors and moments refer to the outgoing photons. It is important to estimate the corrections to these expressions from the one loop graphs. The effective Lagrangian of eq. (40) implies that there are no divergent loop contributions to on-shell transitions of the form (41): The only term in eq. (40) which contributes to eq. (41) (with one neutral meson) is the one of the form

$$-2\epsilon^{\mu\nu\alpha\beta} \partial^\lambda F_{\lambda\mu} F_{\alpha\beta} \text{Tr}((Q + \tilde{Q})^2 \mathcal{L}_\nu); \quad (42)$$

but it vanishes for an on-shell photon.

In fig. 1 are shown the diagrams which give the order p^4 corrections. The K loops have been calculated in ref. [10] for on-shell photons with the result that the π^0 decay amplitude is again represented by eq. (41a) if F is replaced by the physical renormalized value F_π for π decay and for the η_8 amplitude by the physical η_8 decay constant F_8 , predicted to be (at the leading log level)

$$F_8 = F_\pi \left(1 - \frac{m_K^2}{(4\pi F_\pi)^2} \ln \frac{m_K^2}{\mu^2} + \frac{m_\pi^2}{(4\pi F_\pi)^2} \ln \frac{m_\pi^2}{\mu^2} \right), \quad (43)$$

where F_π is the physical pion decay constant.

Fig. 1. One loop renormalization to the $\pi_0\gamma\gamma$ vertex.

Eq. (41) also contributes to off-shell photon processes, such as $\pi^0 \rightarrow \gamma e^+ e^-$, or $\eta \rightarrow \gamma e^+ e^-$. Lately, it is also becoming possible to measure the inverse reactions $e^+ e^- \rightarrow \gamma \gamma \rightarrow e^+ e^- \pi^0(\eta)$. One of the two photons may be observed to be off-shell by measuring the recoil of the final state electron or positron (single aged events). This corresponds to the processes $\gamma \gamma^* \rightarrow \pi^0(\eta)$ for large negative q^2 (4-momentum of γ^*).

At order p^2 there may be modifications to the effective meson- γ - γ amplitude for both on- and off-shell photons. The one loop corrections discussed only contribute to the latter, but in the appendix we detail the $O(p^2)$ Lagrangian which can contribute to the meson- γ - γ amplitude, in particular the terms which make finite contributions to the decay of a neutral meson into two photons.

All corrections can be included in the amplitude by means of a form factor $G(m^2, q^2) = 1 + G_0(m^2) + G_1(q^2)$:

$$\begin{aligned} A(\pi^0 \rightarrow \gamma \gamma) &= \frac{iN_c \alpha}{3\pi F_\pi} G_\pi \cdot \epsilon^{\mu\nu\alpha\beta} \epsilon_\mu^* q_\nu \epsilon_\alpha^* q'_\beta \\ A(\eta_8 \rightarrow \gamma \gamma) &= \frac{iN_c \alpha}{3\sqrt{3} \pi F_{\eta_8}} G_{\eta_8} \cdot \epsilon^{\mu\nu\alpha\beta} \epsilon_\mu^* q_\nu \epsilon_\alpha^* q'_\beta, \end{aligned} \quad (44)$$

where $q'^2 = 0$. The contributions of the counterterms to G_0 and G_1 (denoted by G_1^c) are calculated in the appendix and are given by

$$\begin{aligned} \pi \left\{ \begin{aligned} G_0 &= 16B(m_u - m_d)(c_1 + c_3)/F^2 \\ &\quad + \frac{8}{3}B(4m_u - m_d)(c_2 + c_4)/F^2 - m_\pi^2(c_5 + c_6)/F^2, \\ G_1^c &= q^2(c_5 + c_6 - c_7 - c_8)/F^2; \end{aligned} \right. \\ \eta \left\{ \begin{aligned} G_0 &= 16B(m_u + m_d - 2m_s)(c_1 + c_3)/F^2 \\ &\quad + 8B(4m_u + m_d - m_d - 2m_s)(c_2 + c_4)/F^2 - m_\eta^2(c_5 + c_6)/F^2, \\ G_1^c &= q^2(c_5 + c_6 - c_7 - c_8)/F^2; \end{aligned} \right. \end{aligned} \quad (45)$$

where the c_i are arbitrary constants.

The one-loop q^2 corrections stem from fig. 1b. The inclusion of the contributions of fig. 1a as well as of renormalization graphs [10] yields

$$\begin{aligned} & \left[-\frac{5}{3}I_K - \frac{4}{3}I_\pi \right] + \tilde{I}_K(q^2) + \tilde{I}_K(q'^2) + \tilde{I}_\pi(q^2) + \tilde{I}_\pi(q'^2) \\ & + \frac{1}{6}(I_K + 2I_\pi) - \frac{1}{2}(I_K + 2I_\pi). \end{aligned} \quad (46)$$

Here the first and second contributions are from fig. 1a and 1b respectively, and the

third and fourth contributions stem from replacing in the lowest order expression of eq. (41) the external unrenormalized pion field and the decay constant F by the renormalized field and by the physical value at one loop, respectively. (Eq. (46) is written for $q'^2 \neq 0$; in the following $q'^2 = 0$). In eq. (46) we have

$$\tilde{I}_i(q^2) = \int_0^1 dx \int \frac{d^d p}{(p^2 - Q_i^2)}, \quad i = \pi, K$$

$$Q_i^2 = m_i^2 + x(x-1)q^2, \quad I_i = \tilde{I}_i(q^2 = 0), \quad (47)$$

where we used dimensional regularization, with space-time dimension d . Clearly, for $q^2 = 0$, $G = 1$ [10].

In view of this cancellation we split the integrals I_i into a q^2 independent term and a remainder; then

$$G_1 = q^2 \left(G_1^c + \sum_{i=\pi, K} \int \frac{dx dy x(x-1) d^d p}{(p^2 - R_i^2)^2} \right),$$

$$R_i^2 = m_i^2 + yx(x-1)q^2. \quad (48)$$

Evaluating the integral yields

$$G_1 = q^2 \left[G_1^c + \frac{1}{6} \frac{1}{(4\pi F)^2} \sum_{i=\pi, K} \left(\frac{2}{d-4} - \ln 4\pi - \Gamma'(1) \right) \right]$$

$$+ \frac{1}{6(4\pi F)^2} \sum_i \left((q^2 - 4m_i^2) K_i(q^2) - q^2 \ln \frac{m_i^2}{\mu^2} - \frac{q^2}{3} \right), \quad (49a)$$

$$K_i(q^2) = -2z \operatorname{ctg}^{-1} z + 2 \left\{ \begin{array}{l} z = (4m_i^2/q^2 - 1)^{1/2} \end{array} \right\} \quad 0 \leq q^2 \leq 4m^2, \quad (49b)$$

$$K_i(q^2) = z \ln \frac{z-1}{z+1} + 2 \left\{ \begin{array}{l} z = (1 - 4m_i^2/q^2)^{1/2} \end{array} \right\} \quad q^2 > 4m^2. \quad (49c)$$

The factor between the brackets in eq. (49a) is called $G_{1 \text{ ren}}$.

We note that for $q^2 \rightarrow 0$ ($G - 1$) vanishes; the functions K_i (eq. (49b)) are then zero. μ^2 is the renormalization scale and can be absorbed into β_1 , if desired. As for the η_8 , one obtains

$$G_{(\eta_8)_1}(q^2) = G_{\pi_1}(q^2). \quad (50)$$

For $q^2 > 4m_i^2$ (which is the case of the pion loop in η -decay) $G_{\eta\pi}(q^2)$ acquires an imaginary part, as expected from the existence of the $\eta \rightarrow \pi^+ \pi^- \gamma$ intermediate state.

The loop corrections have some interesting theoretical aspects, even if they turn out to be small phenomenologically. In particular, the effective “charge radius”, defined by

$$G_\pi = 1 + G_0^c + \frac{1}{6} \langle r^2 \rangle q^2 \dots \equiv 1 + q^2/\Lambda^2 \quad (51)$$

will diverge in the chiral limit, as $m_\pi \rightarrow 0$. We find

$$\begin{aligned} \frac{1}{\Lambda^2} &= \left. \frac{d}{dq^2} G_\pi(q^2) \right|_{q^2=0} \\ &= -\frac{1}{96\pi^2 F_\pi^2} \left(\ln m_\pi^2/\mu^2 + \ln m_K^2/\mu^2 + \frac{2}{3} - 6G_{1\text{ren}} \right) \end{aligned} \quad (52)$$

which has a singularity as $m_\pi \rightarrow 0$. The same behavior was found for the electromagnetic charge radii of hadrons, and it is due to the long range pion tail in this limit. However, it does not mean that the form factor as a whole is ill-behaved. Indeed, we find that if we take $m_\pi \rightarrow 0$, the form factor becomes

$$G_\pi = 1 + G_0^c - \frac{q^2}{96\pi^2 F_\pi^2} \left(\ln \left(\frac{-q^2}{\mu^2} \right) - \frac{4}{3} + \text{kaon terms} + 6G_{1\text{ren}} \right), \quad (53)$$

i.e. the behavior is of the form $q^2 \ln q^2$ rather than purely quadratic. It is in this sense only that the charge radius diverges.

The nonpolynomial aspect of the form factor also implies that there will be a different effective slope parameters in different regions of q^2 if the experimenters fit to a purely quadratic form. Unfortunately, all of these interesting non-analytic features seem to be small on the scale of the experimental slope. The status of these difficult measurements is that $1/\Lambda^2$ defined in eq. (52) is measured as

$$\begin{aligned} 1/\Lambda^2 &= (0.5 \pm 6.1) \text{ GeV}^{-2}, & \text{ref. [13]}, & \quad \pi^0 \rightarrow e^+ e^- \gamma, \\ 1/\Lambda^2 &= (5.5 \pm 1.7) \text{ GeV}^{-2}, & \text{ref. [14]}, & \quad \pi^0 \rightarrow e^+ e^- \gamma, \\ 1/\Lambda^2 &= -(2.3 \pm 5.5) \text{ GeV}^{-2}, & \text{ref. [15]}, & \quad \eta \rightarrow e^+ e^- \gamma, \\ 1/\Lambda^2 &= -(0.8 \pm 0.9) \text{ GeV}^{-2}, & \text{ref. [16]}, & \quad \eta \rightarrow e^+ e^- \gamma, \\ 1/\Lambda^2 &= (1.9 \pm 0.4) \text{ GeV}^{-2}, & \text{ref. [17]}, & \quad \eta \rightarrow \mu^+ \mu^- \gamma. \end{aligned} \quad (54)$$

In the case of the electron final states, radiative corrections are very important, and shift the value by a factor of two. The experiments have no agreement among themselves. However the non-analytic corrections enter at the level of (difference below and above threshold)

$$\delta(1/\Lambda^2) = 0.5 \text{ GeV}^{-2}, \quad (55)$$

and hence are not likely to be visible with present experimental accuracy.

In addition to the q^2 dependence, G_0 adds interesting mass dependences in the amplitudes. In order to estimate them, we assume that the coefficients of the mass dependent terms are of the same order as those of the q^2 piece. This is suggested by the counterterm coefficients c_5 and c_6 which contribute to both pieces. Using this as our guide, we would expect the corrections to the rate $\pi^0 \rightarrow \gamma\gamma$ to be of the size

$$\frac{\delta\Gamma_\pi}{\Gamma_\pi} \cong \frac{2m_\pi^2}{\Lambda^2} \sim 10^{-2}, \quad (56)$$

where Λ^2 is the slope described above. This change is below the present experimental uncertainty [18] in the rate

$$\Gamma_{\text{exp}} = (7.34 \pm 0.18 \pm 0.11) \text{ eV},$$

which compares favorably with the lowest order theoretical prediction

$$\Gamma_{\text{TH}} = \frac{1}{64\pi} \left(\frac{\alpha}{\pi} \right)^2 \left(\frac{m_\pi}{F_\pi} \right)^2 m_\pi \simeq 7.47 \text{ eV}.$$

The corrections to the decay rate of the η could be much larger, namely up to

$$\frac{\delta\Gamma'_\eta}{\Gamma_\eta} \cong \frac{2m_\eta^2}{\Lambda^2} \approx 0.5. \quad (57)$$

Despite its large size, it is more difficult to obtain a firm handle on this effect, due to $\eta_8 - \eta_0$ mixing, which strongly influences the rate of $\eta \rightarrow \gamma\gamma$. Using the notation of eq. (45) and adding $\eta_0\eta_8$ mixing, one extracts from the data [10]

$$\begin{aligned} \left(\frac{F_\pi \cos \theta (1 + G_0(\eta_8))}{F_8(1 + G_0(\pi))} - \frac{\sqrt{8} F_\pi \sin \theta}{F_0(1 + G_0(\pi))} \right)^2 &= 3.4 \pm 0.3, \\ \left(\frac{F_\pi \sin \theta (1 + G_0(\eta_8))}{F_8(1 + G_0(\pi))\sqrt{8}} + \frac{F_\pi \cos \theta}{F_0(1 + G_0(\pi))} \right)^2 &= 0.60 \pm 0.05, \end{aligned} \quad (58)$$

where F_0 is defined through

$$A(\eta_0 \rightarrow \gamma(q) + \gamma'(q')) = \sqrt{\frac{8}{3}} \frac{iN_c\alpha}{3F_0\pi} \epsilon^{\mu\nu\alpha\beta} \epsilon_\mu \cdot \epsilon'_\alpha q_\nu q'_\beta.$$

We have not included a possible term $G_0(\eta_0)$ since it can be reabsorbed into the free constant F_0 .

The data is well understood by η - η' mixing alone, with $\theta = -20^\circ$, $F_0 \simeq F_\pi$. This mixing angle is quite consistent with other determinations [19]. The mass dependent correction then would appear mainly in a shift in the mixing angle. A range

$$G_0(m_\eta^2) = \pm m_\eta^2/\Lambda^2 \simeq \pm 0.25$$

implies an uncertainty in θ of about 5° , consistent with the allowed range. To conclude, none of the mass dependent terms are at present required by experiment; however their existence would make a further improvement of the experimental data very desirable.

5. Conclusion

We have calculated the one loop modifications to the anomalous sector of the theory. This procedure preserves the quantization of the WZW anomaly Lagrangian in terms of the number of colors. In usual perturbation theory, this is a manifestation of the Adler-Bardeen theorem [11], which states that the anomaly is not renormalized in higher order. It is also an indication that the anomaly is a unique object in the chiral Lagrangian framework. Not all Lagrangians involving an $\epsilon^{\mu\nu\alpha\beta}$ tensor are "anomalous" in any way. Indeed the E^6 Lagrangians (sixth power in energy) which we have found are perfectly standard terms in the energy expansion. If one were to write out the full chiral Lagrangian without any knowledge of the anomaly, one would have included these terms at order E^6 , but would not have found any local Lagrangian with an $\epsilon^{\mu\nu\alpha\beta}$ tensor at order E^4 . The Sutherland-Veltman theorem [12] which states that the $\pi^0 \rightarrow \gamma\gamma$ amplitude vanishes as $m_\pi^2 \rightarrow 0$, would follow immediately from this construction (as kinematically any E^6 $\pi^0 \rightarrow \gamma\gamma$ amplitude is required to have an overall factor of m_π^2 for dimensional reasons). Thus these E^6 Lagrangians exhibit the expected, non-anomalous behavior. The WZW anomaly construction is the only way to violate the Sutherland-Veltman theorems by producing an E^4 amplitude. Chiral perturbation theory preserves this distinction by not modifying the WZW actions in any way.

It is, however, interesting that the loop expansion generates the E^6 Lagrangians starting from the WZW action. All are of the local four dimensional Lagrangian form; no five dimensional constructions are present. This is related to the inability to form five dimensional forms which are sensitive to only the four dimensional

boundary and are independent of the fifth coordinate. There is likely an elegant topological proof of this fact.

Phenomenologically, the main manifestations of the anomaly in the physical world is in the processes $\pi^0 \rightarrow \gamma\gamma$, $\eta^0 \rightarrow \gamma\gamma$, $\pi^0, \eta \rightarrow \gamma e^+ e^-$, $\gamma^* \rightarrow \pi^+ \pi^- \pi^0$, and in processes related to these by CVC such as $K \rightarrow \pi\pi e\nu$. We have extended the work of ref. [10] to show that although the $\pi^0 \rightarrow \gamma\gamma$ and $\eta \rightarrow \gamma\gamma$ rates have no divergences at order E^6 , the off-shell processes $\eta, \pi^0 \rightarrow \gamma\gamma^* \rightarrow \gamma e^+ e^-$ do have such divergences. This requires the introduction of a counterterm at order E^6 , of the form found by our general loop calculation. The interesting non-analytic structure of the renormalized one loop form factor is beyond the reach of present experiments. Therefore, the only modification to the physics of $\pi^0, \eta \rightarrow \gamma\gamma$ and $\pi^0, \eta \rightarrow \gamma e^+ e^-$, $\gamma\mu^+ \mu^-$ which is required by one loop correction is to use the physical values of F_π and F_8 in the two photon amplitudes. This puts the standard phenomenology of the anomaly on firmer grounds.

This work was partially supported by the Swiss and by the U.S. National Science Foundation. J.F.D. thanks the Eidg. Techn. Hochschule for hospitality during the course of this work. Related work has been done by H. Bijmans, F. Cornet and A. Bramon [20]. We thank them for discussions on the material in the paper and for pointing out the importance of the equation of motion, and A. Pich for useful comments.

Appendix A

ORDER p^6 COUNTERTERMS TO THE AMPLITUDE MESON- γ - γ

The possible terms of order p^6 which contribute to the transition meson- γ - γ are the following:

$$\begin{aligned} \mathcal{L}_{M\gamma\gamma}^c = \frac{N_c \alpha}{4\pi F^2} \bigg\{ & \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \left[(Um^2 - m^2 U^\dagger)(Q^2) c_1 + c_2 (Um^2 Q^2 - m^2 U^\dagger Q^2) \right. \\ & + c_3 (m^2 U - U^\dagger m^2)(UQU^\dagger Q) + c_4 (Um^2 UQU^\dagger Q - U^\dagger m^2 UQUQ) \big] \\ & + \epsilon^{\mu\nu\alpha\beta} \left[c_5 \partial_\lambda F_{\mu\nu} F_{\alpha\lambda} (Q^2 U \mathcal{D}_\beta U^\dagger - Q^2 U^\dagger \mathcal{D}_\beta U) \right. \\ & + c_6 \partial_\lambda F_{\mu\nu} F_{\alpha\lambda} (QUQ \mathcal{D}_\beta U^\dagger - QU^\dagger Q \mathcal{D}_\beta U) \\ & + c_7 \partial_\lambda F_{\lambda\mu} F_{\alpha\beta} (Q^2 U^\dagger \mathcal{D}_\mu U - Q^2 U \mathcal{D}_\mu U^\dagger) \\ & \left. + c_8 \partial_\lambda F_{\lambda\mu} F_{\alpha\beta} (QU^+ Q \mathcal{D}_\mu U - QUQ \mathcal{D}_\mu U^+) \right] \bigg\}, \quad (\text{A.1}) \end{aligned}$$

where the parentheses (...) stands for a trace.

These terms do not include all terms with two photons, the restriction to one meson has considerably reduced the number of terms. The notation is slightly different from that in eq. (3); we have set

$$m^2 = 2B \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}, \quad (\text{A.2})$$

where B is a proportionality constant [4]. In the isospin limit $m_u = m_d$, this definition coincides with the one given in eq. (3). The factor $\text{Tr} Q^2$ in the first term could be omitted, we leave it in order to make the two photon coupling explicit.

(The guiding principle for constructing eq. (A.1) was, besides chiral symmetry, parity and the equation of motion, the U(1) gauge invariance, which allows the photon to enter only as $F_{\mu\nu}$ or in a covariant derivative. But if a photon originates from the latter, the meson must come from the expansion of U in the commutator $[Q, U]$ of the covariant derivative. But this vanishes for neutral mesons. Therefore, the relevant two photon terms must contain two $F_{\mu\nu}$).

Eq. (A.1) contributes as follows to the amplitude $M \rightarrow \gamma(q) + \gamma(q')$:

$$A(\pi \rightarrow \gamma\gamma) = \frac{i\alpha N_c}{3\pi F^3} \left\{ 16B(m_u - m_d)(c_1 + c_3) + {}^8_3B(4m_u - m_d)(c_2 + c_4) - m_\pi^2(c_5 + c_6) + (q^2 + q'^2)(c_5 + c_6 - c_7 - c_8) \right\}, \quad (\text{A.3})$$

$$A(\eta_8 \rightarrow \gamma\gamma) = \frac{i\alpha N_c}{3\sqrt{3}\pi F^3} \left\{ 16B(m_u + m_d - 2m_s)(c_1 + c_3) + {}^8_3B(4m_u + m_d - 2m_s)(c_2 + c_4) - m_\eta^2(c_5 + c_6) + (q^2 + q'^2)(c_5 + c_6 - c_7 - c_8) \right\}. \quad (\text{A.4})$$

Appendix B

EQUATION OF MOTION

For completeness we give the variation of S which vanishes:

$$\begin{aligned} \delta S = \int d^4x \left\{ \frac{F^2}{2} \text{Tr} \left(\left(\Delta - \frac{\Delta^2}{2} \right) \left(\partial_\mu \mathcal{L}^\mu + \frac{m^2 U - U^\dagger m^2}{2} \right. \right. \right. \\ \left. \left. - i\epsilon^{\mu\nu\alpha\beta} \frac{N_c}{24\pi^2 F^2} (\mathcal{L}_\mu \mathcal{L}_\nu \mathcal{L}_\alpha \mathcal{L}_\beta \right. \right. \\ \left. \left. + ie \partial_\mu A_\nu (\mathcal{L}_\alpha \mathcal{L}_\beta (Q + \tilde{Q}) + \mathcal{L}_\alpha (Q + \tilde{Q}) \mathcal{L}_\beta + (Q + \tilde{Q}) \mathcal{L}_\alpha \mathcal{L}_\beta) \right. \right. \\ \left. \left. - 2e^2 \partial_\mu A_\nu \partial_\alpha A_\beta (Q^2 + \tilde{Q}^2 + \frac{1}{2}(\tilde{Q}Q + Q\tilde{Q})) \right) \right\}. \quad (\text{B.1}) \end{aligned}$$

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