Spectrum of QCD and chiral lagrangians of the strong and weak interaction

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Chiral Lagrangians contain in their coefficients information concerning the underlying fundamental theory. We show that to a large measure this structure is determined by a duality with the low-mass spectrum of the theory. This is tested in some model theories and then applied to QCD. In the real world, only the vector mesons are light enough to pass our criteria for being low mass, and in practice they do seem to dominate the structure of the phenomenological Lagrangians. This raises the possibility of a fusion between vector-dominance ideas and rigorous chiral Lagrangian methods.

I. INTRODUCTION

The theory of effective chiral Lagrangians\(^1-3\) provides a compact and elegant method for dealing with the interactions of the Goldstone modes of a theory. These effective theories can describe all of the couplings in terms of a relatively small number of parameters. However, in the case of a complex theory such as QCD these parameters are not yet capable of being predicted directly from the theory, and instead are obtained phenomenologically. This is unfortunate, as the parameters contain a good deal of information on the structure of the theory. In this paper, we will explore the idea that the form of the effective Lagrangians is constrained by the low-lying resonance (non-Goldstone) spectrum. The result will be that in most cases the parameters of the chiral Lagrangian, which in principle can be determined from very precise, very-low-energy data, will at somewhat higher energy contain information relating to the spectrum. For lack of a better name, we will refer to this idea as chiral duality, and will apply it to QCD in order to obtain constraints on the predictions of the chiral Lagrangian.

The effective Lagrangians are organized in an expansion in terms of the energy. The lowest-order Lagrangian in chiral SU(2) is

\[
L_0 = \frac{F_\pi^2}{4} \text{Tr}(\partial_\mu M \partial^\mu M^\dagger) + \frac{m_\pi^2 F_\pi^2}{4} \text{Tr}(M + M^\dagger),
\]

where

\[
M = \exp \left[ \frac{i \pi^i}{F_\pi} \right],
\]

with \(\pi^i, i = 1, 2, 3\), being the pion fields. The parameters here are only \(F_\pi\) and \(m_\pi\), where \(F_\pi\) is the pion decay constant normalized so that the experimental value is \(F_\pi = 94\) MeV. At this order, there is not much information in the form of the Lagrangian. This is because all theories with a slightly broken chiral SU(2) symmetry must have this same form. For example, the linear \(\sigma\) model, the nonlinear \(\sigma\) model, and QCD would all reduce to this form at lowest order, despite being very different theories. It is at the next order in the energy expansion (called order \(E^4\) throughout the paper) that the theories may be distinguished. The generic structure at order \(E^4\) will be reviewed in Sec. II, and will contain two hadronic parameters (in the massless limit) undetermined by any symmetry consideration. These will be the subjects of this paper.

The idea in its simplest form follows from the dual role of scattering processes as being a probe of the particle spectrum while also being governed by the underlying chiral Lagrangian. Consider for definiteness \(\pi^+\pi^-\) scattering in the \(I=1, J=1\) channel. We know that if this experiment is performed we will see the \(\rho\) resonance at \(E = 770\) MeV, with the result displayed as idealized data in Fig. 1. The lowest-order chiral Lagrangian "knows" nothing of this physics. It predicts scattering as shown by the solid line in Fig. 1, given entirely in terms of \(F_\pi\) and \(m_\pi\). However, the chiral Lagrangians at order \(E^4\) govern the deviations from the lowest-order prediction and the information is contained in the two new parameters. These

![FIG. 1. Scattering of \(\pi^+\pi^-\) in a channel where a resonance occurs. The figure is similar to the \(I=1, J=1\) channel containing the \(\rho(770)\). The low-energy region I is not very sensitive to order \(E^4\) corrections because \(E\) is small. In the high-energy region III, all powers in \(E\) are required and the perturbative expansion in \(E\) has broken down. It is region II which is most sensitive to the \(E^4\) corrections in a chiral Lagrangian. The solid line is the lowest-order prediction of chiral symmetry, while the dashed curve is one fit to include the order \(E^4\) corrections.](image-url)
could be fit at low energies and, in our idealized situation, would give a total scattering amplitude of the form of the dashed line. Clearly these could never by themselves reproduce the full $\rho$ resonance shape. However, their sign and magnitude are constrained by the need to match the cross section rise for the resonance. When this idea is expanded to include other channels, it seems promising that the low-lying spectrum of the theory can provide constraints and even a determination of the parameters of the chiral Lagrangian.

The main thrust of our interest is theoretical. The information about the spectrum of a theory is often easier to obtain than the corresponding knowledge about the chiral Lagrangians. For example, in QCD, both rigorous (in principle) lattice Monte Carlo techniques and the quark model can address the resonance spectrum, but no comparable method exists for the chiral Lagrangians. However, if this chiral duality holds, then the spectrum information can be used to predict or constrain the chiral parameters of that theory. In the case of QCD, the interest is primarily to learn what physics input is required to solve the low-energy limit of the theory. In other contexts ideas similar to chiral duality could be more useful in other ways.

There are many similarities to older work on pole models and vector dominance and to aspects of investigation into Skyrmions. Closest in intent is the chiral Lagrangian work of Gasser and Leutwyler. Since this work has been completed we have also learned that related ideas are under investigation by de Rafael, Ecker, Gasser, and Pich.

Our prime focus is on general chiral theories. In Sec. II we describe in more detail the proposed connection of chiral Lagrangians with the spectrum. This is then explored in a pair of solvable but nontrivial theories in Sec. III. Section IV is a rather technical interlude where the magnitude of various resonant intermediate states are calculated in order to decide if an effect is "large." This is then applied to the real world and QCD in Sec. V, leading to a result reminiscent of vector dominance. A summary, Sec. VI, recapitulates the conclusion and gives some suggestions for applications within QCD and in extension of the standard model (technicolor, etc.). In an appendix we comment on the information known about $E^4$ terms in the weak interaction.

II. CHIRAL DUALITY

Consider a chiral SU(2) theory in the massless limit. The effective action can be expressed in an expansion in the number of derivatives acting on a nonlinear chiral matrix $M$:

$$L = L_2 + L_4 + \cdots,$$

$$L_2 = \frac{F_\pi^2}{2} \text{Tr}(\partial_\mu M \partial^\mu M^\dagger),$$

$$L_4 = \frac{\alpha_1}{4} [\text{Tr}(\partial_\mu M \partial^\mu M^\dagger)]^2 + \frac{\alpha_2}{4} \text{Tr}(\partial_\mu M \partial^\mu M')\text{Tr}(\partial_\nu M \partial^\nu M'),$$

where $M$ contains the pion field

$$M = \exp \left[ i \frac{\pi^i}{F_\pi} \right],$$

with $i = 1, 2, 3$. If one algebraically expands the lowest order effective action $L_2$, and takes matrix elements, one obtains relations among amplitudes which are the same as obtained by PCAC (partial conservation of axial-vector-current) current-algebra soft-pion theorems, as must be the case since the predictions follow only from the chiral symmetry. The effective Lagrangian allows one to address systematically corrections to these lowest-order relations. Such corrections are given by $L_4$ and by loop diagrams. The reader is referred to Ref. 1 for an exposition of this method and to Refs. 2 and 7 for its implementations.

The parameters $\alpha_1$ and $\alpha_2$ will govern the deviation of $\pi-\pi$ scattering from the lowest-order predictions of Weinberg. Elsewhere we have shown that, for the scattering in a channel with isospin $I$ and angular momentum $J$, the amplitude $T_J^I$ has the form (at the tree level)

$$T_0^I = \frac{1}{16\pi F_\pi^2} \left[ s + (11\alpha_1 + 7\alpha_2) \frac{s^2}{3F_\pi^2} \right] + O(m_\pi^2),$$

$$T_1^I = \frac{1}{96\pi F_\pi^2} \left[ s + (\alpha_2 - 2\alpha_1) \frac{s^2}{F_\pi^2} \right],$$

$$T_2^I = \frac{1}{32\pi F_\pi^2} \left[ s - \frac{s^2}{3} (\alpha_1 + 2\alpha_2) \frac{s^2}{F_\pi^2} \right],$$

where $s$ is the square of the center-of-mass energy. In particular, for $\pi-\pi$ scattering to reveal the presence of a low-lying resonance, the parameters $\alpha_i$ must be such that this deviation reproduces the low-energy effect, or "tail," of the resonance. The notion of chiral duality is that the parameters in the chiral Lagrangians should be such that they are compatible with the low-lying spectrum of the theory. The lower in mass the resonance is, the stronger its effect will be on $\alpha_i$.

When is the resonance the dominant effect? An appreciation of the scales of chiral symmetry is useful in answering this question. We adopt an argument originally given by Georgi. Loop diagrams will naturally generate a scale dependence to the renormalized parameters, of the form

$$\alpha_i(\mu) = \alpha_i(\mu_0) + b_i \ln \mu_0^2/\mu^2,$$

where $\mu$ and $\mu_0$ are two different scales which can be used to define the theory. If one tries to choose $\alpha_i(\mu_0)$ to be much smaller than $b_i$, this choice is not natural, as a different choice of scale would yield $\alpha_i(\mu) - b_i$. However, $\alpha_i \sim b_i$ is natural and is an indication of the level at which quantum effects contribute to $\alpha_i$. These scale depen-
dences have been calculated and in our normalization lead to the following estimates of the size:

\[
\alpha_1 - b_1 = \frac{1}{96\pi^2} \approx 0.001 , \\
\alpha_2 - b_2 = \frac{1}{48\pi^2} \approx 0.002 ,
\]

i.e., a few parts in a thousand. A contribution to the coefficients will be considered "large" if it is much above these values. We will see in Sec. IV that the resonance contribution can be of order

\[
-\alpha_1 = \alpha_2 = 48\pi \frac{F^4}{m^2} \approx 0.0084
\]

using the real \( p \) as an example. The factor of \( \Gamma/m^2 \) is common to all resonances, and it is clear that a resonance with a low enough mass will be dominant, and that the \( \rho \) may well satisfy this in practice.

III. TRIAL MODELS

It is clear if one introduces a single resonance whose only coupling is to pions and then integrates out the field, as we do in the next section, that the resulting parameters \( \alpha_i \) will be compatible with the presence of the resonance in \( \pi-\pi \) scattering. It will work because it is constructed to do so. On the other hand, one cannot test the idea of this duality in a complex theory such as QCD, as one cannot solve the theory. What we can do is to consider a couple of models with a resonance explicitly in the theory, but which also have nontrivial couplings and/or extra fields. In such a case, is the resonance still strongly visible or do the extra fields or interactions produce a major modifications? We will address this in (1) a gauged version of the chiral model with vector and axial-vector fields with Yang-Mills-type coupling among themselves and (2) the linear \( \sigma \) model treated in the perturbative regime. The answer appears to be that it is the resonance which is the dominant effect on the parameters.

A. The gauged chiral model

In this model, vector and axial-vector mesons are added to a chiral Lagrangian by treating them as if they were gauge particles of a local chiral symmetry. However, a mass term must also be added, so that the result is not a true gauge theory. In particular, one introduces a covariant derivative

\[
\partial \bar{M} \rightarrow D \mu M = \partial \bar{M} - i g L_{\mu 0} M + ig M \bar{R}_\mu ,
\]

\[
L_\mu = \frac{i}{2}(V_\mu + A_\mu) = \frac{1}{2} \sigma^\mu (V^\mu + A^\mu) ,
\]

\[
R_\mu = \frac{i}{2}(V_\mu - A_\mu) = \frac{1}{2} \sigma^\mu (V^\mu - A^\mu) ,
\]

where \( V_\mu \) and \( A_\mu \) are SU(2) triplets which would be identified with the \( \rho \) and \( A1 \) mesons in the real world

\[\rho(770) \text{ and } A(1270) \text{ in modern notation}. \]

The Lagrangian then has the form

\[
L_{GCM} = \frac{F^2}{4} \text{Tr}(D_\mu M D^\mu M^\dagger) - \frac{i}{4} \text{Tr}(L_{\mu 0} L^{\mu 0} + R_{\mu 0} R^{\mu 0})
\]

\[
+ M_0^2 \text{Tr}(L_\mu L^{\dagger \mu} + R_\mu R^{\dagger \mu}) + B \text{Tr}(L_\mu M R^{\dagger} R^{\mu}) ,
\]

\[
L_{\mu 0} = \partial_\mu L_v - \partial_v L_\mu - i g [L_{\mu 1} L_v] ,
\]

\[
R_{\mu 0} = \partial_\mu R_v - \partial_v R_\mu - i g [R_{\mu 1} R_v] .
\]

The model then contains two sets of spin-1 particles \( (L_\mu, R_\mu \text{ or } V_\mu, A_\mu) \) with triple and quartic self-couplings, plus couplings to the chiral field \( M \). The last term, with coefficient \( B \), is not in principle essential, but allows the model to be reasonably compatible with phenomenology.

By its construction this model must have a pole in the \( I=1, J=1 \) \( \pi-\pi \) scattering channel. The axial-vector particle is pushed to higher mass by mixing with the pion. Our prime question is whether the \( O(E^2) \) coefficients of the Lagrangian clearly reflect the effect of the \( \rho \), or whether the extra axial field and Yang-Mills-type self-interactions obscure this feature at low energy. In order to do this we integrate out the spin-1 fields and obtain the resulting low-energy Lagrangian. Formally,

\[
\exp \left[ i \int d^4x L_{\text{eff}}(M) \right] = \int [dL_\mu][dR_\mu] \exp \left[ i \int d^4x L_{\text{eff}}(L_\mu, R_\mu, M) \right] .
\]

The quadratic portion may, of course, be done exactly. Because the coupling \( g \) need not be small, a usual perturbation series is not applicable. However, we are after an expansion in the number of derivatives, and working to order \( E^4 \) truncates the tree diagram at those of Fig. 2. We do not include the loop effect for \( L_\mu \) and \( R_\mu \), partially because we view this only as an effective theory and partially because of the problems of massive gauge particles without the Higgs mechanism. The result, after a significant amount of algebra, is

![Diagram](attachment:image.png)

FIG. 2. Diagrams which generate higher-order corrections to chiral Lagrangians. The × represents vertices containing the chiral matrix \( M \), while the lines are heavy meson propagators.
Let us define the effective Lagrangian by integrating out the scalar field

\[
\exp \left\{ i \int d^4x L_{\text{eff}}(M) \right\} = \int [ds] \exp \left\{ i \int d^4x L(s, M) \right\} .
\]

This can be organized in a perturbation expansion. The quartic coupling \( \lambda \) must be taken small enough for perturbation theory to apply. The diagrams which contribute are also given in Fig. 2. The \( x \) in the graphs is of order \( E^2 \), so that in order to obtain the Lagrangian at order \( E^4 \) only two \( x \) factors need to be considered. At the tree level this gives

\[
L(u) = \frac{\mu^2}{4} \text{Tr}(\partial_\mu M \partial^\mu M^\dagger) + \frac{v^2}{4} \int d^4y \text{Tr}(\partial_\mu M \partial^\mu M^\dagger)_x D(x - y)
\]

\[
\times \text{Tr}(\partial_\mu M \partial^\mu M^\dagger)_y + \cdots
\]

\[
= \frac{\mu^2}{4} \text{Tr}(\partial_\mu M \partial^\mu M^\dagger)
\]

\[
+ \frac{v^2}{4m^2} \left[ \text{Tr}(\partial_\mu M \partial^\mu M^\dagger) \right]^2 + \cdots ,
\]

where in the last step we have Taylor expanded

\[
\text{Tr}(\partial_\mu M \partial^\mu M^\dagger)_x \quad \text{Tr}(\partial_\mu M \partial^\mu M^\dagger)_y + (y - x)^\mu \partial_\mu \left[ \text{Tr}(\partial_\mu M \partial^\mu M^\dagger) \right]_x + \cdots ,
\]

dropped higher-order terms, and integrated over the \( s \) propagator. Written in terms of the width of the \( s \),

\[
\Gamma(s \rightarrow \pi\pi) = \frac{3m^3}{64\pi v^2} ,
\]

this yields

\[
\alpha_1 = \frac{16\pi\Gamma_s}{3} \frac{v^4}{m^5} = \frac{1}{8\lambda} , \quad \alpha_2 = 0 ,
\]

exactly the same result as our non-self-interacting scalar of the next sections.

It is interesting that the self-interactions of the \( s \) modify the tree-level Lagrangian only starting at order \( E^6 \). Moreover it is clear that pion loops will not dominate for \( \lambda \) small enough. This is because the only scale for pion interactions is \( v = \sqrt{\mu^2 / \lambda} \), and pion loops cannot involve \( \mu \) or \( \lambda \) separately. Therefore the dimensionless constants \( \alpha_1 \) and \( \alpha_2 \) can only be a pure number (estimated of order \( 1/48\pi^2 \) in Sec. II) and cannot have a factor of \( 1/\lambda \) as given in Eq. (22). In perturbation theory the \( 1/\lambda \) factor is larger than the pure number. Loops of \( s \) particles appear in a way as to renormalize \( \lambda \) and \( v \). The full one-loop renormalization is only modified by a pure number, resulting in
\[\alpha_1(\mu) = \frac{1}{8\lambda} + \frac{1}{96\pi^2} \left[ \frac{m^2}{\mu^2} - \frac{35}{6} \right],\]
\[\alpha_2(\mu) = \frac{1}{48\pi^2} \left[ \frac{m^2}{\mu^2} - 11 \right].\]

(23)

For a scale \( \mu = m_s \), the resonance contribution will be dominant for \( \lambda < 40 \), i.e., throughout the perturbative regime.

These trial models illustrate how different theories lead to distinctive predictions for the \( E^4 \) chiral Lagrangian. In each case, the lowest-energy resonances are the prime determinate in the chiral coefficients, with results essentially the same as found for simple fields with no self-interaction.

IV. RESONANCES AND THEIR CHIRAL COUPLINGS

In this section we consider particles whose quantum numbers match those of \( \pi-\pi \) scattering, so that their spectrums could in principle be revealed in these reactions. We couple these to chiral fields and obtain their low-energy effect on chiral Lagrangians. This will allow us to estimate how low in mass a resonance must lie in order to strongly influence the spectrum.

The five channels of relevant for \( \pi-\pi \) scattering below 1 GeV are \( (I=0,J=0) \), \( (I=2,J=0) \), \( (I=1,J=1) \), \( (I=0,J=2) \), and \( (I=2,J=2) \). We will use particles with these quantum numbers, named, respectively, \( \sigma, \phi, \rho, f_{\mu\nu} \), and \( h^{ij}_{\mu\nu} \), where \( i,j=1,2,3 \) are isospin indices. The chiral couplings of \( \sigma \) and \( f_{\mu\nu} \) are easy to write down, and we will use

\[L_\sigma = g_\sigma \sigma \tau(\partial_{\mu} M \partial^\mu M^\dagger),\]
\[L_f = g_f f_{\mu\nu} \tau(\partial_{\mu} M \partial^\nu M^\dagger),\]

(24)

where \( g_\sigma \) and \( g_f \) are some coupling strengths which will later be eliminated in favor of the particle widths. For isotensor fields, the interaction is a bit more complicated. \(^9\) The field \( M \) transforms under \( SU(2)_L \times SU(2)_R \) rotations as

\[M \rightarrow M' = LMR^\dagger.\]

(25)

However the matter fields \( \phi^{ij} \) and \( h^{ij}_{\mu\nu} \) should not have chiral transformation properties, but should transform vectorially. The method for constructing this involves the fields \( \xi \) defined as

\[\xi^\dagger = M.\]

(26)

These may have the transformation properties of

\[\xi \rightarrow U\xi U^\dagger = U\xi R^\dagger,\]

(27)

where \( U \) is nonlinear matrix, \( U = U(L,R,\pi) \) in vectorial \( SU(2) \). The isotensor fields would then transform as

\[\phi^{ij} \rightarrow U^{ij}(U)\phi^{ij}(U)\phi^{ij},\]
\[D^{ij}(U) = \frac{1}{2}\tau(\partial^i U \partial^j U^\dagger).\]

(28)

This allows the chirally invariant couplings

\[L_\sigma = g_\sigma \phi^{ij}\tau(\partial^i M \partial^j M^\dagger),\]
\[L_f = g_f h^{ij}_{\mu\nu} \tau(\partial^i M \partial^j M^\dagger).\]

(29)

The case of the \( \rho \) has already been treated in this framework by Gasser and Leutwyler. \(^2\)

At low energies the effect of the heavy mesons on the low-energy physics can be found by “integrating out” the heavy fields. The result is equivalent to one-meson exchange with the propagators evaluated at \( q^2 = 0 \).

Specifically we obtain a Lagrangian of the form of Eq. (2), with the five cases contributing:

\[\sigma: \quad \alpha_1 = \frac{16\pi}{3} \frac{\Gamma_{\rho_F^\sigma}}{m_\sigma}, \quad \alpha_2 = 0;\]
\[\phi: \quad \alpha_1 = -\frac{1}{4} \alpha_2 = -4\pi \frac{\Gamma_{\rho_F^\phi}}{m_\phi};\]
\[\rho: \quad \alpha_1 = \alpha_2 = -48\pi \frac{\Gamma_{\rho_F^\rho}}{m_\rho};\]
\[f: \quad \alpha_1 = -\frac{1}{3} \alpha_2 = -\frac{160\pi}{3} \frac{\Gamma_{\rho_F^f}}{m_f};\]
\[h: \quad \alpha_1 = -7\alpha_2 = 280\pi \frac{\Gamma_{\rho_F^h}}{m_h}.\]

Here we have eliminated the coupling constants in favor of the particle's width. Note that all are proportional to the same combination of widths and masses, up to overall numerical factors.

For what masses will these contributions be most important? If one is dealing with a general chiral theory, rather than QCD, which we feel is the real theory of pions, this answer will depend on the scale of \( F_\pi \). However, let us make the estimate using a width which has the same relation to \( F_\pi \) as does the width of the \( \rho \), i.e., \( \Gamma/F_\pi \approx 1.6 \). If we require that the resonance contributions is a few times our estimate of the “natural” size of \( \alpha_i \), the resonance must generate \( \alpha_i \geq 0.004 \); this says that the resonance will dominate for masses

\[\sigma: \quad m \leq 6F_\pi;\]
\[\phi: \quad m \leq 7F_\pi;\]
\[\rho: \quad m \leq 9F_\pi;\]
\[f: \quad m \leq 12F_\pi;\]
\[h: \quad m \leq 13F_\pi.\]

(31)

(The scalar \( s \) field in Sec. III had a higher bound because it's width was larger.) These make more definite what we mean by “low-lying” resonance.

V. PION LOOPS

When pion loops are considered, the coefficients \( \alpha_i \) need to be renormalized and become functions of the scale used to define the theory. In what sense then can the renormalized values be predicted in terms of the spectrum? In order to answer this, we turn to the operational
procedure for phenomenologically determining the parameters. The data and the chiral model are compared over range of energy and the deviations from the lowest-order result tell us the best phenomenological parameters. This is depicted schematically, but not unrealistically, in Fig. 1. At low energies, the effect of $s^2$ terms are too small to be visible, while at $\sqrt{s}$ equal to the resonance mass $m_R$ the expansion in the energy has become nonlinear and yet higher powers of $s$ are required. The greatest strength in determining the parameters will occur at $\sqrt{s} \approx m_R/2$. Therefore, physically we will want to adjust the parameters in order to match the tail of the resonance at this energy.

Let us demonstrate this by using the dimensionally regularized form of the amplitude in the chiral limit.\textsuperscript{2} Here

$$M(\pi^a \pi^b \to \pi^c \pi^d) = A(s, t, u) \delta_{ap} \delta_{bD} + A(t, s, u) \delta_{at} \delta_{bs}$$

$$+ A(u, t, s) \delta_{au} \delta_{bt},$$

(32)

where $s, t, u$ are the Mandelstam variables

$$s = (p_a + p_b)^2,$$

$$t = (p_a - p_t)^2,$$

$$u = (p_a - p_u)^2.$$\textsuperscript{23}

The amplitude $A(s, t, u)$ can be decomposed into a lowest-order result, loop corrections, and terms due to the chiral parameters in $L_4$:

$$A(s, t, u) = \frac{s}{F_\pi^2} + B(s, t, u) + C(s, t, u),$$

$$B(s, t, u) = \frac{1}{6F_\pi^2} \left[ 3s^2 I(s) + t(u - t) I(u) + u(t - u) I(t) ight. - \frac{1}{96\pi^2} \left[ 21s^2 + 5(t - u)^2 \right],$$

$$C(s, t, u) = \frac{1}{2F_\pi^2} \left[ 4s^2 \alpha_1 + \alpha_2 [s^2 + (t - u)^2] \right],$$

$$I(a) = -\frac{1}{16a^2} \ln \left( \frac{-a}{\mu^2} - 2 \right).$$

(34)

If it were not for the logarithmic factors in $I$, the effect of loops could all be absorbed into a new set of coefficients $\alpha_i$. However, in the moderate energy region $\sqrt{s} = m_R/2$, the logarithmic factors do not vary too rapidly, and we can approximate the loop function by setting $\sqrt{s}$ equal to $m_R/2$ inside the logarithm. If we do this then the resulting amplitude will again be a polynomial and the effect of the loops can be absorbed into a shift of the chiral parameters. In particular, in this approximation

$$A(s, t, u) = \frac{s}{F_\pi^2} + C_{\text{off}}(s, t, u),$$

(35)

where

$$C_{\text{eff}} = \frac{1}{2F_\pi^2} \left[ 4s^2 \left( \alpha_1(\mu) + \frac{1}{144\pi^2} - \frac{1}{96\pi^2} \ln \frac{m_R^2}{4\mu^2} \right) ight.$$

$$\left. + [s^2 + (t - u)^2] \right]$$

$$\times \left( \alpha_2(\mu) + \frac{7}{288\pi^2} - \frac{1}{48\pi^2} \ln \frac{m_R^2}{4\mu^2} \right).$$

(36)

This method is designed to match the amplitude in the region most sensitive to the chiral coefficients. Operationally then, the parameters will be adjusted to match the effect of the tail of the resonance in this region. This tells us that there is shift of the parameters such that

$$\alpha_1(\mu) + \frac{1}{144\pi^2} - \frac{1}{96\pi^2} \ln \frac{m_R^2}{4\mu^2} \approx \alpha_1^{\text{tree}},$$

(37)

$$\alpha_2(\mu) + \frac{7}{288\pi^2} - \frac{1}{48\pi^2} \ln \frac{m_R^2}{4\mu^2} = -\frac{44}{m_R^4}. \quad (38)$$

It would be these parameters which would be identified with the effect of integrating out the resonances, such as was given in Eq. (31), i.e., for the case of the $\rho$:

$$\alpha_1(\mu) + \frac{1}{144\pi^2} - \frac{1}{96\pi^2} \ln \frac{m_R^2}{4\mu^2} = \frac{\Gamma_{\rho} F_\rho^4}{m_\rho^5}. \quad (38)$$

Note that this procedure correctly gives the $\mu$ dependence of the chiral parameters.

VI. THE CHIRAL LAGRANGIAN FOR QCD

The best chiral Lagrangian compatible with data is of course known phenomenologically, and we expect QCD to describe the real world, so that in this sense the chiral Lagrangian of QCD is known. However, what we are trying to do is to see to what extent a Lagrangian can be obtained from purely theoretical considerations. Here we imagine that we know about the QCD spectrum from lattice gauge theory and/or the quark model, but have not yet performed $\pi-\pi$ scattering experiments. What would we predict?

The low-energy spectrum has two potentially relevant features. One is the $\rho$ meson, which appears as the lightest non-Goldstone particle. The other is the possibility of a scalar glueball. Present lattice calculations put this at 1.2 GeV or slightly higher. Other resonance states of appropriate quantum numbers are above 1.2 GeV, such as the $J^{PC} = 2^{--}, I = 0$ $F_2(1270)$. Because of the $1/m^5$ feature in the chiral couplings, the $\rho$ is the only clear resonance which should dominate the low-energy constants. It would yield

$$-\alpha_1 = \alpha_2 = \frac{44}{m_\rho^4} \frac{\Gamma_{\rho} F_\rho^4}{m_\rho^5} \approx 0.0084. \quad (39)$$

(Here we have reinstated the pion mass into the decay width to properly account for phase space. This usage is technically yet higher order in the chiral expansion, and
could be optional. This gives a minimal estimate of the theoretical uncertainty in the procedure of 20%. If we use the procedure advocated in Sec. V for predicting the scale-dependent one-loop renormalized coupling, this would yield

$$\alpha_i(m_p) = -0.010, \quad \alpha_2(m_p) = 0.003.$$  \hspace{1cm} (40)

Theoretically, one might also include an effect from the scalar glueball but this would only change $\alpha$ by $+0.0004$ for $\Gamma = 300$ MeV, $m = 1$ GeV.

These estimates are in remarkably good agreement with the phenomenological fits, where we previously found

$$\alpha_1 = -0.0092, \quad \alpha_2 = +0.0080$$  \hspace{1cm} (41)

as the best tree-level parameters, and

$$\alpha_1(m_p) = -0.011, \quad \alpha_2(m_p) = +0.00057$$  \hspace{1cm} (42)

at the one-loop level. The results are better than we would have expected, and do seem to indicate that the $\rho$ is the dominant feature of the spectrum.

This result is very similar to the ideas of vector-meson dominance which have been heavily used since being introduced by Sakurai4 in the early 1960s. In fact, a look at the chiral Lagrangian treatment of the pion electromagnetic form factor will emphasize the similarities. In vector-meson dominance the pion form factor is determined by the coupling to the $\rho$ meson to have the form

$$F_{\pi}(q^2) = \frac{1}{1 - \frac{q^2}{m_\rho^2}} \approx 1 + \frac{q^2}{m_\rho^2} + \cdots.$$  \hspace{1cm} (43)

In a chiral Lagrangian, on the other hand, the $q^2$ behavior is controlled by a coefficient in the $O(E^4)$ Lagrangian

$$L_4 = \cdots + \frac{i\alpha_0}{F^2} \text{Tr}(\partial \rho U Q \partial \rho U^\dagger + \partial \rho U^\dagger Q \partial \rho U),$$  \hspace{1cm} (44)

such that, at the tree level,

$$F_{\pi}(q^2) = 1 + \frac{-\alpha_0}{F^2} q^2.$$  \hspace{1cm} (45)

In order to match the vector-dominance result one chooses

$$\alpha_0 = \frac{F^2}{m_\rho^2} = 0.014.$$  \hspace{1cm} (46)

This is in fact the result which Gasser and Leutwyler2 found for this coefficient when coupling the $\rho$ to chiral fields and integrating out the $\rho$.

The criteria proposed in Sec. II, for determining when a contribution is large, is well satisfied by the $\rho$ in this case. When chiral loops are considered, they introduce a scale dependence of the form

$$\alpha_0(\mu) = \alpha_0(\mu_0) + b_9 \ln \frac{\mu^2}{\mu_0^2},$$

$$b_9 = \frac{1}{96\pi^2} \approx 0.001.$$  \hspace{1cm} (47)

Thus the effect of the $\rho$ on the $\rho$ is much larger than the scale dependence. The data are in good agreement with the vector-dominance prediction.

A similar situation occurs in the $q^2$ variation of the $\pi^0 \to \gamma \gamma$ amplitude. The chiral loop results for this have recently been described in Ref. 10. At one loop, the Wess-Zumino-Witten anomaly Lagrangian generates a $q^2$ variation for $M^{10} \to \gamma \gamma$ via the Lagrangian. The finite remainder and the $\mu^2$ variation are both very small compared to the prediction obtained by integrating out the $\rho$.

In this case the data are not very precise, but are consistent with the vector-dominance prediction.

Overall, the parameters in the QCD Lagrangians seem to reflect primarily the presence of the $\rho$ meson and its SU(3) partners.

VII. SUMMARY

We have explored, phenomenologically and through solvable trial models, the notion that the low-lying spectrum of a theory gives the major contribution to coefficients of the chiral Lagrangian at order $E^4$. In all cases studied the idea proved correct. Criteria were developed for deciding whether a given resonance should make a "large" contribution. When applied to QCD the $\rho$ contribution appears to be the only significant one, leading to a connection of chiral Lagrangian with the older idea of vector-meson dominance.

Besides the intrinsic interest in the origin of the chiral Lagrangian of QCD, there may be applications of this idea. For example, integrating out the $\rho$ resonance also implies a set of effects at order $E^6$ and higher. One can sum these by use of the full $\rho$ propagator. Perhaps in applications of chiral Lagrangians this separation of the $\rho$ effect, treated to all orders in the energy expansion, plus the remaining (small) $O(E^4)$ effects can lead to an improved phenomenology. Formally no error is made in doing this as the extra terms are of order $E^5$ and higher. However, numerically this may be an improvement. For example, in the pion electromagnetic form factor it would lead to the full monopole form rather than simply the linear term, which in fact is a favorable improvement.

In a different context, these ideas may be profitably applied to extensions of the standard model. All proposed extensions, even without the fundamental Higgs mechanism, lead to the same lowest-order predictions for $W_L W_L$ scattering as would the standard model with a heavy Higgs boson.11 This is a consequence of the underlying symmetry. However, deviations from this lowest-order result may depend on new physics, or on the true underlying theory. In an analogy with the above work, the standard fundamental Higgs mechanism is most like the linear $\sigma$ model, while a theory such as technicolor12 would more likely be similar to QCD. These could be distinguished by correction to the lowest-order result. A theory such as the strongly coupled standard model13 would have yet a third pattern. These issues are under investigation.

APPENDIX: THE WEAK INTERACTIONS

In the strong interactions one can convert the resonance coupling strength involving the chiral field into the
width of the resonance. In the case of the weak interaction, there is no way to normalize the weak coupling of the resonance to chiral fields. Here the criterion for a "large" contribution to the Lagrangian coefficient is lost. Nevertheless, one would expect that whatever plays a major role in generating the $\Delta I = \frac{1}{2}$ rule would leave its imprint on the chiral Lagrangians as well. Since there is very little consensus on the origin of this puzzling enhancement in nonleptonic transitions, it seems worthwhile to determine phenomenologically what the known features are. This appendix is meant as a simple exploration of the data.

Chiral symmetry relates $K \to 3\pi$ to $K \to 2\pi$. Typically the $K \to 3\pi$ amplitude is expanded about the center of the Dalitz plot:

$$A(K_L \to \pi^+ \pi^- \pi^0) = \alpha + \beta \bar{Y}$$

$$+ \gamma \left[ \bar{V}^2 - \frac{\bar{V}^2}{3} \right] + \delta \left[ \bar{V}^2 + \frac{\bar{V}^2}{3} \right],$$

$$A(K_L \to \pi^0 \pi^0 \pi^0) = 3 \left[ \alpha + \delta \left( \bar{V}^2 + \frac{\bar{V}^2}{3} \right) \right],$$

$$\bar{Y} = (s_3 - s_0)/s_0 \ ,$$

$$\bar{V} = (s_1 - s_2)/s_0 \ ,$$

$$s_i = (k - p_i)^2 ,$$

$$s_0 = \frac{1}{3}(s_1 + s_2 + s_3) = m_K^2 + m_\pi^2 .$$

Focusing on the $\Delta I = \frac{1}{2}$ amplitude only, the data are

$$\alpha = 9.15 \times 10^{-7} ,$$

$$\beta = 13.46 \times 10^{-7} ,$$

$$\gamma = -3.40 \times 10^{-7} ,$$

$$\delta = -1.01 \times 10^{-7} ,$$

(A2)

taken from the review of Devlin and Dickey.\textsuperscript{14}

In Ref. 15, it was shown that any chiral Lagrangian which does not lead to quadratic terms in the amplitude, $\gamma$ and $\delta$, has the same prediction for $\alpha$ and $\beta$:

$$\alpha = 7.5 \times 10^{-7} ,$$

$$\beta = 9.4 \times 10^{-7} ,$$

(A3)

up to possible corrections of order $m_\rho^2$. This includes the unique order $E^2$ form

$$L_{\omega} = g_\omega \text{Tr}(\lambda_8 \delta_\rho M \bar{\rho} M^\dagger) ,$$

(A4)

plus many $E^4$ Lagrangians. Of the four\textsuperscript{16} remaining ones, only two kinematic invariants are possible, which can be written as

$$A_{\text{quad}} = \gamma \frac{2}{3s_0^2} \left[ -8k \cdot p_0 p + p \cdot p + 4(k \cdot p + p_0 \cdot p + k \cdot p_0 p + ) \right]$$

$$- \delta \frac{2}{3s_0^2} \left[ 4k \cdot p + p_0 \cdot p + k \cdot p_0 p + \right]$$

$$+ 4(k \cdot p + p_0 \cdot p + k \cdot p_0 p + ) .$$

(A5)

Besides generating the $X^2$ and $Y^2$ terms these significantly improve the predictions of $\alpha$ and $\beta$. It is clear that our information on the $E^4$ terms in the weak interaction is limited, which will hinder our ability to draw firm conclusions.

One general feature which is easy to extract from these amplitudes is that they are more compatible with a strong $\rho$ presence rather than a scalar particle such as $\sigma$. This is clear because a $\rho$ never couples to two $\pi^0$s and hence can only contribute to the parameter $\gamma$ but not $\delta$. An isoscalar in contrast would generate $\gamma = \delta$. Since $\gamma$ is significantly larger than $\delta$, the $\rho$ contribution could be dominant.

This conclusion will turn out, under further inspection, to be both true and potentially misleading. This is because some of the contributions to $\gamma$ and $\delta$ come from the strong-interaction $E^4$ Lagrangian. A statement to the contrary in Ref. 15 is erroneous. We already know that the $\rho$ is a dominant feature in the strong interaction. However, a relevant question is, once the strong-interaction effect is subtracted off, what is the character of the purely weak-interaction remainder? We will see that the remainder is much smaller than the original size, so that we are cautious about drawing any conclusions.

The strong interactions enter through the pole diagram of Fig. 3(a). That of Fig. 3(b) is suppressed by a factor of $m_\pi^2/m_K^2$, and will be neglected. If we write

$$\gamma = \gamma_\omega + \gamma_\pi , \quad \delta = \delta_\omega + \delta_\pi ,$$

the strong contribution is

$$\gamma_\omega = \frac{g_\omega s_0^2}{F^6_\pi} (2\alpha_1 - \alpha_2) = -2.8 \times 10^{-7} ,$$

$$\delta_\omega = \frac{g_\omega s_0^2}{F^6_\pi} 2(\alpha_1 + \alpha_2) = -0.26 \times 10^{-7} ,$$

FIG. 3. Pole diagrams for $K \to 3\pi$. In these, the order $E^4$ effects of the strong interaction generate $E^4$ effects in $K \to 3\pi$.\textsuperscript{30}
which leaves, for the weak-interaction contribution,  
\[ \gamma_w = -0.6 \times 10^{-7}, \; \delta_w = -0.75 \times 10^{-7}. \]
This remainder is not much larger than scale dependence 
induced by the strong interactions alone:  
\[ \Delta \delta \approx \frac{g^* \Sigma_0^2}{F_\pi} \ln \frac{l^2}{\mu_0^2} \sim 0.64 \times 10^{-7}. \]  
(A6) 
Thus it is not clear if there is much which an be learned 
from the weak terms. 
It is of interest to ask whether the one model with a 
definite prediction for these coefficients—the gauged 
chiral model—produces these coefficients. Here the mod-
el is  
\[ L = g_0^2 \operatorname{Tr}(\lambda_\mu D_\mu M D^\mu M^\dagger), \]  
(A7) 
with \( D_\mu \) is given before [Eq. (8)]. In terms of the renor-
amalized weak coefficient \( A_0 \), this model predicts  
\[ \gamma_w = +1.4 \times 10^{-7} = g_0^2 \frac{3 \Sigma_0^2}{F_\pi^2} \frac{g_0^4}{m_p^2} \frac{1}{m_p^2 + m_A^2}, \]  
(A8) 
using methods such as described in Sec. III. This gives a 
numerically small contribution (as was observed above) 
but of the wrong sign.

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