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Although functional methods are common in the research literature, they are not generally seen by students until a course in field theory. Nevertheless, it is straightforward and pedagogically useful to treat a familiar problem—the harmonic oscillator—via such techniques, as shown in the present work.

I. INTRODUCTION

Quantum mechanics is usually taught within the context of the standard Schrödinger picture, wherein one deals with a wavefunction whose absolute square gives the relative probability density and whose time dependence is generated by the quantum mechanical Hamiltonian $\hat{H}$ (we set $\hbar = 1$)

$$i \frac{\partial}{\partial t} |\psi\rangle = \hat{H} |\psi\rangle. \quad (1)$$

On the other hand, in the research literature quantization is often performed via path integral methods, leading naturally to modern quantum field theory, which is generally done via functional integrals. Here connection to experimental quantities is provided in terms of the Green's functions, which are the vacuum expectation values of time-ordered products of fields.¹

Having learned the subject in the conventional Schrödinger fashion students often subsequently find the path integral-field theory approach to be somewhat forbidding. This transition can be somewhat eased by presenting a familiar problem—the harmonic oscillator—within the context of a functional integral approach, and that is the purpose of this article.

Below we shall calculate familiar oscillator matrix elements by three different procedures—the standard Dirac approach in Sec. II, the path integral method in Sec. III, and the functional integral procedure in Sec. IV. The first two techniques are standard but the third is an interesting pedagogical exercise not described in textbooks. In Sec. V we show how the driven quantum oscillator may be addressed in a similar way. Finally, we summarize our results in Sec. VI.

II. HARMONIC OSCILLATOR: SCHRÖDINGER APPROACH

Probably the most familiar of the exactly soluble problems is that of the harmonic oscillator. For a mass m and fundamental frequency ω , the Hamiltonian

$$H = -\frac{1}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2} m\omega^2 x^2 \quad (2)$$

is known to possess eigenvalues

$$E_n = \omega(n + \frac{1}{2}), \quad n = 0, 1, 2, \dots \quad (3)$$

with associated eigenfunctions²

$$\psi_n(x) = \left(\frac{m^2 \omega^2}{2^n \pi n!} \right)^{1/4} H_n(\sqrt{m\omega} x) e^{-(1/2)m\omega x^2}. \quad (4)$$

Here $H_n(x)$ are the usual Hermite polynomials.

While one could straightforwardly calculate matrix ele-

ments using these wavefunctions, it is easier to use the elegant formalism of Dirac, in which one defines creation, annihilation operators³

$$a = \sqrt{\frac{m\omega}{2}} \left(x + i \frac{1}{m\omega} p \right) = \sqrt{\frac{m\omega}{2}} \left(x + \frac{1}{m\omega} \frac{\partial}{\partial x} \right), \quad (5)$$

$$a^\dagger = \sqrt{\frac{m\omega}{2}} \left(x - i \frac{1}{m\omega} p \right) = \sqrt{\frac{m\omega}{2}} \left(x - \frac{1}{m\omega} \frac{\partial}{\partial x} \right),$$

that satisfy the commutation relations

$$[a, a^\dagger] = 1, \quad [a, a] = [a^\dagger, a^\dagger] = 0. \quad (6)$$

The Hamiltonian then can be written as

$$H = \omega(a^\dagger a + \frac{1}{2}), \quad (7)$$

with a ground state $|0\rangle$ such that $a|0\rangle = 0$ and higher eigenstates $|n\rangle$ satisfying

$$|n\rangle = [(a^\dagger)^n / \sqrt{n!}] |0\rangle, \quad (8)$$

$$a^\dagger a |n\rangle = n |n\rangle, \quad n = 0, 1, 2, \dots$$

The usual Schrödinger wavefunctions are found by projecting onto coordinate space

$$\psi_n(x) = \langle x | n \rangle. \quad (9)$$

Using the relations

$$[a, a^\dagger a] = a, \quad [a^\dagger, a^\dagger a] = -a^\dagger, \quad (10)$$

we easily derive

$$a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle, \quad a |n\rangle = \sqrt{n} |n-1\rangle, \quad (11)$$

that enable us to calculate arbitrary matrix elements. Thus, for example, since we have

$$x = (1/\sqrt{2m\omega})(a + a^\dagger), \quad (12)$$

we easily find

$$\begin{aligned} \langle n-1 | x | n \rangle &= \int dx \psi_{n-1}(x) x \psi_n(x) \\ &= \frac{1}{\sqrt{2m\omega}} \langle n-1 | a + a^\dagger | n \rangle = \sqrt{\frac{n}{2m\omega}}, \end{aligned} \quad (13)$$

$$\begin{aligned} \langle n-1 | p | n \rangle &= -i \int dx \psi_{n-1} \frac{\partial \psi_n}{\partial x} \\ &= -i \sqrt{\frac{m\omega}{2}} \langle n-1 | a - a^\dagger | n \rangle \\ &= -i \sqrt{\frac{nm\omega}{2}}. \end{aligned} \quad (14)$$

Higher-order forms can also be determined, e.g.,

$$\begin{aligned}\langle n|x^2|n\rangle &= \int dx \psi_n(x) x^2 \psi_n(x) \\ &= \frac{1}{2m\omega} \langle n|(a+a^\dagger)(a+a^\dagger)|n\rangle = \frac{2n+1}{2m\omega}.\end{aligned}\quad (15)$$

In a similar fashion, arbitrary matrix elements can be calculated. This is a familiar procedure and, having these results in hand, we move to the less familiar path integral method.

III. HARMONIC OSCILLATOR VIA THE PATH INTEGRAL

We begin our discussion by defining the propagator as the amplitude for the particle to travel between space-time points x_1, t_1 and x_2, t_2 ,

$$\begin{aligned}D(x_2, t_2; x_1, t_1) &= \langle x_2 | \exp -iH(t_2 - t_1) | x_1 \rangle \\ &= \int D[x(t)] \exp \{iS[x(t)]\}.\end{aligned}\quad (16)$$

The first form $[\langle x_2 | \exp -iH(t_2 - t_1) | x_1 \rangle]$ is the familiar Schrödinger version that expresses the time development in terms of the Hamiltonian. The second form for the propagator is the remarkable path integral prescription, a detailed presentation of which appears in the book by Feynman and Hibbs.⁴ Here the notation

$$\int D[x(t)]$$

implies a summation over *all* paths $x(t)$ that connect the two space-time points x_1, t_1 , and x_2, t_2 , each path being weighted by a phase factor determined by the classical action S associated with this path

$$S[x(t)] = \int_{t_1}^{t_2} dt \left(\frac{1}{2} m \dot{x}^2 - V(x(t)) \right). \quad (17)$$

It is highly nonobvious, but true, as shown in Ref. 2, that this sum over paths is totally equivalent to the more traditional Schrödinger prescription. Of course, it is no more possible, in general, to sum over a nondenumerable set of paths than it is to solve exactly the Schrödinger equation. However, as is well known, one *can* perform the sum exactly for the case of a harmonic oscillator by breaking up the time interval into finite steps Δt and integrating over all possible values of x at each time $t_1 + n\Delta t$. As shown in Feynman and Hibbs, this procedure allows one to find the usual expression for the harmonic oscillator propagator⁴

$$\begin{aligned}D(x_2, t_2; x_1, t_1) &= \left(\frac{m\omega}{2\pi i \sin \omega(t_2 - t_1)} \right)^{1/2} \exp \left(\frac{im\omega}{2 \sin \omega(t_2 - t_1)} \right. \\ &\quad \times [(x_2^2 + x_1^2) \cos \omega(t_2 - t_1) - 2x_2 x_1] \Big). \quad (18)\end{aligned}$$

(A simpler but less graphical derivation is given in Appendix A.) An expression involving the usual wavefunctions and eigenvalues can be found by inserting a complete set of oscillator states into the Hamiltonian expression for the

propagator, yielding

$$\begin{aligned}D(x_2, t_2; x_1, t_1) &= \sum_{n=0}^{\infty} \langle x_2 | n \rangle e^{-iE_n(t_2 - t_1)} \langle n | x_1 \rangle, \\ &= \sum_{n=0}^{\infty} \psi_n(x_2) \psi_n^*(x_1) e^{-iE_n(t_2 - t_1)}.\end{aligned}\quad (19)$$

The connection between these two forms, Eqs. (18) and (19), may most easily be seen by making a transformation to imaginary time $t = -i\tau$. Then Eq. (18) can be expanded in terms of the small quantity $\exp(-\omega\sigma)$, where $\sigma = \tau_f - \tau_i$

$$\begin{aligned}D(x_2, t_2; x_1, t_1) &= \left(\frac{m\omega}{\pi(e^{\omega\sigma} - e^{-\omega\sigma})} \right)^{1/2} \exp \left[- \frac{m\omega}{(e^{\omega\sigma} - e^{-\omega\sigma})} \right. \\ &\quad \times \left(\frac{1}{2} (x_2^2 + x_1^2) (e^{\omega\sigma} + e^{-\omega\sigma}) - 2x_2 x_1 \right) \Big] \\ &= \left(\frac{m\omega}{\pi} \right)^{1/2} \exp \left(- \frac{m\omega}{2} (x_2^2 + x_1^2) \right) \\ &\quad \times e^{-\omega\sigma/2} (1 + 2m\omega x_2 x_1 e^{-\omega\sigma} + \cdots).\end{aligned}\quad (20)$$

We can thus identify, by comparing with Eq. (19)

$$\begin{aligned}\psi_0(x) &= \left(\frac{m\omega}{\pi} \right)^{1/4} \exp \left(- \frac{1}{2} m\omega x^2 \right), \quad E_0 = \frac{1}{2} \omega, \\ \psi_1(x) &= \left(\frac{m\omega}{\pi} \right)^{1/4} \sqrt{2m\omega} x \exp \left(- \frac{1}{2} m\omega x^2 \right), \quad E_1 = \frac{3}{2} \omega, \\ &\vdots\end{aligned}\quad (21)$$

which, of course, agree with the usual forms of Eqs. (3) and (4).

Matrix elements can easily be calculated. Thus, for example, we find

$$\begin{aligned}\int_{-\infty}^{\infty} dx x^2 D(x, t_2; x, t_1) &= \left(\frac{m\omega}{2\pi i \sin \omega(t_2 - t_1)} \right)^{1/2} \left(\frac{i\pi \cot(\omega/2)(t_2 - t_1)}{4m^3\omega^3} \right)^{1/2} \\ &= \frac{1}{i - i\tau} \frac{1}{2m\omega} e^{-(1/2)\omega\sigma} (1 + 3e^{-\omega\sigma} + 5e^{-2\omega\sigma} + \cdots).\end{aligned}\quad (22)$$

Comparison with the Schrödinger form

$$\begin{aligned}\int_{-\infty}^{\infty} dx x^2 D(x, t_2; x, t_1) &= \sum_{n=0}^{\infty} e^{-\sigma\omega(n+1/2)} \int dx x^2 \psi_n^2(x)\end{aligned}\quad (23)$$

then yields

$$\langle n | x^2 | n \rangle = \int dx x^2 \psi_n^2(x) = \frac{2n+1}{2m\omega} \quad (24)$$

in agreement with Eq. (15). Likewise, nondiagonal matrix elements can be found by using creation or annihilation

operators. Thus, for example, we find

$$\begin{aligned} & \int_{-\infty}^{\infty} dx x \sqrt{\frac{m\omega}{2}} \left(x + \frac{1}{m\omega} \frac{\partial}{\partial x} \right) D(x, t_2; x', t_1) \Big|_{x'=x} \\ &= \left(\frac{m\omega}{2\pi i \sin \omega(t_2 - t_1)} \right)^{1/2} \left(\frac{i\pi \cot^3(\omega/2)(t_2 - t_1)}{8m^2\omega^2} \right)^{1/2} \\ & \quad \times \{1 + i[\cot \omega(t_2 - t_1) - \csc \omega(t_2 - t_1)]\} \\ &= \frac{1}{\sqrt{2m\omega}} e^{-i(3\omega\sigma/2)} (1 + 2e^{-\omega\sigma} + 3e^{-2\omega\sigma} + \dots) \end{aligned} \quad (25)$$

Comparison with the corresponding Schrödinger form

$$\begin{aligned} & \int dx x \sqrt{\frac{m\omega}{2}} \left(x + \frac{1}{m\omega} \frac{\partial}{\partial x} \right) D(x, t_2; x', t_1) \Big|_{x'=x} \\ &= \sum_{n=1}^{\infty} e^{-\omega(n+1/2)\sigma} \sqrt{n} \int_{-\infty}^{\infty} dx \psi_{n-1}(x) x \psi_n(x) \end{aligned} \quad (26)$$

then yields

$$\langle n-1|x|n \rangle = \int dx x \psi_{n-1}(x) \psi_n(x) = \sqrt{\frac{n}{2m\omega}} \quad (27)$$

in agreement with Eq. (13). In a similar fashion one can calculate arbitrary matrix elements. However, we move on to consider the functional derivation of these results, which is our primary goal.

IV. HARMONIC OSCILLATOR VIA THE FUNCTIONAL INTEGRAL

As noted earlier the Green's functions of quantum field theory are defined as vacuum expectation values of products of field operators.¹ The corresponding limit that projects out the vacuum, or ground state, configuration of the propagator is that of large imaginary time—large σ . Thus we find

$$D(x_2, t_2; x_1, t_1) \underset{\sigma \rightarrow \infty}{\sim} \psi_0(x_2) \psi_0^*(x_1) e^{-(1/2)\omega\sigma} [1 + O(e^{-\omega\sigma})]. \quad (28)$$

Similarly in the large real-time limit $t_2 - t_1 \rightarrow \infty$, the ground state is isolated, since higher excited states are associated with more rapid time fluctuations

$$\begin{aligned} D(x_2, t_2; x_1, t_1) & \underset{t_2 - t_1 \rightarrow \infty}{\sim} \psi_0(x_2) \psi_0^*(x_1) e^{-(i/2)\omega(t_2 - t_1)} \\ & \quad \times [1 + O(e^{-i\omega(t_2 - t_1)})]. \end{aligned} \quad (29)$$

We begin our transition to field theoretic techniques by inserting a driving term $xj(t)$ into the Lagrangian. The equation of motion then becomes

$$m \frac{d^2}{dt^2} x + m\omega^2 x = j(t), \quad (30)$$

so that $j(t)$ represents an external force (or “current,” hence the symbol j) that acts upon the oscillator. The modified path integral becomes

$$\begin{aligned} & D[x_2, t_2; x_1, t_1; j(t)] \\ &= \int D[x(t)] \exp i \int_{t_1}^{t_2} dt \left(\frac{1}{2} m \dot{x}^2 \right. \\ & \quad \left. - \frac{1}{2} m \omega^2 x^2 + xj(t) \right), \end{aligned} \quad (31)$$

where we assume that $j(t)$ vanishes at the endpoints. While it is possible to evaluate explicitly the path integral, it is sufficient for our purposes to determine its value only in the limit as $t_2 \rightarrow \infty$ and $t_1 \rightarrow -\infty$ which, as we pointed out above, isolates the ground-state contribution. We have then the so-called “generating functional”

$$\begin{aligned} W[j(t)] &= \int D[x(t)] \exp i \int_{-\infty}^{\infty} dt \left(\frac{1}{2} m \dot{x}^2 \right. \\ & \quad \left. - \frac{1}{2} m \omega^2 x^2 + xj(t) \right). \end{aligned} \quad (32)$$

[The term “functional” is used since the quantity W depends not just upon the value of a single variable but rather upon the shape of a function $j(t)$ over the entire real axis.] The basic idea here is that the properties of the oscillator can be revealed provided that one knows their responses to an arbitrary external probe.

The key to this procedure is the proof given in Appendix B that

$$\begin{aligned} & (-i)^n \frac{\delta^n}{\delta j(t_1) \cdots \delta j(t_n)} W[j(t)] \Big|_{j=0} \\ &= \lim_{\substack{t_f \rightarrow \infty \\ t_i \rightarrow -\infty}} \int D[x(t)] x(t_n) \cdots x(t_1) \\ & \quad \times \exp i \int_{t_i}^{t_f} dt \left(\frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2 \right) \\ &= \lim_{\substack{t_f \rightarrow \infty \\ t_i \rightarrow -\infty}} \langle x_{t_f}, t_f | T[x(t_n) \cdots x(t_1)] | x_{t_i}, t_i \rangle, \end{aligned} \quad (33)$$

where T denotes the usual time-ordered product

$$\begin{aligned} T[A(t_2)B(t_1)] &= \theta(t_2 - t_1) A(t_2) B(t_1) \\ & \quad + \theta(t_1 - t_2) B(t_1) A(t_2) \end{aligned} \quad (34)$$

and $\delta/\delta j(t)$ describes functional differentiation as indicated in Appendix B. Thus the generating functional $W[j]$ describes arbitrary matrix elements of harmonic oscillator ground states. We shall make this connection more precise below.

We can evaluate the generating functional by taking the Fourier transform

$$x(t) = \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{-iEt} \tilde{x}(E) \quad (35)$$

so that

$$\begin{aligned} S[x(t)] &= \int_{-\infty}^{\infty} dt \left(\frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2 + xj(t) \right) \\ &= \int_{-\infty}^{\infty} \frac{dE}{2\pi} \left(\frac{m}{2} (E^2 - \omega^2) \tilde{x}(E) \tilde{x}(-E) \right. \\ & \quad \left. + \frac{1}{2} \tilde{j}(E) \tilde{x}(-E) + \frac{1}{2} \tilde{j}(-E) \tilde{x}(E) \right). \end{aligned} \quad (36)$$

By completing the square, one finds

$S[x(t)]$

$$= \int_{-\infty}^{\infty} \frac{dE}{2\pi} \left[\frac{m}{2} (E^2 - \omega^2) \left(\tilde{x}(E) + \frac{1}{m(E^2 - \omega^2)} \tilde{j}(E) \right) \right. \\ \left. \times \left(\tilde{x}(-E) + \frac{1}{m(E^2 - \omega^2)} \tilde{j}(-E) \right) \right. \\ \left. - \frac{1}{2m} \tilde{j}(E) \frac{1}{E^2 - \omega^2} \tilde{j}(-E) \right]. \quad (37)$$

Defining the Green's function

$$D(t-t') = \int \frac{dE}{2\pi} e^{-iE(t-t')} \frac{1}{E^2 - \omega^2 + i\epsilon} \\ = -\frac{i}{2\omega} \theta(t-t') e^{-i\omega(t-t')} \\ - \frac{i}{2\omega} \theta(t'-t) e^{-i\omega(t'-t)} \\ = -\frac{i}{2\omega} e^{-i\omega|t-t'|}, \quad (38)$$

which satisfies

$$\left(\frac{d^2}{dt^2} + \omega^2 \right) D(t-t') = -\delta(t'-t), \quad (39)$$

we have then

$$S = \int_{-\infty}^{\infty} dt \left(\frac{1}{2} m \dot{x}'^2 - \frac{1}{2} m \omega^2 x'^2 \right) \\ - \frac{1}{2m} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' j(t) D(t-t') j(t'), \quad (40)$$

where

$$x'(t) = x(t) + \frac{1}{m} \int_{-\infty}^{\infty} dt' D(t-t') j(t'). \quad (41)$$

Now change variables from $x(t)$ to $x'(t)$. Because the path integral involves an integral over all $x(t)$ the Jacobian of the change of variables is unity, i.e., $\int D[x'(t)] = \int D[x(t)]$. This yields then

$$W[j(t)] \\ = \exp \left(-\frac{i}{2m} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' j(t) D(t-t') j(t') \right) \\ \times \left[\int D[x'(t)] \exp i \int_{-\infty}^{\infty} dt \left(\frac{1}{2} m \dot{x}'^2 \right. \right. \\ \left. \left. - \frac{1}{2} m \omega^2 x'^2 \right) \right]. \quad (42)$$

Finally, we note that the quantity in brackets is just the propagator for the harmonic oscillator in the absence of a driving force $j(t)$ so that our final result for the generating functional is

$$W[j] = W[0] \exp \left(-\frac{i}{2m} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' j(t) \right. \\ \left. \times D(t-t') j(t') \right). \quad (43)$$

Having now evaluated $W[j]$ we can begin to calculate matrix elements. Thus, e.g., for the ground state we can use Eq. (33) to obtain (for $t_2 > t_1$)

$$\frac{\langle \psi_0 | T[x(t_2)x(t_1)] | \psi_0 \rangle}{\langle \psi_0 | \psi_0 \rangle} \\ = (-i)^2 \left| \frac{1}{W[0]} \frac{\delta^2 W[j]}{\delta j(t_2) \delta j(t_1)} \right|_{j=0} \\ = \frac{i}{m} D(t_2 - t_1) = \frac{1}{2m\omega} e^{-i\omega(t_2 - t_1)}, \quad (44)$$

that reproduce the familiar result

$$\langle \psi_0 | x^2 | \psi_0 \rangle = 1/2m\omega \quad (45)$$

in the final limit $t_2 \rightarrow t_1$. Further applications of this method will reproduce arbitrary ground-state matrix elements.

Despite the apparent restriction to the ground state in the generating functional, excited state matrix elements can also be obtained. For example, we note that matrix elements involving the first excited state can be "reduced" to a ground-state matrix element by use of the usual creation operator

$$|1\rangle = a^\dagger |0\rangle = \sqrt{\frac{m\omega}{2}} \left(x - i \frac{p}{m\omega} \right) |0\rangle, \quad (46)$$

so that

$$\langle 0 | O | 1 \rangle = \sqrt{\frac{m\omega}{2}} \langle 0 | O \left(x - i \frac{p}{m\omega} \right) | 0 \rangle, \quad (47)$$

where O is any operator. Using the classical expression $p = m\dot{x}$ we find then

$$\langle 0 | O | 1 \rangle = \sqrt{\frac{m\omega}{2}} \left(1 - \frac{i}{\omega} \frac{\partial}{\partial t} \right) \langle 0 | O x(t) | 0 \rangle. \quad (48)$$

Thus, for example,

$$\langle 0 | x | 1 \rangle = \lim_{t_2 \rightarrow t_1} (-i)^2 \sqrt{\frac{m\omega}{2}} \left(1 - \frac{i}{\omega} \frac{\partial}{\partial t_1} \right) \\ \times \frac{\delta^2}{\delta j(t_2) \delta j(t_1)} \frac{W[j]}{W[0]} \Big|_{j=0} \\ = \lim_{t_2 \rightarrow t_1} \sqrt{\frac{m\omega}{2}} \left(1 - \frac{i}{\omega} \frac{\partial}{\partial t_1} \right) \frac{1}{m} D(t_2 - t_1) \\ = \frac{1}{\sqrt{2m\omega}}. \quad (49)$$

Similarly

$$\langle 0 | p | 1 \rangle = \lim_{t_2 \rightarrow t_1} m \frac{\partial}{\partial t_2} \sqrt{\frac{m\omega}{2}} \left(1 - \frac{i}{\omega} \frac{\partial}{\partial t_1} \right) \frac{1}{m} D(t_2 - t_1) \\ = -i \sqrt{\frac{m\omega}{2}} \quad (50)$$

and

$$\begin{aligned}
 \langle 1|x^2|1\rangle &= \frac{m\omega}{2} \lim_{\substack{t_1 \rightarrow t^+ \\ t_2 \rightarrow t^-}} \left(1 + \frac{i}{\omega} \frac{\partial}{\partial t_1}\right) \left(1 - \frac{i}{\omega} \frac{\partial}{\partial t_2}\right) \\
 &\quad \times \langle 0|x(t_1)x^2(t)x(t_2)|0\rangle \\
 &= (-i)^4 \lim_{t_2, t_3, t_4 \rightarrow t_1} \left(\frac{m\omega}{2}\right) \left(1 + \frac{i}{\omega} \frac{\partial}{\partial t_1}\right) \left(1 - \frac{i}{\omega} \frac{\partial}{\partial t_2}\right) \\
 &\quad \times \frac{\delta^2}{\delta j(t_1)\delta j(t_2)\delta j(t_3)\delta j(t_4)} \frac{W[j]}{W[0]} \Big|_{j=0} \\
 &= \left(\frac{m\omega}{2}\right) \lim_{t_1, t_2 \rightarrow t} \left(1 + \frac{i}{\omega} \frac{\partial}{\partial t_1}\right) \left(1 - \frac{i}{\omega} \frac{\partial}{\partial t_2}\right) \left(-\frac{i}{m}\right)^2 \\
 &\quad \times [D(t_1 - t_2)D(0) + 2D(t_1 - t)D(t - t_2)] \\
 &= \frac{3}{2m\omega}. \tag{51}
 \end{aligned}$$

In like fashion arbitrary harmonic oscillator matrix elements can be reduced to ground-state expectation values, which in turn can be determined using the generating functional $W[j]$. The ground-state amplitude in the presence of an arbitrary source $j(t)$ contains *all* the information about the harmonic oscillator.

In field theory, of course, the ground state is the no-particle state—the vacuum—and excited states correspond to the presence of one or more particles. Transition matrix elements can all be expressed in terms of vacuum expectation values by use of the LSZ (Lehmann, Symansik, and Zimmerman) reduction formulas.⁵ In this procedure particle matrix elements governing transitions in field theory are “reduced” to vacuum matrix elements by a procedure analogous to Eq. (47). These vacuum matrix elements have a corresponding path integral representation in terms of a generating function in the presence of an arbitrary external source.

V. THE FORCED OSCILLATOR

Finally, we note that simple oscillator matrix elements are not all that we can calculate using this formalism. Thus considering an oscillator with an externally applied force $F(t)$, the amplitude to begin and end in the ground state is given by

$$\begin{aligned}
 \langle 0(\text{out})|0(\text{in})\rangle &= \frac{W[F]}{W[0]} \\
 &= \exp - \frac{i}{2m} \int dt \int dt' F(t) D(t - t') F(t') \\
 &\equiv \exp - \frac{1}{2} G, \tag{52}
 \end{aligned}$$

where

$$G = \int dt \int dt' F(t) e^{-i\omega|t-t'|} F(t') \frac{1}{2m\omega}. \tag{53}$$

Similarly we can determine the amplitude to make a transi-

tion to excited states, e.g.,

$$\begin{aligned}
 \langle 0(\text{out})|1(\text{in})\rangle &= \lim_{t \rightarrow -\infty} -i \sqrt{\frac{m\omega}{2}} \left(1 - \frac{i}{\omega} \frac{\partial}{\partial t}\right) \frac{1}{W(0)} \frac{\delta W[F]}{\delta F(t)} \\
 &= \lim_{t \rightarrow -\infty} -\sqrt{\frac{\omega}{2m}} \left(1 - \frac{i}{\omega} \frac{\partial}{\partial t}\right) \\
 &\quad \times \int_{-\infty}^{\infty} dt' D(t' - t) F(t') e^{-G/2} \\
 &= -\frac{\gamma}{\sqrt{2m\omega}} e^{-G/2} e^{-i\omega t}, \tag{54}
 \end{aligned}$$

where

$$\gamma = \int dt e^{i\omega t} F(t) \tag{55}$$

is the Fourier transform of the force. Alternatively, for matrix elements between two excited states, one obtains, e.g.,

$$\begin{aligned}
 \langle 1(\text{out})|1(\text{in})\rangle &= (-i)^2 \lim_{\substack{t_2 \rightarrow \infty \\ t_1 \rightarrow -\infty}} \frac{m\omega}{2} \left(1 + \frac{i}{\omega} \frac{\partial}{\partial t_2}\right) \left(1 - \frac{i}{\omega} \frac{\partial}{\partial t_1}\right) \\
 &\quad \times \frac{\delta^2 W[F]}{\delta F(t_2)\delta F(t_1)} \frac{1}{W[0]} \\
 &= (-i)^2 \lim_{\substack{t_2 \rightarrow \infty \\ t_1 \rightarrow -\infty}} \frac{m\omega}{2} \left(1 + \frac{i}{\omega} \frac{\partial}{\partial t_2}\right) \left(1 - \frac{i}{\omega} \frac{\partial}{\partial t_1}\right) e^{-G/2} \\
 &\quad \times \left[\frac{i}{m} F(t_2 - t_1) + \left(\frac{i}{m}\right)^2 \right. \\
 &\quad \times \left. \int dt' D(t' - t_1) F(t') \int dt D(t_2 - t) F(t) \right] \\
 &= (-i)^2 e^{-G/2} \left(1 + \frac{|\gamma|^2}{2m\omega}\right) e^{-i\omega(t_2 - t_1)}. \tag{56}
 \end{aligned}$$

In all cases results obtained by use of the generating functional are found to coincide with those obtained via alternative means.⁶

VI. SUMMARY

We have seen how matrix elements of the quantum harmonic oscillator can be evaluated in three very different procedures—via Dirac operator methods, Feynman path integrals, and using the so-called generating functional. Obviously the quickest and most efficient method is the straightforward operator technique. However, use of the path integral is an interesting and useful alternative. Finally, the generating functional procedure, while somewhat more unwieldy than the first two, is pedagogically useful in that it is similar in many ways to the methods of quantum field theory. Thus the writing of the matrix elements in terms of a series of two-particle Green's functions $D(t_2 - t_1)$ is related to the expansion of field theory Green's functions via Wick's theorem, while the writing of arbitrary matrix elements in terms of vacuum expectation values of T products of operators in many ways parallels the LSZ reduction.⁵ We have shown how free oscillator matrix elements and forced oscillator transition amplitudes can be evaluated in this fashion. Hopefully, after be-

ing introduced to these techniques in a nonthreatening situation such as this, students will find a future meeting with quantum field theory to be a somewhat less forbidding one.

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APPENDIX A

One can derive the harmonic oscillator in the standard "time slice" way, as shown elsewhere or, equivalently, can use a quicker, less intuitive approach, as shown here.⁷ Thus write an arbitrary path $x(t)$ in terms of the classical path connecting points x_1, t_1 and x_2, t_2 and a deviation $\delta x(t)$

$$x(t) = x_{cl}(t) + \delta x(t). \quad (A1)$$

Here the classical path is that which satisfies the usual Newtonian equation of motion

$$m \frac{d^2}{dt^2} x = -m\omega^2 x(t) \quad (A2)$$

and is easily found to be

$$x_{cl}(t) = x_1 \cos \omega(t - t_1) + \frac{x_2 - x_1 \cos \omega(t_2 - t_1)}{\sin \omega(t_2 - t_1)} \times \sin \omega(t - t_1). \quad (A3)$$

The action for an arbitrary path $x(t)$ then becomes

$$S[x(t)] = S[x_{cl}(t)] + \int dt \left(\frac{1}{2} m \delta \dot{x}^2 - \frac{1}{2} m\omega^2 \delta x^2 \right), \quad (A4)$$

where

$$S[x_{cl}(t)] = \frac{m\omega}{2 \sin \omega(t_2 - t_1)} [(x_2^2 + x_1^2) \times \cos \omega(t_2 - t_1) - 2x_2 x_1] \quad (A5)$$

and where linear terms in $\delta x, \delta \dot{x}$ vanish by Hamilton's equation $\delta S = 0$. We find then

$$D(x_2, t_2; x_1, t_1) = A(t_2 - t_1) \exp iS[x_{cl}(t)], \quad (A6)$$

where

$$A(t_2 - t_1) = \int D[\delta x] \exp -i \int dt \left(\frac{1}{2} m \delta \dot{x}^2 - \frac{1}{2} m\omega^2 \delta x^2 \right). \quad (A7)$$

Now $A(t_2 - t_1)$ can be evaluated by conventional path integral methods, or we can use a trick based upon the completeness relation

$$\begin{aligned} D(x_2, t_2; x_1, t_1) &= \langle x_2 | x_1 \rangle = \delta(x_2 - x_1) \\ &= \int dx \langle x_2 | e^{-iHt} | x \rangle \langle x | e^{iHt} | x_1 \rangle \\ &= \int dx D(x_2, t_2; x, 0) D^*(x, t_1; x_1, 0). \end{aligned} \quad (A8)$$

Using the expression for the propagator from Eq. (A6) this

becomes

$$\begin{aligned} \delta(x_2 - x_1) &= \int dx |A(t)|^2 \exp i[S_{cl}(x_2, t_2; x, 0) - S_{cl}(x_1, t_1; x, 0)] \\ &= |A(t)|^2 \int dx \exp i\gamma(x)(x_2 - x_1) \\ &= |A(t)|^2 \int \frac{d\gamma}{2\pi} e^{i\gamma(x_2 - x_1)} \frac{1}{|(1/2\pi)(d\gamma/dx)|}, \end{aligned} \quad (A9)$$

where here

$$\gamma(x) = \frac{\partial S_{cl}(x_1, t_1; x, 0)}{\partial x_1}. \quad (A10)$$

Using the usual representation of the delta function

$$\delta(x) = \int \frac{d\gamma}{2\pi} e^{i\gamma x} \quad (A11)$$

we find then

$$|A(t)| = \left| \frac{1}{2\pi} \frac{\partial^2 S_{cl}(x_2, t_2; x_1, 0)}{\partial x_2 \partial x_1} \right|^{1/2}. \quad (A12)$$

The phase can be found by comparing with the well-known free particle case, yielding

$$\begin{aligned} A(t) &= \left(-\frac{1}{2\pi i} \frac{\partial^2 S_{cl}(x_2, t_2; x_1, 0)}{\partial x_2 \partial x_1} \right)^{1/2} \\ &= \left(\frac{m\omega}{2\pi i \sin \omega(t_2 - t_1)} \right)^{1/2} \end{aligned} \quad (A13)$$

as used in Eq. (18).

APPENDIX B

Here we derive the relation between functional differentiation of the generating functional and time-ordered Green's functions. The basic property of functional derivatives that is required is

$$\frac{\delta}{\delta j(t_i)} \int_{t_i}^{t_f} dt A(t) j(t) = A(t_i) \quad (B1)$$

if t_1 lies between t_i and t_f . Using this, we note that

$$\begin{aligned} (-i)^n \frac{\delta^n}{\delta j(t_1) \cdots \delta j(t_n)} \int D[x(t)] \\ \times \exp i \int_{t_i}^{t_f} dt \left(\frac{1}{2} m \dot{x}^2 - \frac{1}{2} m\omega^2 x^2 + jx \right) \Big|_{j=0} \\ = \int D[x(t)] x(t_1) \cdots x(t_n) \\ \times \exp i \int_{t_i}^{t_f} dt \left(\frac{1}{2} m \dot{x}^2 - \frac{1}{2} m\omega^2 x^2 \right). \end{aligned} \quad (B2)$$

Also we recall the connection of the path integral with the propagator, Eq. (2), and the completeness relation

$$D(x_f, t_f; x_i, t_i) = \int dx' D(x_f, t_f; x', t) D(x', t; x_i, t_i). \quad (B3)$$

Then if $t_i < t_1 < \cdots < t_n < t_f$ we have

$$\begin{aligned}
& (-i)^n \frac{\delta^n}{\delta j(t_1) \cdots \delta j(t_n)} W[j] \Big|_{j=0} \\
&= \int dx_n \cdots \int dx_1 D(x_f, t_f; x_n, t_n) x_n \\
&\quad \times D(x_n, t_n; x_{n-1}, t_{n-1}) x_{n-1} \cdots D(x_2, t_2; x_1, t_1) x_1 \\
&\quad \times D(x_1, t_1; x_i, t_i) \\
&= \langle x_f, t_f | x(t_n) x(t_{n-1}) \cdots x(t_1) | x_i, t_i \rangle. \quad (B4)
\end{aligned}$$

On the other hand, if there is a different time ordering, a similar result clearly holds, but with the n x 's always ordered such that later times appear to the left. Thus we have, in general,

$$\begin{aligned}
& (-i)^n \frac{\delta^n}{\delta j(t_1) \cdots \delta j(t_n)} W[j] \Big|_{j=0} \\
&= \langle x_f, t_f | T[x(t_n) \cdots x(t_1)] | x_i, t_i \rangle \quad (B5)
\end{aligned}$$

as promised.

¹See, e.g., P. Ramond, *Field Theory, A Modern Primer* (Benjamin/Cummings, Reading, MA, 1982); J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1964).

²See, e.g., E. Merzbacher, *Quantum Mechanics* (Wiley, New York, 1970), Chap. 5; R. Shankar, *Principles of Quantum Mechanics* (Plenum, New York, 1980), Chap. 7.

³R. Shankar, Ref. 3, Sec. 7.4; E. Merzbacher, Ref. 3, Chap. 15.

⁴R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965); another very readable introduction to path integral methods is provided in L. S. Schulman, *Techniques and Applications of Path Integration* (Wiley, New York, 1981); finally, a somewhat more formal treatment is given by J. Glimm and A. Jaffe, *Quantum Physics: A Functional Integral Point of View* (Springer, New York, 1981).

⁵J. D. Bjorken and S. D. Drell, Ref. 1; C. Itzykson and J. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).

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The electrodynamic tether

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The electrodynamic tether project being developed by NASA will use a conducting tether attached to a spacecraft in low Earth orbit to interact with the Earth's magnetic field. This project represents a novel application of Faraday's and Lenz's laws of classical electrodynamics. The dynamics of the tether also provide an elegant application of classical mechanics. To emphasize this point, the dynamics of the tether are considered and solved exactly in a reasonable model. The solution for the tether's configurations has the rather interesting feature that it bifurcates for a certain range of the electrical current in the tether.

I. INTRODUCTION

The rapidly developing area of space science offers many novel applications of classical physics. One such example is the electrodynamic tether system currently being researched by NASA for spacecraft operating in low Earth orbits.¹ It has been nearly 20 years since Drell and Foley² proposed the adaptation of the "solar-wind sail" of Alfvén,³ which used the magnetic fields of the solar winds, to a system utilizing the magnetic field of the Earth. As presently envisioned (see Fig. 1), the tether will be a conducting wire, extending several kilometers out from a spacecraft and continually crossing magnetic field lines. In a low Earth orbit, the induced (motional) emf, equal to the product of the speed of the spacecraft and the normal component of the Earth's magnetic field, is on the order of a few tenths of a volt/meter. The induced voltage for 10 km of wire would be on the order of a few kilovolts. Assuming

sufficient electrical contact can be maintained with the ionosphere, e.g., by plasma generators, it is estimated that a 10-km tether will provide several kilowatts of electrical power.¹ The extraction of electrical power via an associated current I results in an $I \times B$ force that reduces the orbital speed of the spacecraft. Conversely, if a sufficiently large reverse voltage is applied to cause a reversal of the current, then the $I \times B$ force will increase the orbital speed and electrical energy will be converted into orbital energy.

Consequently, the electrodynamic tether system is expected to perform as a motor/generator system providing a simple, reliable, and economic means of efficiently converting mechanical energy of orbital motion into electrical energy in a fully reversible process.

The dynamics of the tether itself are an elegant application of classical mechanics. For noncurrent-carrying tethers whose anticipated usage include (1) connection for spacecraft constellations, (2) transportation devices, and