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Aristotle was right: heavier objects fall faster†

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Abstract According to the weak form of the equivalence principle all objects fall at the same rate in a gravitational field. However, recent calculations in finite-temperature quantum field theory have revealed that at \( T > 0 \) heavier and/or colder objects actually fall faster than their lighter and/or warmer counterparts. This unexpected result is demonstrated here using elementary quantum mechanical arguments.

1. Introduction

Anyone who has taught introductory physics knows well that Aristotelian ideas about motion are far from dead (Clement 1982). Thus, one notion which students sometimes bring with them is that heavier objects fall faster than their lighter counterparts. However, one of the first things an elementary physics student learns is that all objects, regardless of their mass, fall at the same rate, an assertion which is often presented in terms of Galileo’s probably apocryphal experiment performed from the top of the Leaning Tower of Pisa. In a more advanced course, students may learn the same concept from the principle of equivalence which, in its weak form, requires the strict equality of gravitational and inertial masses so that all bodies must have identical gravitational accelerations (Ohanian 1976, Feynman 1965a).

Experimentally, there exists remarkably strong support for this principle, with no violation detected at the level of a part in \( 10^{12} \) (Roll 1964, Braginsky 1971). Since in a quantum theory a portion of a particle’s mass (formally infinite!) arises through quantum radiative corrections (Messiah 1965, Sakurai 1967, Feynman 1965b), these too must presumably obey the equivalence principle and this has been explicitly demonstrated within the context of relativistic quantum field theory (Adler 1977, Brown 1977). However, in quantum systems at non-zero temperature a portion of a particle’s mass arises through finite-temperature radiative corrections (Weidon 1982, Peresutti and Skagerstam 1982), and it has recently been demonstrated that these contributions to a particle’s mass do not respect the equivalence principle (Donoghue et al 1985a, b, 1986): if \( \Delta m_T \) is the mass shift produced by finite-temperature radiative corrections then

\[
\delta m_T\text{\textit{inertial}} = -\delta m_T\text{\textit{gravitational}}. \tag{1}
\]

Although this result was demonstrated via techniques of relativistic field theory, one can also derive equation (1) rigorously simply via ‘old-fashioned’ time-dependent perturbation theory and that is the purpose of this article. It is rare that a current research topic in quantum field theory can be pedagogically efficacious. This calculation, however, is one which involves aspects of general relativity, renormalisation and finite-temperature quantum mechanics and might prove of real interest in an advanced or honours course.

The manipulations which we employ in the following two sections require a familiarity with relativistic quantum mechanics. However, a reader who is uneasy with the Dirac equation or with the quantised radiation field may skip to § 4 where the results of our calculation are summarised and where a more intuitive derivation of the physics is presented.

2. Mass renormalisation at \( T = 0 \)

Before launching our discussion of finite-temperature renormalisation, it is useful to review the way in which
negative energy states play a critical role in non-relativistic quantum mechanics (Sakurai 1967). We start with the Hamiltonian for an electron interacting with an electromagnetic field

\[ H = \frac{1}{2m} \sigma \cdot (p - eA) \sigma \cdot (p - eA) + \epsilon \phi \]  

(2)

where \( \phi \) is an external electromagnetic potential and

\[ A = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_i} \sum_k (a(k, \lambda) e^{-ik \cdot x} \delta_{k,i} + a^\dagger(k, \lambda) e^{ik \cdot x} \delta_{k,i}) \]  

(3)

is the quantised radiation field, with \( a_{k,i} \) and \( a^\dagger_{k,i} \) being the conventional creation and annihilation operators

\[ a_{k,i} |n_{k,i}\rangle = \sqrt{(n + 1)_{k,i}} |(n + 1)_{k,i}\rangle \]  

(4)

\[ a^\dagger_{k,i} |n_{k,i}\rangle = \sqrt{n_{k,i}} |(n - 1)_{k,i}\rangle \]

Note that in this and all succeeding equations, we shall employ 'natural' units so that we set \( h = c = k \) (the Boltzmann constant) = 1.

If one calculates Compton scattering from a free electron (\( \phi = 0 \)) in this formalism, one finds (in the electric dipole approximation and neglecting the \( -p \cdot B \) interaction)

\[ \langle p_2, s_2; k_2, \lambda_2 | T | p_1, s_1; k_1, \lambda_1 \rangle \approx \left( \frac{e^2}{2m} \right)^2 \delta_{s_1, s_2} \delta_{k_1, k_2} \]  

[6]

\[ \approx \left( \frac{e^2}{m} \right) \epsilon^{\mu \nu \rho \sigma} \epsilon_{\mu \lambda_1} \epsilon_{\sigma \lambda_2} \]  

(5)

Here the first ('seagull') term arises from the diagram (figure 1(a)) involving \( A \cdot A \), while the remaining two ('pole') terms are represented by the diagrams shown in figure 1(b). In the case where the photon energy \( \omega_1 \) is much greater than the electron kinetic energy \( (\omega_1 > p_1^2/2m) \) but is still nonrelativistic \( (\omega_1 \ll c) \) we can drop all but the first term, yielding the well known Thomson scattering amplitude (Feynman 1965a, Sakurai 1967)

If, however, one examines the same scattering process from the point of view of the Dirac equation an apparent paradox results. We use the conventional Dirac Hamiltonian

\[ H = \alpha \cdot (p - eA) + \beta m + \epsilon \phi \]  

(7)

with

\[ \alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]  

(8)

being the usual Dirac matrices. Now consider the relativistic analogue of figure 1(b) and work in the nonrelativistic limit so that, for positive energy states (again neglecting the M1 interaction energy associated with the magnetic moment)

\[ \langle p_2, s_2; k_2, \lambda_2 | T | p_1, s_1; k_1, \lambda_1 \rangle \approx \frac{1}{2m} (p_1 + p_2) \chi_{s_2} \chi_{s_1}. \]  

(9)

The transition amplitude corresponding to figure 1(b) can then be written, using the transversality condition \( \epsilon_{\mu \lambda_1} \cdot k = 0 \), as

\[ \langle p_2, s_2; k_2, \lambda_2 | T | p_1, s_1; k_1, \lambda_1 \rangle \approx \left( \frac{e^2}{m} \right)^2 \left( \frac{\epsilon^{\mu \nu \rho \sigma} \epsilon_{\mu \lambda_1} \epsilon_{\sigma \lambda_2}}{(p_1^2/2m) + \omega_1 - (p_1 + k_1)^2/2m} \right) \]  

[10]

\[ + \left( \frac{\epsilon^{\mu \nu \rho \sigma} \epsilon_{\mu \lambda_1} \epsilon_{\sigma \lambda_2}}{(p_1^2/2m) + \omega_2 - (p_1 - k_2)^2/2m} \right) (\chi_{s_2} \chi_{s_1}), \]

which agrees precisely with its nonrelativistic analogue, equation (5).

However, there is no term in the relativistic Hamiltonian involving \( A \cdot A \) and thus the seagull amplitude, which generates the Thomson scattering process, appears to be missing. The resolution of this mystery lies in our mistaken neglect of negative energy solutions to the Dirac equation, which yield the relativistic Feynman diagrams in figure 2. Thus in figure 2(b) an incident photon can knock a negative energy electron out of the Dirac sea, creating a positive energy state and leaving behind a 'hole' which is interpreted as a positron. The corresponding matrix

Figure 1 Diagrams representing nonrelativistic Compton scattering. Here the full lines represent electrons, while wiggly lines are photons.

Figure 2 Diagrams involving negative energy and representing Compton scattering in a Dirac picture.
Aristotle was right: heavier objects fall faster.

Element in the nonrelativistic limit is

\[ \langle \mathbf{p}_2, s_2 | \mathbf{p}', s'(-) \rangle \approx \chi_2 \left( \mathbf{p}_2 - \mathbf{p}' \right) \chi_1, \]

\[ \approx \chi_2 \sigma \chi_1. \]  

For both figures 2(a) and 2(b) the energy of the initial state is

\[ E_s = E_{p_1} + \omega_1 + E_{\text{Dirac sea}} \]  

and the intermediate state energies are (respectively)

\[ E_{\text{int}} = E_{p_1} + E_{p_2} + (E_{\text{Dirac sea}} - (|E_p|)) \]

\[ E_{\text{fin}} = E_{p_1} + E_{p_2} + \omega_1 + \omega_2 + (E_{\text{Dirac sea}} - (|E_p|)) \]

so the transition amplitude corresponding to figure 2 becomes (in the electric dipole approximation)

\[ \langle \mathbf{p}_2, s_2; \mathbf{k}_2, \lambda_2 | T | \mathbf{p}_1, s_1; \mathbf{k}_1, \lambda_1 \rangle \]

\[ \approx e^i \chi_2 \left( \mathbf{p}_2 - \mathbf{p}' \right) \chi_1. \]

Here \( E \), of course, represents the relativistic energy

\[ E = m + p^2/2m + \ldots \]  

so that if \( \omega \ll m \) we can neglect recoil and approximate \( E \approx m \), yielding

\[ \langle \mathbf{p}_2, s_2; \mathbf{k}_2, \lambda_2 | T | \mathbf{p}_1, s_1; \mathbf{k}_1, \lambda_1 \rangle \]

\[ \approx (e^i/2m) 2 \delta_{\mathbf{k}_1, \mathbf{k}_2} \delta_{\lambda_1, \lambda_2} \chi_2 \chi_1. \]

which reproduces the seagull diagram – equation (6). Thus, the use of negative energy states is essential, even though the process we are treating is a non-relativistic one.

The importance of negative energy solutions and of the Dirac sea can also be seen in a different context: electromagnetic self-energy or mass renormalisation. Thus, if one considers the Feynman propagator \( G_F(t_2 - t_1) \) which takes a state forward in time

\[ G_F(t_2 - t_1) \psi(t_1) = \begin{cases} \psi(t_2) & t_2 > t_1, \\ 0 & t_2 < t_1 \end{cases} \]

then

\[ G_F(i) = \begin{cases} e^{-Ht} & t > 0, \\ 0 & t < 0 \end{cases} \]

where

\[ G_F(s) = \frac{i}{s - H + i\epsilon} \]

is the Fourier transform of the propagator. Writing

\[ H = H_0 + V \]

and expanding the propagator as a geometric series we have then

\[ -iG_F(s) = \frac{1}{s - H_0 + i\epsilon} + \frac{1}{s - H_0 + i\epsilon} \]

\[ + \ldots \equiv \sum_{n=0}^\infty G_F^n(s) \]

which provides a perturbative expansion of \( G_F(s) \) in powers of \( V \).

Consider now a free electron with momentum \( p \) and we have

\[ H_0 |p\rangle = E_p |p\rangle \]

where \( E_p = (p^2 + m^2)^{1/2} \). Since there is no potential \( (V=0) \) then

\[ G_F(s) = \frac{i}{s - H_0 + i\epsilon} \]

so the amplitude to find the electron in the same state at time \( t > 0 \) is

\[ \text{Amp}(t) = \frac{s}{2\pi} e^{-i\omega t} \langle p | G_F(s) | p \rangle \]

\[ = \frac{s}{2\pi} e^{-i\omega t} \frac{i}{s - E_p + i\epsilon} \]

\[ = e^{-iEt}. \]

The time rate of change of this phase, i.e. the location of the pole in \( \langle p | G_F(s) | p \rangle \), then measures the electron energy. However, in the presence of interactions, this amplitude will be modified. Thus, including interaction with the radiation field to second order, there exist two types of modifications to the propagator, as shown in figure 3 – one involving positive energy intermediate states, one involving negative energy. Expanding \( G_F(s) \) to second order in \( V \), we have then

\[ \langle p | G_F^{(0)}(s) | p \rangle = \frac{i}{s - E_p + i\epsilon} \]

\[ \langle p | G_F^{(1)}(s) | p \rangle = 0 \]

\[ \langle p | G_F^{(2)}(s) | p \rangle = \frac{i}{s - E_p + i\epsilon} \gamma(s) \frac{1}{s - E_p + i\epsilon} \]

Figure 3 Diagrams representing mass renormalisation in a Dirac approach.
where, in the electric dipole approximation and neglecting recoil

\[ \gamma(s) = e^2 \int \frac{d^4k}{(2\pi)^3} \frac{1}{2\omega_\gamma} \sum_\tau \delta_{\tau,\gamma} \cdot \mathbf{p} \cdot \frac{1}{m^2} \delta_{\tau,\gamma} \cdot \mathbf{p} \cdot \frac{1}{m^2} \delta_{\tau,\gamma} \cdot \mathbf{p} \cdot \frac{1}{m^2} \delta_{\tau,\gamma} \cdot \mathbf{p} \]

\[ \cdot \left( \sigma + \frac{\sigma \cdot \mathbf{p}}{2m} - \frac{\sigma \cdot \mathbf{p}}{2m} \right) \left( \sigma + \frac{\sigma \cdot \mathbf{p}}{2m} - \frac{\sigma \cdot \mathbf{p}}{2m} \right) \left( \sigma + \frac{\sigma \cdot \mathbf{p}}{2m} - \frac{\sigma \cdot \mathbf{p}}{2m} \right) \left( \sigma + \frac{\sigma \cdot \mathbf{p}}{2m} - \frac{\sigma \cdot \mathbf{p}}{2m} \right) \]

\[ \cdot \left( \frac{1}{s-\omega_\gamma \pm i\epsilon} \right) \left( \frac{1}{s-\omega_\gamma \pm i\epsilon} \right) \left( \frac{1}{s-\omega_\gamma \pm i\epsilon} \right) \left( \frac{1}{s-\omega_\gamma \pm i\epsilon} \right) \]

\[ \approx \frac{2\alpha}{\pi} \int \frac{dk}{s-3m+i\epsilon} + O(\alpha p^2/m^2). \] (26)

In accord with the static approximation we have in the last line dropped terms in \( k \) relative to the large mass \( m \). The same \( \phi(s) \) also appears in higher-order diagrams (cf figure 4) and the resulting series can be summed to all orders, yielding

\[ \langle p|G_\tau(s)p\rangle \approx \frac{i}{s-\omega_\gamma \pm i\epsilon} \left( 1 + \left( \frac{\gamma(s)}{s-\omega_\gamma \pm i\epsilon} \right) \gamma(s) + \ldots \right) \]

\[ \approx \frac{i}{s-\omega_\gamma \pm i\epsilon} \left( 1 + \left( \frac{\gamma(s)}{s-\omega_\gamma \pm i\epsilon} \right) \gamma(s) + \ldots \right) \] (27)

Comparing with equation (24) and keeping only the leading term we see that the energy \( E_\gamma \) has been replaced by the renormalised value \( E'_\gamma \)

\[ E'_\gamma = E_\gamma + \gamma(E_\gamma) \approx E_\gamma + \frac{\alpha}{\pi m} \int dk \, k + O(\alpha p^2/m^2). \] (28)

Since

\[ E'_\gamma = m_0 + (p^2/2m_0) + \ldots \] (29)

we can identify the renormalised mass as

\[ m_0 = m + \delta m_0 \] (30)

with \( \delta m_0 = (\alpha/\pi m) \int dk \, k \). Thus the nonrelativistic renormalised mass appears to be quadratically divergent. This infinity is well known and is associated with the tadpole bubble diagram in nonrelativistic perturbation theory. Since the shift is identical for all states of a given system this term is usually dropped when energy differences are involved (Feynman 1965b). However, we shall see that this piece leads to a finite energy shift at non-zero temperature.

One suspects that this renormalised mass \( m_0 \) is the inertial mass. In order to confirm this, we need to insert a probe, for example an external scalar potential \( \phi \), into the system. To lowest order, this produces an additional potential energy

\[ V_2 = \phi \cdot \phi. \] (31)

Electromagnetic corrections could in principle modify this potential energy. However, because of charge conservation and gauge invariance, the combined effects of vertex correction and wavefunction renormalisation cancel as shown in the appendix.

Thus the full electromagnetically corrected Hamiltonian can be written as

\[ H = m + \delta m_0 + \frac{p^2}{2(m + \delta m_0)} + \phi \cdot \phi \] (32)

and we may determine the acceleration of the electron via

\[ a = -[H, [H, r]] = e \frac{-\nabla \phi}{m + \delta m_0}. \] (33)

Since \(-\nabla \phi\) is just the electric field \( E \), we find

\[ (m + \delta m_0)a = eE \] (34)

so that we recognise \( m + \delta m_0 \) as the inertial mass. The above derivation then is a brief description of the renormalisation programme of quantum electrodynamics using nonrelativistic time-ordered perturbation theory.

In order to address the question of the quantum corrections to the gravitational mass, one must calculate the equivalent electromagnetic renormalisation of the gravitational potential energy. Recall that gravity couples to the energy contained in a system (Scadron 1979, Ohanian 1977). Thus if \( \phi \) is the gravitational potential, the corresponding potential energy which enters the Schrödinger equation is

\[ \phi = \frac{1}{2} \int d^4x \, \phi \cdot \phi. \]

Unfortunately, there exists no elementary way in which to perform the calculations of the radiative corrections to \( \phi \). Since the energy is carried not only by the electron but also by the photons, by the Coulomb interaction, etc, there exist a large number of additional diagrams. Also, proper relativistic treatments show that misleading results are obtained unless the calculation is performed in \( 4 + \epsilon \) dimensions (Adler 1977, Brown 1977). Nevertheless, when such a relativistic evaluation is performed, it is found that the renormalised gravitational coupling is

\[ V_{\phi} = E_\phi \phi. \] (35)

with \( E_\phi \approx m \). Unfortunately, there exists no elementary way in which to perform the calculations of the radiative corrections to \( V_{\phi} \). Since the energy is carried not only by the electron but also by the photons, by the Coulomb interaction, etc, there exist a large number of additional diagrams. Also, proper relativistic treatments show that misleading results are obtained unless the calculation is performed in \( 4 + \epsilon \) dimensions (Adler 1977, Brown 1977). Nevertheless, when such a relativistic evaluation is performed, it is found that the renormalised gravitational coupling is

\[ V'_{\phi} = E_\phi \phi. \] (36)

\[ \text{Figure 4} \] Diagrams representing the 'full' electron propagator.
Aristotle was right: heavier objects fall faster

with $E_F = [(m + \delta m_0)^2 + p^2]^{1/2}$ so that in the non-relativistic limit the acceleration of an electron in a gravitational field becomes

$$a = -[H, [H, r]] = -\nabla \varphi_k.$$ (37)

Thus the acceleration is independent of mass and the quantum equivalence principle is valid.

3. Mass renormalisation at $T > 0$

Now consider the case that the electron finds itself in thermal equilibrium with a heat bath at temperature $T < m$. Then the only evidence of the temperature is that there exists a black-body spectrum of photons with the usual Planck number density (Eisberg and Resnick 1974, Tipler 1979)

$$n(\omega) = \frac{1}{\exp(\omega/T) - 1}$$ (38)

Although this might seem to be a relatively minor change from the $T=0$ situation, it has some important consequences.

Firstly, one would expect the heat bath to increase the electron’s inertial mass. This is due to the fact that continued interaction with the black-body photons will presumably impede its motion, which can be attributed to an effective increase in inertial mass.

In order to study this effect quantitatively we consider the diagrams in figure 5, which contribute to the electron self-energy. In figure 5(a) a real photon is absorbed from the heat bath and later is re-emitted. The corresponding amplitude is (in the electric dipole approximation and the non-relativistic limit)

$$\delta m(5a) = -e^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \sum \hat{e}_{k,i} \cdot \sigma \frac{1}{-2m+k} \hat{e}_{k,i} \cdot \sigma n(\omega_k)$$ (39)

while for figure 5(b) a photon is first emitted into the heat bath and later an identical photon is absorbed.

The amplitude is

$$\delta m(5b) = -e^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \sum \hat{e}_{k,i} \cdot \sigma \frac{1}{-2m-k} \hat{e}_{k,i} \cdot \sigma n(\omega_k) + 1.$$ (40)

In equation (40) the term independent of $n(\omega_k)$ simply reproduces our previous estimate of the mass shift and corresponds to the case that the absorbed and emitted photon is the same one, while $n(\omega_k)$ accounts for the feature that, because of the heat bath, stimulated emission and absorption are also possible. We find then

$$\delta m_{T=0} = \delta m_0 + \delta m_\beta$$ (41)

where $\delta m_\beta$ is given in equation (30) and is independent of temperature, while $\delta m_0$ is the temperature-dependent mass shift given by equations (39) and (40) as

$$\delta m_\beta = \frac{2\pi}{\pi} \int_0^\infty dk k n(k) \left( \frac{1}{2m+k} + \frac{1}{2m-k} \right).$$ (42)

However, since $T < m$ the Planck function $n(k)$ is damped exponentially unless $k < T < m$. Thus we can neglect the terms $k$ in the denominators, leaving a convergent result

$$\delta m_\beta = \frac{2\pi}{\pi} \int_0^\infty dk k n(k) \frac{1}{e^{\frac{m}{T}} - 1} = \frac{\pi \alpha T^2}{3m}$$ (43)

which agrees precisely with the answer calculated in relativistic finite-temperature field theory (Weldon 1982, Peresutti and Skagerstam 1982). This is to be expected since, because of the exponential damping introduced by the Planck function, this finite-temperature correction is basically a nonrelativistic calculation.

That $\delta m_\beta$ represents also the inertial mass shift can be seen by repeating the renormalisation of the external electromagnetic vertex at finite temperature. We show in the appendix that at $T \neq 0$

$$V' = ev.$$ (44)

Thus even at non-zero temperature, the electromagnetic vertex is unrenormalised, as expected from gauge invariance. The finite-temperature Hamiltonian in the presence of an external electric field then becomes

$$H = m + \delta m_0 + \delta m_\beta + \frac{p^2}{2(m + \delta m_0 + \delta m_\beta)} + ev.$$ (45)

Then

$$a = -[H, [H, r]] = \frac{-e \nabla \varphi}{m + \delta m_0 + \delta m_\beta}$$ (46)

whence we recognise the inertial mass as

$$m_{\text{inertial}} = m + \delta m_0 + \delta m_\beta.$$ (47)

Surprisingly, although we were unable to present the $T=0$ correction to the gravitational mass within a
simple nonrelativistic framework, the finite-temperature modifications do lend themselves to such a treatment. This is due to the fact that one need only study the effect of physical (real) photons present in the heat bath, thus reducing the number and subtlety of the corrections which must be considered, and also because the Planck distribution function suppresses the very high momentum contributions and thereby does not lead to the divergences which plague the \( T=0 \) calculations. The procedure is identical to that used to renormalise the Coulomb vertex except that now gravity, coupling to the energy, has interaction both with the electron (with energy \( E_p \omega \)) and to the photon (with energy \( k \)). The calculation is described in the appendix, where we show that

\[
V' = \varphi_0 \left( m + e^\alpha \right) \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega} \sum_{k,\epsilon} \delta_{k,\epsilon} \cdot \sigma \left( n(\omega_\epsilon) + 1 \right) + e^\alpha \int \frac{d^3k}{(2\pi)^3} \frac{2m - k}{2\omega_k} \sum_{k,\epsilon} \delta_{k,\epsilon} \cdot \sigma n(\omega_\epsilon) \frac{2m - k}{2\omega_k} \delta_{k,\epsilon} \cdot \sigma n(\omega_\epsilon)
\]

This gives only the finite-temperature contribution, since, as discussed above, there exist additional \( T=0 \) diagrams which contribute an amount \( 2\beta m_0 \varphi_0 \), yielding our final answer

\[
V' = \varphi_0 (m + \delta m_0 - \delta m_\beta).
\]  (48)

If we now calculate the gravitational acceleration we find that

\[
a = -\left[H, \sigma, r\right] = -\nabla \varphi_0 \frac{m + \delta m_0 - \delta m_\beta}{m + \delta m_0 - \delta m_\beta}
\]  (50)

so that the gravitational mass is

\[
m_{\text{gravitational}} = m + \delta m_0 - \delta m_\beta
\]  (51)

and is not equal to the inertial mass.

4. Conclusion

We now summarise our results and will show how the curious conclusions derived via formal manipulations in the preceding sections can be understood in a more intuitive fashion.

We have calculated the energy shift which is due to the interaction of a charged particle with the radiation field. That such an energy change should exist follows straightforwardly from use of formal perturbation theory with the quantised radiation field, as shown in §§ 2 and 3. However, it is also useful to look at the problem from a different and less formal perspective, as suggested by Feynman (Feynman 1961, Power 1966). Thus, suppose that we are at zero temperature and imagine a single charged particle to be present within an otherwise empty box. Although there are no photons inside the box (since \( T=0 \)) there does exist, of course, a zero-point energy \( \frac{1}{2} \omega_p \) for each allowed photon mode \( p \). Associated with the 'matter distribution' inside the box (in our case the single charged particle of mass \( m \)) there is an index of refraction \( r(\omega) \)† which is given by (Sakurai 1967)

\[
r(\omega) = 1 + \frac{2\pi}{\omega} f(\omega).
\]  (52)

Here \( f(\omega) \) is the forward photon scattering amplitude, which in our approximation is given (cf equation (16)) by

\[
f(\omega) \approx -\alpha/m.
\]  (53)

Because of the relation

\[
\omega = \frac{1}{r(\omega)} k
\]  (54)

between wavenumber and frequency, the presence of the charged particle induces a shift in the zero-point energy in the amount

\[
\Delta E_\omega = \frac{1}{2} \omega_p \left( \frac{1}{r(\omega_p)} - 1 \right)
\]  (55)

for each mode compared to its value in the empty box. Summing over all modes, we find then the total energy shift

\[
\Delta E_{\text{tot}}(T=0) = \int \frac{d^3k_p}{(2\pi)^3} \sum_{\epsilon} \omega_\epsilon \left( \frac{1}{r(\omega_\epsilon)} - 1 \right)
\]  (56)

\[
\approx \frac{4\pi}{2m} \int \frac{d^3k_p}{(2\pi)^3} \sum_{\epsilon} \frac{1}{2\omega_\epsilon}
\]

which agrees precisely with the result derived via formal methods in equation (30). At non-zero temperature, of course, there exists an additional energy

\[
\Delta E_\omega = n(\omega_p) \omega_p
\]

for each mode and the total energy change at non-zero temperature becomes

\[
\Delta E(T) = \Delta E(T=0) + \int \frac{d^3k_p}{(2\pi)^3} \sum_{\epsilon} n(\omega_\epsilon) \omega_\epsilon \left( \frac{1}{r(\omega_\epsilon)} - 1 \right)
\]

\[
= \Delta E(T=0) + \frac{\pi \alpha T^2}{3m} = \delta m_0 - \delta m_\beta
\]  (57)

in agreement with the finite-temperature energy shift derived in equation (43). Thus one can look on this change in energy either as due to the emission and absorption of photons by the charged particle or as the shift in the energy of the radiation due to the presence of the particle.

† We use the symbol \( r \) for the index of refraction rather than the conventional symbol \( n \) in order to avoid confusion with the Bose distribution function \( n(\omega) \) defined in equation (38).
This energy shift modifies the effective Hamiltonian for the system and hence, we would expect, represents an increase in the inertial mass of the system, as verified in equation (46). Although at zero temperature one expects an identical increase in gravitational mass, in accordance with the weak equivalence principle, a difference between the two masses should be anticipated at non-zero temperature. The point is that while the Hamiltonian/inertial mass is identified with the free energy of the system 

\[ \langle H \rangle = F \]  
(58)

the gravitational field is a spin-2 quantity and couples to the energy–momentum tensor \( T_{\mu \nu} \), whose time–time component is a measure of the internal energy of the system \( U \)

\[ \langle T_{\infty} \rangle = U. \]  
(59)

These energies are related by thermodynamics

\[ U = F - T \frac{\partial F}{\partial T} \]
or

\[ m_s = m_T - T \frac{\partial m_T}{\partial T}. \]  
(60)

Thus at \( T = 0 \), the two masses are identical. However, at non-zero temperature

\[ m_s = m_T - T \frac{\partial m_T}{\partial T} m_b = m_T - 2m_s \]  
(61)

since the finite-temperature mass shift \( \delta m_s \) is quadratic in the temperature. This relation between gravitational and inertial mass is precisely what was found earlier using more formal techniques. We see, however, that this mass inequality should not be regarded as a surprise but rather is required by the strictures of equilibrium thermodynamics.

The acceleration of a falling object is then given by

\[ a = g \frac{m_T}{m_s} = g \left( 1 - \frac{2\pi T^2}{m^2} \right) \]  
(62)

as given in equation (50). We see that the gravitational acceleration depends upon both \( m \) and \( T \) and that heavier and/or cooler objects will indeed fall at a faster rate than their lighter and/or hotter counterparts.† Remarkably, this is the pattern asserted by Aristotle 2300 years before the advent of quantum field theory. Of course, this is merely a curiosity and has no bearing on the simplistic earth, air, fire, water view of the world on which its 'derivation' was based. Experimentally, we note that there is no consistency with the Eotvos experiment, since if we take \( T = 300 \) K and \( m \) to be the mass of an electron, we find

\[ 2\pi T^2/3m^2 \sim 3 \times 10^{-17} \]  
(63)

† Another amusing curiosity is that if two mirror nuclei are dropped together the one of lesser charge will reach the ground first. However, this asymmetry should not be unexpected since electromagnetism breaks charge symmetry anyway.

which is much below the one part in \( 10^{12} \) accuracy of state-of-the-art experiments.

The violation of the equivalence principle might be expected also since we have violated one of the fundamental notions which led to its postulation – the impossibility of defining absolute motion through the vacuum. However, one can measure absolute velocity and acceleration relative to the heat bath, since there exists a preferred frame – the frame in which the black-body radiation is isotropic. For example, the absolute velocity of the earth has been detected with respect to the 3 K distribution left over from the early universe. Thus the conditions under which the equivalence principle was formulated are not met at finite temperature. The fact that we live in a universe at non-zero temperature could in principle have led to unexpected results in the Eotvos experiment, were it not for the fact that the correction is too small to be detected at presently achievable temperatures. However, we may ascribe this result not to any intrinsic violation of the equivalence principle in the fundamental Hamiltonian describing the universe, but rather to the particular physical state in which we exist.

Appendix

The same electromagnetic effects which renormalise the mass affect the normalisation of the bare wavefunction and can also change the scale of couplings to external potentials. Thus the zeroth-order Coulomb interaction

\[ V_2 = \alpha \]  
(A.1)

is modified, as shown in figure 6, to become the effective interaction

\[ (ep)^{\text{eff}} = \alpha + \frac{1}{E_0 - H_0 + i\epsilon} \frac{1}{E_0 - H_0 + i\epsilon} \]  
\[ \times \frac{1}{(2\pi)^2} \sum \frac{\delta \epsilon k \cdot \sigma}{(2m-k)^2} \epsilon \delta k \cdot \sigma \]  
\[ = \alpha \left( 1 + 2\pi \int \frac{dk}{(2\pi)^2} \right). \]  
(A.2)

Figure 6 Diagram representing renormalisation of the Coulomb interaction by electromagnetic effects.

Here \( \times \) denotes the coupling to the external potential.
The same non-renormalisation obtains at \( T > 0 \). In this case the correction to the Coulomb vertex comes from the diagrams in figure 8, yielding

\[
\text{eq}^{\text{eff}} = \text{eq} \left( 1 + e^2 \int \frac{d^4 k}{(2\pi)^4} \frac{1}{2\omega_k} \sum_i \delta_{k,i} \cdot \sigma \cdot \frac{1}{(2m + k)^2} \delta_{k,i} \right) 
\]

\[
\cdot \sigma (n(\omega_k) + 1) + e^2 \int \frac{d^4 k}{(2\pi)^4} \frac{1}{2\omega_k} \sum_i \delta_{k,i} \cdot \sigma \cdot \frac{1}{(2m - k)^2} \delta_{k,i} \cdot \sigma (n(\omega_k)) 
\]

\[
\cdot \sigma \cdot \left( \frac{1}{(2m + k)^2} \delta_{k,i} \cdot \sigma (n(\omega_k)) \right) \quad \text{(A.9)}
\]

However, the finite-temperature wavefunction renormalisation is given by

\[
Z = 1 - e^2 \int \frac{d^4 k}{(2\pi)^4} \frac{1}{2\omega_k} \sum_i \delta_{k,i} \cdot \sigma \cdot \left( \frac{1}{(2m + k)^2} \delta_{k,i} \cdot \sigma (n(\omega_k)) \right) \quad \text{(A.10)}
\]

so that

\[
\text{eq}^{\text{eff}} = \text{eq} \cdot Z = \text{eq} \cdot \text{eq}^{\text{eff}} \quad \text{(A.11)}
\]

as before.

Finally, we apply the same techniques to the renormalisation of the gravitational vertex. The difference in this case is that we must include also the gravitational coupling to the photon energy. Then

\[
\nu^{\text{eff}}_k = g_k \left( m + e^2 \int \frac{d^4 k}{(2\pi)^4} \frac{1}{2\omega_k} \sum_i \delta_{k,i} \cdot \sigma \cdot \left( \frac{m + k}{(2m + k)^2} \delta_{k,i} \cdot \sigma \cdot \left( \sigma (n(\omega_k) + 1) \right) \right) \right) \quad \text{(A.12)}
\]

so that the renormalised gravitational vertex is given by

\[
\nu_k^{\text{eff}} = Z \nu_k^{\text{eff}} = g_k \left( m + e^2 \int \frac{d^4 k}{(2\pi)^4} \frac{1}{2\omega_k} \sum_i \delta_{k,i} \cdot \sigma \cdot \left( \frac{m + k}{(2m + k)^2} \delta_{k,i} \cdot \sigma \cdot \left( \sigma (n(\omega_k) + 1) \right) \right) \right) \quad \text{(A.13)}
\]

The reader who is interested in the full relativistic treatment should see Donoghue et al (1985a).

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