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Quantum Electrodynamics at Finite Temperature*

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We present a systematic examination of finite temperature effects in quantum electrodynamics at one loop order. We calculate mass, charge, and wavefunction renormalization, demonstrate the running of the coupling constant at high temperatures, and study the renormalized vertex function and the energy momentum tensor. The confusion in the literature concerning the finite temperature corrections to the electron's magnetic moment is resolved. We also present the finite temperature effects in scalar electrodynamics. Throughout we stress the need to provide a well-defined method to observe a given quantity when interpreting the results of a calculation, and we suggest new techniques which incorporate the novel features of finite temperature theories. © 1985 Academic Press, Inc.

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I. INTRODUCTION

Although finite temperature field theory was formulated nearly a decade ago, detailed applications to quantum electrodynamics have appeared only recently. Thus the first formulation of relativistic finite T field theory was presented in papers by Dolan and Jackiw [1], Weinberg [2], and Bernard [3], and the first applications of it were to the calculations of temperature behavior of the effective potential in Higgs theories. However, more recently some of the basic properties of the theory have been elucidated by Weldon [4], and general procedures have been developed for calculating renormalization and radiative corrections [5]. Most notably, there have been many applications of finite temperature theory to a variety of interesting processes [6, 7]. The study of a familiar theory, such as QED, but at a nonzero temperature, allows one to illustrate in a clear manner some of the many interesting and novel features of finite temperature field theory. In addition, there may well be observable applications to physical situations where temperature is

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involved, such as the early universe or the quark-gluon plasma. Most previous studies have focused on a particular portion of the theory. It is the purpose of the present paper to give a comprehensive and thorough treatment of the one loop radiative corrections in finite temperature QED.

An additional motivation for this work is found by noting that both the methods and the results of many of the specific calculations in the literature do not always agree with each other.¹ Such problems appear to arise because a finite temperature theory is not Lorentz covariant due to the preferred rest frame of the heat bath. Thus, many of the standard methods of conventional field theory are not applicable, and new techniques must be developed. It is then essential to be very cautious in the interpretation of any calculated quantity, to be certain that there is an operational procedure which connects that quantity with a realistic physical measurement. This is particularly true of the magnetic moment, and we will devote considerable discussion to the methods of measuring a magnetic moment. Throughout the paper we hope in this manner to resolve the present disagreements which appear in the literature.

There are two limits to the theory which are natural to consider— $T \ll m_e$ and $T \gg m_e$. In the former the only temperature-dependent effects are due to the heat bath of photons, with all effects of the fermions being exponentially suppressed, i.e., $O(e^{-m/T})$. The interpretation of the theory is simplest in this case, since the motion of electrons in a photon heat bath can be studied using conventional methods. In particular it is sensible to consider, among other situations, the case in which the motion of the electron is nonrelativistic. At high temperatures, ($T \gg m_e$), the interpretation can become more difficult, as there is a background heat bath of electron-positron pairs. All available energy states are filled up to an energy $E \sim T$ (we work with units such that Boltzmann's constant is unity), which implies that nonrelativistic treatments are inapplicable—the Pauli principle prevents the electron under consideration from occupying any momentum state with a nonrelativistic velocity. Even for relativistic electrons there can be confusion caused by the Pauli principle for $E \gtrsim T$. Despite these difficulties, the radiative corrections at high T are mathematically well defined and we shall quote the results. However, most of our discussion on the interpretation of our calculations will be focused on the intuitively clearer case when $T \ll m_e$.

In Section II we introduce the finite temperature formalism and compute the self energy. This yields mass and wavefunction renormalization constants and brings out the unusual kinematic properties of particles which are immersed in a heat bath. In Section III, we study the one loop corrections to the vertex function at $T \ll m_e$ and discuss the interpretation of the magnetic moment in the nonrelativistic approximation. Section IV is devoted to a treatment of relativistic motion in electromagnetic fields, and we derive a version of the Bargmann-Michell-Telegdi equation which includes finite temperature kinematics. Section V gives the vertex

¹ For example, the anomalous magnetic moment of the electron is evaluated in the papers of Ref. [7], each with different results.

correction when $T \gg m_e$, while Section VI discusses the vacuum polarization and temperature dependence of the coupling constant. We find that the coupling varies logarithmically with a behavior consistent with that which one could expect from renormalization group considerations. For completeness in Section VII we recount some results on the renormalization of the energy momentum tensor. One loop calculations in electrodynamics of scalar particles are presented in Section VIII. The validity of the Dirac quantization condition for magnetic monopoles at finite temperature is examined in Section IX. Two applications to experiment—the Lamb shift and the Casimir effect—are discussed in Sections X and XI. Finally, our conclusions are presented in Section XII.

II. THE SELF-ENERGY AND FINITE TEMPERATURE SPINORS

In this section we study the one loop self-energy and the associated kinematic modifications which it introduces. Much of the material in this section is not new, but it is important to review the methods of finite temperature field theory and to introduce our notation. Moreover, as we shall demonstrate in later sections, the kinematic features of particle propagation in a heat bath play a large (and previously neglected) role in the interpretation of physical quantities. In this section we demonstrate how these modifications are generated. A very useful tool in handling the kinematics are the finite temperature spinors described below.

For simplicity consider a spin- $\frac{1}{2}$ particle—an electron—with unrenormalized (bare) mass m_0 , so that the unrenormalized Lagrangian is

$$\mathcal{L} = \bar{\psi}_0(x)(i\nabla - m_0)\psi_0(x). \quad (1)$$

First, work at zero temperature. Then, as is well known, radiative corrections will modify this simple theory. Thus, calculating the so-called self-energy diagram, as shown in Fig. 1, we find

$$\Sigma(p) = -ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \frac{\gamma_\mu (\not{p} - \not{k} + m) \gamma^\mu}{k^2 - 2p \cdot k + p^2 - m^2}. \quad (2)$$

The required integration is performed in the Appendix using dimensional regularization, yielding

$$\Sigma(p) = -\frac{\alpha}{4\pi} [m_0 - (\not{p} - m_0)] \left(\frac{3}{\epsilon} - 4 \right) + \dots \quad (3)$$

FIG. 1. The standard electron self-energy diagram.

where

$$\frac{1}{\varepsilon} = \frac{2}{d-4} + \gamma - \ln 4\pi + \ln m^2 \quad (4)$$

with d representing the number of spacetime dimensions. Then writing the inverse propagator as

$$\begin{aligned} G^{-1}(p) &= \not{p} - m_0 - \Sigma(p) \\ &\cong Z_2^{-1}(\not{p} - m_0 - \delta m_0), \end{aligned} \quad (5)$$

we read off the mass renormalization at zero temperature

$$\delta m_0 = -\frac{\alpha}{4\pi} \left(\frac{3}{\varepsilon} - 4 \right) m_R \quad (6)$$

and the wavefunction renormalization constant

$$Z_2^{-1} = 1 - \frac{\alpha}{4\pi} \left(\frac{3}{\varepsilon} - 4 \right). \quad (7)$$

One can now rewrite the free Lagrangian as

$$\mathcal{L} = Z_2(\bar{\psi}_R(x)(i\not{\nabla} - m_R)\psi_R(x) + \bar{\psi}_R(x)\delta m_0\psi_R(x)) \quad (8)$$

where

$$\psi_R(x) = \frac{1}{\sqrt{Z_2}} \psi_0(x) \quad (9)$$

is the renormalized field operator and $m_R = m_0 + \delta m$ is the renormalized mass which corresponds to the observed mass of the particle.

The calculation until this point is textbook material and similar derivations of the renormalized vertex can be found in numerous sources [8]. Now, however, consider what happens when the equivalent calculations is performed at nonzero temperature. We imagine our fermion to be situated in a heated vacuum (not a plasma) so that the chemical potential vanishes. In this circumstance the particle propagators must be modified to take into account the effects of motion through this background heat bath. Thus, for example, in the “real time” formulation of finite temperature field theory, the photon propagator becomes [9] (in Feynman gauge)

$$D_{\mu\nu}(k) = g_{\mu\nu} \left(-\frac{i}{k^2 + i\varepsilon} - 2\pi\delta(k^2) n_B(k) \right) \quad (10)$$

with $n_B(k)$ being the Bose–Einstein distribution function

$$n_B(k) = \frac{1}{e^{\beta k} - 1}. \quad (11)$$

This latter term in $D_{\mu\nu}(k)$ represents the emission and absorption of real photons, which are present in the heat bath.

It is also possible to calculate finite temperature effects in the so-called imaginary time formalism, wherein the propagators are described in terms of a discrete sum over an imaginary "energy" variable rather than an integration over the real k_0 axis. Thus the photon propagator can be written as

$$D_{\mu\nu}(x) = 2T \int \frac{d^3k}{(2\pi)^3} \sum_{n=-\infty}^{\infty} e^{i(\omega_n t - \mathbf{k} \cdot \mathbf{x})} \frac{-ig_{\mu\nu}}{\omega_n^2 - \mathbf{k}^2} \quad (12)$$

where $\omega_n = i2\pi nT$ (recall we are employing units where the Boltzmann constant is set equal to unity). The sum over n may then be performed by noting that

$$\frac{1}{2T} \int_c \frac{dz}{2\pi i} \operatorname{ctnh} \frac{z}{2T} f(z) = \sum_n f(2\pi i n T) \quad (13)$$

where the contour encloses the poles in an obvious manner. Thus we find

$$\begin{aligned} D_{\mu\nu}(x) &= \int \frac{d^3k}{(2\pi)^3} \int_c \frac{dz}{2\pi i} \operatorname{ctnh} \frac{z}{2T} e^{i(zt - \mathbf{k} \cdot \mathbf{x})} \frac{-ig_{\mu\nu}}{z^2 - \mathbf{k}^2} \\ &= \int \frac{d^3k}{(2\pi)^3} \left(\operatorname{ctnh} \frac{|\mathbf{k}|}{2T} e^{i|\mathbf{k}|t - i\mathbf{k} \cdot \mathbf{x}} + \operatorname{ctnh} \frac{-|\mathbf{k}|}{2T} e^{-i|\mathbf{k}|t - i\mathbf{k} \cdot \mathbf{x}} \right) \frac{1}{2|\mathbf{k}|}. \end{aligned} \quad (14)$$

But since

$$\operatorname{ctnh} \frac{|\mathbf{k}|}{2T} = 1 + 2 \frac{1}{\exp(|\mathbf{k}|/T) - 1} \quad (15)$$

we can write this as

$$D_{\mu\nu}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2|\mathbf{k}|} (1 + 2n_B(k)) (e^{i|\mathbf{k}|t - i\mathbf{k} \cdot \mathbf{x}} + e^{-i|\mathbf{k}|t - i\mathbf{k} \cdot \mathbf{x}}) \quad (16)$$

which is identical to the result obtained using the real-time technique. Since the real-time formalism explicitly separates the $T=0$ and $T \neq 0$ components and is considerably simpler to use, we shall perform our calculations in real time. However, reversion to the imaginary-time technique is sometimes required to properly define certain integrals, as we shall see.

We shall at first restrict our attention to low temperature— $T \ll m_e$. Since the mass of an electron is about 10^{10}K , while the temperature at the center of even a very hot star is no higher than several times 10^7K , this situation nearly always obtains except in the *very* early universe ($t \gtrsim 10 \text{ s}$). A convenient feature of this limit is that although in principle both photon and fermion propagators are effected

by temperatures, the fermion terms are suppressed with respect to those of the photon by factors of order

$$\exp\left(-\frac{m_e}{T}\right)$$

and can thus be neglected. It is then straightforward (in the real-time formalism) to evaluate the finite temperature corrections to the self-energy calculated previously. In fact, because of the Bose-Einstein distribution function there exist no additional ultraviolet divergences and dimensional regularization is not required [10]. Thus we find

$$\Sigma(E, \mathbf{p}) = \Sigma_{T=0}(E, \mathbf{p}) + \Sigma_\beta(E, \mathbf{p}) \quad (17)$$

where

$$\begin{aligned} \Sigma_\beta(E, \mathbf{p}) = & \frac{\alpha}{4\pi^2} [I_A(\not{p} - m_R) + I|_{\not{p}=m_R} + L|_{\not{p}=m_R}(p^2 - m_R^2) + \cdots] \\ & + O(\exp(-m/T)). \end{aligned} \quad (18)$$

Here

$$\begin{aligned} I_A &= 8\pi \int \frac{dk}{k} n_B(k) \\ I_\mu &= 2 \int \frac{d^3k}{k_0} n_B(k) \frac{(k_0, \mathbf{k})}{E_p k_0 - \mathbf{p} \cdot \mathbf{k}} = \frac{2\pi^3}{3} \frac{T^2}{p} \left(\ln \frac{E_p + p}{E_p - p}, \hat{p} \left(\frac{E_p}{p} \ln \frac{E_p + p}{E_p - p} - 2 \right) \right) \\ L_\mu &= -\frac{1}{E_p} \int d^3k n_B(k) \frac{(k_0, \mathbf{k})}{(E_p k_0 - \mathbf{p} \cdot \mathbf{k})^2} \\ &= -\frac{\pi^3}{3\omega_p} T^2 \left(\frac{2}{m_R^2}, \hat{p} \left(\frac{2E_p}{m_R p} - \frac{1}{p^2} \ln \frac{E_p + p}{E_p - p} \right) \right) \end{aligned} \quad (19)$$

with

$$E_p = \sqrt{p^2 + m_R^2}.$$

Thus the standard decomposition into a Lorentz invariant mass shift δm and a wavefunction renormalization proportional to $\not{p} - m$ is not obtained. Lacking such covariance one must be very cautious in defining what is meant by the term "mass."

One possible definition is provided by the location of the pole in the propagator. This occurs when

$$E^2 = \mathbf{p}^2 + m_R^2 + 2p \cdot I \frac{\alpha}{4\pi^2} = \mathbf{p}^2 + m_R^2 + \frac{2\pi\alpha T^2}{3} \equiv \mathbf{p}^2 + m_p^2(T) \quad (20)$$

which determines the mass shift to be

$$\delta m_\beta = \frac{\pi \alpha T^2}{3m_R}. \quad (21)$$

The feature that $p \cdot I$ is independent of momentum is somewhat of a surprise. In principle one would expect some dispersion and this does indeed show up when the finite temperature corrections due to the fermion propagator are considered. In any case, we shall refer to $m_p(T) = m_R + \delta m_\beta$ as the “phase space mass” since it is associated directly with the kinematics of the particle and can be given a clear operational definition in terms of threshold and phase space behavior for particle reactions. For example, the decay of a neutral boson (H^0) into an e^+e^- pair cannot take place if the H^0 mass is below $2m_p(T)$ —even if m_{H^0} is larger than $2m_R$. One can thus imagine measuring the phase space mass by examining the threshold of various reactions. It could also be determined via very careful study of phase space distributions of a particular reaction. (Both techniques are currently being utilized in the search for the possible existence of neutrino mass.) In principle, the phase space mass could be distinct from either the inertial or gravitational masses.² As an aside, we note that a fourth definition of mass is sometimes utilized in particle physics studies of chiral symmetry breaking. This “chiral” mass is defined to be the component of the inverse propagator which commutes with the Dirac matrix γ_5 ($\not{p}$ anticommutes with γ_5). At finite temperature this chiral mass is equal to the renormalized $T=0$ mass m_R and *not* the phase space mass $m_p(T)$. At $T=0$ all four definitions coincide, but this is *not* the case at finite temperature and one must be careful to specify which definition of the mass is being employed.

In defining the kinematics and propagation of a particle at $T=0$, one uses spinors containing the renormalized mass. At finite temperature the characterization of the kinematics is more complicated, due to the lack of Lorentz covariance of the self energy. We follow Ref. [5] and introduce “finite temperature spinors,” which satisfy

$$[\not{p} - m_R - \Sigma_\beta(p)] u_\beta(p) = 0 \quad (22)$$

with the normalization

$$u_\beta^+(p) u_\beta(p) = 1. \quad (23)$$

For notational convenience, we define

$$\tilde{p} - \tilde{m} \equiv \not{p} - m_R - \Sigma_\beta(p), \quad (24)$$

which leads to a spinor projection operator of the form

$$\sum_{\text{spin}} u_\beta(p) \bar{u}_\beta(p) = \frac{\tilde{p} + \tilde{m}}{2\tilde{E}}. \quad (25)$$

² The gravitational and inertial masses are discussed in Ref. [11].

These spinors can now be used to define the wavefunction renormalization constant. The renormalized propagator is expanded in terms of these solutions to yield,

$$S^R(x-y) = \int \frac{d^3p}{(2\pi)^3} \left[\theta(x_0 - y_0) \frac{\not{p} + \not{m}}{2\tilde{E}} e^{-ip \cdot (x-y)} - \theta(y_0 - x_0) \frac{\not{p} - \not{m}}{2\tilde{E}} e^{ip \cdot (x-y)} \right]. \quad (26)$$

The definition of the wavefunction renormalization constant is such that this should be equal to

$$\begin{aligned} S^R(x-y) &= -i \int \frac{d^4p}{(2\pi)^4} \frac{Z_2^{-1} e^{-ip \cdot (x-y)}}{\not{p} - m_R - \Sigma_\beta(p) + i\epsilon} \\ &= -i \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot (x-y)} \frac{Z_2^{-1} (\not{p} + \not{m})}{\tilde{p}^2 - \tilde{m}^2 + i\epsilon}. \end{aligned} \quad (27)$$

At low temperatures we find

$$\tilde{p}^2 - \tilde{m}^2 + i\epsilon = \left(1 - \frac{\alpha}{4\pi^2} I_A \right)^2 \left(p^2 - m_R^2 - \frac{2\pi\alpha T^2}{3} + i\epsilon \right). \quad (28)$$

The integral over p_0 can be easily performed, yielding

$$\begin{aligned} S^R(x-y) &= \int \frac{d^3p}{(2\pi)^3} \frac{Z_2^{-1}}{(1 - (\alpha/4\pi^2) I_A)^2} \\ &\times \left[\theta(x_0 - y_0) \frac{\not{p} + \not{m}}{2E} e^{-ip \cdot (x-y)} + \theta(y_0 - x_0) \frac{-\not{p} + \not{m}}{2E} e^{ip \cdot (x-y)} \right]. \end{aligned} \quad (29)$$

Finally the relation

$$\tilde{E} = E \left(1 - \frac{\alpha}{4\pi^2} (I_A + I_0/E) \right) \quad (30)$$

yields

$$Z_2^{-1} = \left(1 - \frac{\alpha}{4\pi^2} I_A + \frac{\alpha}{4\pi^2} I_0/E \right) \quad (31)$$

which is the form of the wavefunction renormalization to be employed at low temperatures.

In a more general context we can write

$$\Sigma_\beta(p) = B(p) + C(p)(\not{p} - m) + D(p) \quad (32)$$

where B , C , D are general (and not necessarily covariant) functions of E and $\mathbf{p}$. The applications of the above procedure leads then to a mass shift

$$m^2 = m_0^2 + 2m \left(B + \frac{1}{m} \mathbf{p} \cdot D \right) \quad (33)$$

and the wavefunction renormalization constant

$$Z_2^{-1} = \left(1 - C - \frac{m}{E} \frac{d}{dE} \left(B + \frac{p \cdot D}{m} \right) + D_0/E \right). \quad (34)$$

Note that the term involving differentiation with respect to the energy arises in doing the p_0 integration in those cases where $\hat{p}^2 - \hat{m}^2$ depends nontrivially on p_0 .

At high temperatures ($T > m_e$) the fermion heat bath must also be accounted for in the self energy. The fermion propagator is modified to become

$$S_\beta(p) = \frac{i}{\not{p} - m + i\epsilon} - 2\pi\delta(p^2 - m^2)(\not{p} + m) n_F(E_p) \quad (35)$$

where

$$n_F(E_p) = \frac{1}{e^{\beta E_p} + 1} \quad (36)$$

is the Fermi-Dirac distribution function. The self-energy becomes then

$$\Sigma_\beta(p) = \frac{\alpha}{4\pi^2} [I_A(\not{p} - m) + I|_{\not{p}=m} + L|_{\not{p}=m}(p^2 - m^2) + 2mJ_A + J] \quad (37)$$

where the new integrals J_A and J_μ are defined by

$$J_A = \int \frac{d^3l}{E_l} n_F(E_l) \left[\frac{1}{E_p E_l + ((m^2 + p^2)/2) - \mathbf{p} \cdot \mathbf{l}} - \frac{1}{E_p E_l - ((m^2 + p^2)/2) + \mathbf{p} \cdot \mathbf{l}} \right] \quad (38)$$

$$J_\mu = \int \frac{d^3l}{E_l} n_F(E_l) l_\mu \left[\frac{1}{E_p E_l + ((m^2 + p^2)/2) - \mathbf{p} \cdot \mathbf{l}} + \frac{1}{E_p E_l - ((m^2 + p^2)/2) + \mathbf{p} \cdot \mathbf{l}} \right].$$

The physical mass now contains some dependence on the particles' three momentum through $J_A(p)$ (i.e., there exists dispersion)

$$m_p^2(T) = m_R^2 + \frac{2}{3} \alpha \pi T^2 + \frac{\alpha}{2\pi^2} m_R^2 J_A(p) + \frac{4\alpha}{\pi} \int_0^\infty \frac{l^2 dl}{E_l} \eta_F(E_l). \quad (39)$$

However, at very high temperatures, $T \gg m_e$, $J_A(p)$ becomes small compared to the other corrections, and a Lorentz invariant result is obtained once more

$$m_{\text{phys}}^2 \xrightarrow{T \gg m_e} m_R^2 + \alpha \pi T^2. \quad (40)$$

Finally, the explicit renormalization constant at high temperature can be written as

$$Z_2^{-1} = 1 - \frac{\alpha}{4\pi} \left(\frac{3}{\epsilon} - 4 \right) + \frac{\alpha}{4\pi^2} \left\{ -I_A + 2J_A + \frac{I_0}{E} + \frac{J_0}{E} - \frac{m^2}{E} \frac{dJ_A}{dE} \right\}. \quad (41)$$

At very high temperatures $T \gg m_e$ this becomes

$$Z_2^{-1} \xrightarrow{T \gg m_e} 1 - \frac{\alpha}{4\pi} \left(\frac{3}{\epsilon} - 4 \right) + \frac{\alpha}{4\pi^2} I_A + \frac{\alpha\pi}{4} T^2 \frac{1}{p} \ln \frac{1 + p/E_p}{1 - p/E_p} \quad (42)$$

III. THE ELECTROMAGNETIC VERTEX AT LOW TEMPERATURE

In this section we calculate the vertex corrections and vacuum polarization, first at $T=0$ then when $T \neq 0$. Our goals are to see how the electromagnetic vertex renormalizes, to evaluate the effective charge, and to compute the finite temperature shift in the magnetic moment. We find that for $T \ll m_e$, there is no charge renormalization. The vertex function $L_\mu(p, p')$ will be seen to have a complicated form, but this structure appears to be primarily due to the kinematic modifications to the electron propagation. Use of either the nonrelativistic limit or finite temperature spinors leads to the expected result that the charge vertex is not modified at $q=0$

$$M_\mu = \bar{u}_\beta(p) A_\mu(p, p) u_\beta(p) = \frac{p_\mu}{E}. \quad (43)$$

In discussing the magnetic moment, these kinematic modifications are crucial. At high temperatures, on the other hand, the effective charge will be seen to depend on temperature, as expected from the renormalization group arguments.

Consider first the zeroth order interaction Lagrangian

$$\mathcal{L}_{\text{int}}^{(0)} = -e_0 \bar{\psi}_0 \gamma_\mu \psi_0 A^\mu \quad (44)$$

where $-e_0$ is the (bare) charge of the electron and A^μ is an externally generated electromagnetic field. Radiative corrections, of course, renormalize this vertex. Thus the vertex diagram indicated in Fig. 2d yields

$$\begin{aligned} A_\mu(p', p) \equiv A_\mu(p, p) + A_\mu^{(c)}(p', p) &= -ie^2 \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2} \frac{1}{(k^2 - 2p \cdot k)(k^2 - 2p' \cdot k)} \\ &\times \gamma_\alpha (\not{p}' - \not{k} + m) \gamma_\mu (\not{p} - \not{k} + m) \gamma^\alpha. \end{aligned} \quad (45)$$

Here the piece $A_\mu(p, p)$ can be determined via the Ward identity

$$\bar{u}(p') A_\mu(p, p) u(p) = \bar{u}(p') \frac{\partial}{\partial p^\mu} \Sigma(p) u(p) = \bar{u}(p') \gamma_\mu u(p) \frac{\alpha}{4\pi} \left(\frac{3}{\epsilon} - 4 \right) \quad (46)$$

while $A_\mu^{(c)}(p', p)$ can be calculated directly, as shown in the Appendix, yielding, correct to first order in q/m

$$\bar{u}(p') A_\mu^{(c)}(p', p) u(p) = \frac{\alpha}{2\pi} \bar{u}(p') (-i\sigma_{\mu\nu} q^\nu) \frac{1}{2m_R} u(p) \quad (47)$$

where $q = p - p'$ is the four-momentum transferred to the external field. Then, adding the self-energy and mass renormalization diagrams to the vertex correction and multiplying by $1/\sqrt{Z_2}$ for each electron leg (cf. Fig. 2) we find the results

$$\text{Fig. 2a } M_\mu^{(0)} = -e_0 \bar{u}(p') \gamma_\mu u(p), \quad (48a)$$

$$\text{Fig. 2b } M_\mu^{SE} = -e_0 \bar{u}(p') \left(\gamma_\mu \frac{1}{\not{p} - m} \delta m + \delta m \frac{1}{\not{p} - m} \gamma_\mu - 2(Z_2^{-1} - 1) \gamma_\mu \right) u(p) \quad (48b)$$

$$\text{Fig. 2c } M_\mu^{CT} = +e_0 \bar{u}(p') \left(\gamma_\mu \frac{1}{\not{p} - m} \delta m + \delta m \frac{1}{\not{p} - m} \gamma_\mu \right) u(p) \quad (48c)$$

$$\text{Fig. 2d } M_\mu^\nu = -e_0 \bar{u}(p') \left(\gamma_\mu (Z_2^{-1} - 1) - i \frac{\alpha}{2\pi} \sigma_{\mu\nu} q^\nu \frac{1}{2m} \right) u(p). \quad (48d)$$

Then

$$\begin{aligned} M_\mu &= Z_2^{-1} (M_\mu^{(0)} + M_\mu^{SE} + M_\mu^{CT} + M_\mu^\nu) \\ &= -e_0 \bar{u}(p') \left(\gamma_\mu - \frac{\alpha}{2\pi} i \sigma_{\mu\nu} q^\nu \frac{1}{2m} \right) u(p) + O\left(\frac{q^2}{m^2}\right). \end{aligned} \quad (49)$$

Thus, to this order, we find the familiar result that the charge vertex is unrenormalized. However, the electron picks up an anomalous magnetic moment $\alpha/2\pi$.

FIG. 2. The diagrams used to renormalize the electromagnetic vertex.

FIG. 3. The vacuum polarization diagram.

Finally, we must also calculate the vacuum polarization diagram—Fig. 3. Again dimensionally regularizing we find at zero temperature, as shown in the Appendix

$$\Pi_{\mu\nu}(q) = (q^2 g_{\mu\nu} - q_\mu q_\nu) \frac{\alpha}{3\pi\epsilon} (1 + \cdots). \quad (50)$$

For the vacuum polarization contribution to the vertex—Fig. 4—we find then

$$M_\mu^{VP} = e_0 \bar{u}(p') \gamma_\mu u(p) \frac{\alpha}{3\pi\epsilon} \equiv e_0 \bar{u}(p') \gamma_\mu u(p) (Z_3 - 1) \quad (51)$$

so that the full renormalized vertex becomes, including now a factor $Z_3^{-1/2}$ for the external photon

$$\begin{aligned} T=0 M_\mu^{\text{Total}} &= Z_2^{-1} Z_3^{-1/2} (M_\mu^{(0)} + M_\mu^{SE} + M_\mu^{CT} + M_\mu^V + M_\mu^{VP}) \\ &= -e_R \bar{u}(p') \left(\gamma_\mu - i \frac{\alpha}{2\pi} \sigma_{\mu\nu} q^\nu \frac{1}{2m_R} \right) u(p) \end{aligned} \quad (52)$$

where

$$e_R = e_0 \sqrt{Z_3} \simeq e_0 \left(1 - \frac{\alpha}{6\pi\epsilon} \right) \quad (53)$$

is the renormalized charge and corresponds to the charge observed experimentally (at large r).

Having in the last section identified the appropriate mass and wavefunction renormalizations at zero T , we move to consider the temperature-dependent renor-

FIG. 4. The vertex correction due to vacuum polarization.

malizations of the electromagnetic vertex. Here we choose the zeroth-order Lagrangian to satisfy the equation of motion

$$\left(\not{p} - \frac{\alpha}{4\pi^2} \not{I} - m_R \right) \psi_\beta = 0 \quad (54)$$

so that the finite temperature counter-term becomes

$$\delta \mathcal{L} = \frac{\alpha}{4\pi^2} \not{I}(p) \quad (55)$$

and the temperature-dependent counter-term contribution to the electromagnetic vertex becomes

$${}^\beta M_\mu^{CT} = +e\bar{u}(p') \left(\gamma_\mu \frac{1}{\not{p} - m} \frac{\alpha}{4\pi^2} \not{I}(p) + \frac{\alpha}{4\pi^2} \not{I}(p') \frac{1}{\not{p}' - m} \gamma_\mu \right) u(p) \quad (56)$$

while the self-energy diagram (Fig. 4b) yields

$$\begin{aligned} {}^\beta M_\mu^{SE} = & -e\bar{u}(p') \left(\gamma_\mu \left(\frac{1}{\not{p} - m} \not{I}(p) + I_A + (\not{p} + m) \not{L}(p) \right) \right. \\ & \left. + \left(\not{I}(p') \frac{1}{\not{p}' - m} + I_A + \not{L}(p')(\not{p} + m) \right) \gamma_\mu \right) u(p) \frac{\alpha}{4\pi^2}. \end{aligned} \quad (57)$$

Then

$${}^\beta M_\mu^{CT} + {}^\beta M_\mu^{SE} = -e\bar{u}(p') \gamma_\mu u(p) (2I_A + 2p \cdot \not{L}(p) + 2p' \cdot \not{L}(p')) \frac{\alpha}{4\pi^2} \quad (58)$$

and, using Eq. (19) this becomes

$${}^\beta M_\mu^{CT} + {}^\beta M_\mu^{SE} = -2e\bar{u}(p') \gamma_\mu u(p) \frac{\alpha}{4\pi^2} \left(I_A - \frac{1}{2\omega_p} I_0(p) - \frac{1}{2\omega_{p'}} I_0(p') \right). \quad (59)$$

The finite temperature vertex correction (Fig. 2d) is found to be

$$\begin{aligned} {}^\beta A_\mu(p', p) = & e^3 \int \frac{d^4 k}{(2\pi)^4} 2\pi i n_B(k_0) \delta(k^2) \gamma_\alpha (\not{p}' - \not{k} + m) \gamma_\mu (\not{p} - \not{k} + m) \gamma^\alpha \frac{1}{2p \cdot k 2p' \cdot k} \\ = & e^3 \int \frac{d^4 k}{(2\pi)^4} 2\pi i n_B(k_0) \delta(k^2) \\ & \times \left[\frac{m^2 - \frac{1}{2}q^2}{p \cdot k p' \cdot k} \gamma_\mu - \frac{k_\mu}{2p \cdot k} - \frac{k_\mu}{2p' \cdot k} + i\sigma_{\alpha\nu} q^\nu \frac{1}{2m} \frac{k^\alpha k_\mu}{p \cdot k p' \cdot k} \right]. \end{aligned} \quad (60)$$

Then, dropping terms of order q^2/m^2 as before we find

$$\begin{aligned} {}^\beta M_\mu^\nu = & -e\bar{u}(p') \left(-I_A + \frac{1}{2m} I_\mu(p) + \frac{1}{2m} I_\mu(p') \right. \\ & \left. + i\sigma_{\alpha\nu} q^\nu \frac{1}{2m} I_\mu^\alpha(p) \right) u(p) \frac{\alpha}{4\pi^2} + O\left(\frac{q^2}{m^2}\right) \end{aligned} \quad (61)$$

where

$$\frac{\alpha}{4\pi^2} I_{\mu\nu}(p, p') = e^2 \int \frac{d^4 k}{(2\pi)^4} 2\pi i n_B(k_0) \delta(k^2) \frac{k_\mu k_\nu}{(p \cdot k)(p' \cdot k)}. \quad (62)$$

We note also that, to this order, one need not consider the temperature dependent vacuum polarization, as it involves integration only over fermion loops so that its effects are $O(e^{-m_e/T})$.

Finally, we note also that there exists a temperature dependent contribution to the leading order term. Since we have used $(\alpha/4\pi^2) I(p)$ as a counter term the leading order amplitude must be evaluated with finite temperature spinors $u_\beta(p)$ defined in Eq. (22). Using

$$\tilde{p}_\mu \equiv p_\mu - \frac{\alpha}{4\pi^2} I_\mu(p) \quad (63)$$

we have then

$$u_\beta(p) = u(\tilde{p}) \quad (64)$$

so that the leading order amplitude is

$$M_\mu^{(0)} = -e\bar{u}_\beta(p') \gamma_\mu u_\beta(p). \quad (65)$$

Adding all the contributions (Figs. 2a-d), we find then

$$\begin{aligned} M_\mu^{\text{Total}} = & -e_R \bar{u}_\beta(p') \left[\gamma_\mu \left(1 - \frac{\alpha}{4\pi^2} \left(\frac{1}{2\omega_p} I_0(p) + \frac{1}{2\omega_{p'}} I_0(p') \right) \right) \right. \\ & + \frac{\alpha}{4\pi^2} \left(\frac{1}{2m_p(T)} I_\mu(p) + \frac{1}{2m_{p'}(T)} I_\mu(p') \right) \\ & \left. - i\sigma_{\alpha\nu} q^\nu \frac{1}{2m_p(T)} \left(\frac{\alpha}{2\pi} g_{\alpha\mu} + \frac{\alpha}{4\pi^2} I_{\alpha\mu}(p) \right) \right] u_\beta(p) \\ \equiv & -e_R \bar{u}_\beta(p') V_\mu(p', p) u_\beta(p), \end{aligned} \quad (66)$$

where E_β is related to the momentum $\mathbf{p}$ via

$$E_\beta = \sqrt{\mathbf{p}^2 + m_p^2(T)}. \quad (67)$$

The noncovariant nature of the finite temperature electromagnetic vertex is obvious. It comes about because of the existence of a preferred frame—the one in which the black body spectrum falling on particle is isotropic. This lack of covariance implies that one must be very cautious in analyzing the meaning of our results.

We begin by confirming that our vertex is gauge invariant. Indeed we note that since

$$p u_\beta(p) = \left(m_R + \frac{\alpha}{4\pi^2} I(p) \right) u_\beta(p) \quad \text{and} \quad p^\mu I_\mu(p) = \text{const. independent of } p \quad (68)$$

we have

$$\begin{aligned} q^\mu {}^\beta M_\mu^{\text{Total}} = & -e_R \bar{u}_\beta(p') \frac{\alpha}{4\pi^2} \left(I(p) - I(p') - \frac{1}{2m_p(T)} p^\mu I_\mu(p') \right. \\ & \left. + \frac{1}{2m_p(T)} p'^\mu I_\mu(p) + i\sigma_{\alpha\nu} q^\nu \frac{1}{2m_p(T)} I^{\alpha\mu}(p, p') q_\mu \right) u_\beta(p). \end{aligned} \quad (69)$$

Using the Gordon decomposition this becomes

$$\begin{aligned} q^\mu {}^\beta M_\mu^{\text{Total}} = & -e_R \bar{u}_\beta(p') \frac{\alpha}{4\pi^2} \left(-i\sigma_{\mu\nu} q^\nu \frac{1}{2m_p(T)} \right) \\ & \times (I^\mu(p) - I^\mu(p') + q^\alpha I_\alpha^\mu(p, p')) u_\beta(p) \\ = & 0 \end{aligned} \quad (70)$$

so that gauge invariance is indeed satisfied.

Now examine the vertex function at $q = 0$. In this case we find

$${}^\beta M_\mu^{\text{Total}}(p, p) = -e_R \bar{u}_\beta(p) \left(\gamma_\mu \left(1 - \frac{\alpha}{4\pi^2} \frac{1}{\omega_p} I_0(p) \right) + \frac{\alpha}{4\pi^2} \frac{I_\mu(p)}{m_p(T)} \right) u_\beta(p). \quad (71)$$

This appears to be a complicated renormalization of the vertex. However, it simplifies dramatically if one uses finite-temperature spinors. Normalizing our spinors so that (cf. Eq. (23))

$$\bar{u}_\beta(p) \gamma_\mu u_\beta(p) = \frac{\tilde{p}_\mu}{\tilde{E}} \quad (72)$$

this becomes

$$\begin{aligned} {}^\beta M_\mu^{\text{total}} = & -e_R \left(\frac{\tilde{p}_\mu}{\tilde{E}} \left(1 - \frac{\alpha}{4\pi^2} \frac{1}{E} I_0(p) \right) + \frac{1}{\tilde{E}} I_\mu(p) \frac{\alpha}{4\pi^2} \right) \\ = & -e_R \frac{P_\mu}{E_R} \end{aligned} \quad (73)$$

which is identical to the result obtained at $T=0$ except that now the energy E and momentum $\mathbf{p}$ are related via the T -dependent phase space mass $m_p(T)$.

$$E_\beta = \sqrt{\mathbf{p}^2 + m_p^2(T)}. \quad (74)$$

Thus the time-component gives

$${}^\beta M_0^{\text{Total}} = -e_R \quad (75)$$

so that the charge is unrenormalized by finite temperature effects when $T \ll m_e$. The unexpected terms in the vertex only compensate for the electron's kinematic behavior in the heat bath. This will be demonstrated again below by use of a non-relativistic reduction. To understand the implications of the finite temperature vertex away from $q=0$, it is convenient to make a nonrelativistic reduction. Thus, our equation of motion is

$$\begin{aligned} & \left(\not{p} - \frac{\alpha}{4\pi^2} I(p) - e_R A \left(1 - \frac{\alpha}{4\pi^2} \left(\frac{1}{\omega_p} I_0(p) \right) \right) \right) \\ & + \frac{\alpha}{4\pi^2} \frac{1}{m_R} A^\mu I_\mu(p) + \sigma_{\mu\nu} \frac{1}{2m} \left(\partial^\nu A^\mu \frac{\alpha}{2\pi} + \frac{\alpha}{4\pi^2} \partial^\nu A^\alpha I_\alpha^\mu(p) \right) - m_R \Big) u_\beta(p) = 0. \end{aligned} \quad (76)$$

Using the nonrelativistic approximation ($p \ll E$)

$$\begin{aligned} \frac{\alpha}{4\pi^2} I_\mu & \approx \frac{\pi}{3} T^2 \alpha \left(\frac{1}{m} - \frac{1}{6} \frac{p^2}{m^3}, \mathbf{p} \frac{1}{3m^2} \right) = \frac{\delta m}{m} \left(1 - \frac{1}{6} \frac{p^2}{m^2}, \frac{1}{3} \frac{\mathbf{p}}{m} \right), \\ \frac{\alpha}{4\pi^2} I_{\mu\nu} & \approx (\mu = \nu) \frac{\pi T^2 \alpha}{9m^2} (-g_{\mu\nu} + 4\delta_{\mu 0} \delta_{\nu 0}) = \delta m / 3m (-g_{\mu\nu} + 4\delta_{\mu 0} \delta_{\nu 0}), \\ & \approx_{\substack{\mu=i, \nu=0 \\ \nu=i, \mu=0}} \frac{\pi T^2 \alpha}{9m^2} \frac{(\mathbf{p} + \mathbf{p}')_i}{m} = \frac{\delta m}{3m} \frac{(\mathbf{p} + \mathbf{p}')_i}{m} \end{aligned} \quad (77)$$

so that Eq. 75 becomes

$$\begin{aligned} & \left[\gamma_0 \left(i \frac{\partial}{\partial t} - \delta m \left(1 - \frac{1}{6} \frac{p^2}{m^2} \right) - e_R \phi \left(1 - \frac{\delta m}{m} \right) \right) \right. \\ & \quad \left. - \boldsymbol{\gamma} \cdot \left(\mathbf{p} \left(1 - \frac{1}{3} \frac{\delta m}{m} \right) - e_R \mathbf{A} \left(1 - \frac{\delta m}{m} \right) \right) \right. \\ & \quad \left. - \frac{e_R}{m_R} \frac{\delta m}{m} \left(\dot{\phi} - \frac{1}{3} \frac{\mathbf{p}}{m} \cdot \mathbf{A} \right) - m_R + e_R \frac{1}{2m} \boldsymbol{\sigma} \cdot \mathbf{B} \left(\frac{\alpha}{2\pi} - \frac{\delta m}{3m} \right) \right] u_\beta = 0. \end{aligned} \quad (78)$$

Now rewrite the wavefunction in terms of two component spinors

$$u_\beta(p) = \begin{pmatrix} u \\ l \end{pmatrix} \quad (79)$$

and solve for the leading (upper component) yielding

$$\begin{aligned}
 i \frac{\partial}{\partial t} u &= \left(m_R + \delta m + \frac{p^2}{2(m_R)} \left(1 - \frac{\delta m}{m_R} \right) - e_R \frac{\mathbf{A} \cdot \mathbf{p}}{m_R} \left(1 - \frac{\delta m}{m} \right) + e_R \phi \right. \\
 &\quad \left. - e_R \frac{1}{2m_R} \boldsymbol{\sigma} \cdot \mathbf{B} \left(1 - \frac{5}{3} \frac{\delta m}{m} + \frac{\alpha}{2\pi} \right) \right) u \\
 &\approx \left(m_p(T) + \frac{p^2}{2m_p(T)} - e_R \frac{\mathbf{A} \cdot \mathbf{p}}{m_p(T)} + e_R \phi - \frac{e_R}{2m_p(T)} \right. \\
 &\quad \left. \times \boldsymbol{\sigma} \cdot \mathbf{B} \left(1 - \frac{2}{3} \frac{\delta m}{m} + \frac{\alpha}{2\pi} \right) \right) u.
 \end{aligned} \tag{80}$$

We can then identify the inertial mass by measuring the acceleration of a particle acted upon by an external electric field $\mathbf{E}$

$$\mathbf{a} = -[H, [H, \mathbf{r}]] = \frac{-e\mathbf{E}}{m_R + \delta m_\beta} \quad \text{or} \quad m_I = m_R + \delta m_\beta = m_p(T). \tag{81}$$

Thus, we see that the inertial mass is renormalized to become the phase space mass $m_p(T)$, while the anomalous moment, expressed in finite temperature (phase space mass) magnetons $e_R/2m_p(T)$ is given by

$$\mu_a = \frac{\alpha}{2\pi} - \frac{2}{3} \frac{\delta m}{m_R}. \tag{82}$$

This appears to agree with the result of Fujimoto and Yee (although these authors did not specify which magnetons they were using) and disagrees with the calculations of Peressutti and Skagerstam and Palanques-Mestre and Tarrach [7]. The identification of the finite temperature anomalous moment can be most easily understood by placing the particle in a uniform magnetic field described by the vector potential

$$\mathbf{A} = \frac{1}{2} \mathbf{B} \times \mathbf{r}. \tag{83}$$

Then the interaction potential becomes

$$-\frac{e_R}{2m_p(T)} \mathbf{B} \cdot (\mathbf{L} + 2\mathbf{S}(1 + \mu_a)) \tag{84}$$

so that a standard $g-2$ experiment which measures the spin precession relative to its orbital counterpart is indeed sensitive only to μ_a .

Thus we have developed a fairly complete picture of the quantum mechanics of a nonrelativistic electron which finds itself immersed in a heat bath with temperature

$T \ll m_e$. Due to interactions with the photons of the heat bath the electron finds its inertial mass increased by an amount

$$\delta m_\beta = \frac{1}{3} \frac{\alpha \pi T^2}{m_R} \quad (85)$$

while its magnetic moment is changed by an amount

$$\delta \mu_\beta = -\frac{2}{3} \frac{\delta m}{m_R} \quad (86)$$

measured in finite temperature magnetons. There is no renormalization of the electron charge

$$\delta e_\beta = 0. \quad (87)$$

IV. RELATIVISTIC MOTION IN ELECTROMAGNETIC FIELDS

It is also possible to discuss the interaction of a relativistic electron at finite temperature under the influence of an external electromagnetic field. However, the analysis is complicated by the noncovariance of the wave equation, and we shall employ a Foldy-Wouthuysen transformation to bring out the form of the relativistic interaction.

First, to simplify things and to introduce notation, we suppose the absence of an electromagnetic field— $A_\mu = 0$ —and assume $T = 0$. The wavefunction then satisfies

$$(\not{p} - m_R)\psi = 0 \quad (88)$$

so that the Hamiltonian is given by

$$i \frac{\partial}{\partial t} \psi = H\psi = [\boldsymbol{\alpha} \cdot \mathbf{p} + \gamma_0 m_R] \psi, \quad (89)$$

where $\boldsymbol{\alpha} = \gamma_0 \boldsymbol{\gamma}$ are the usual Dirac matrices. Defining

$$\rho_1 = -\gamma_5, \quad \rho_2 = i\gamma_0 \gamma_5, \quad \rho_3 = \gamma_0 \quad (90)$$

this can be written as

$$H = \rho_1 \boldsymbol{\sigma} \cdot \mathbf{p} + \rho_3 m_R. \quad (91)$$

We note that the matrices ρ_i obey the algebra

$$\rho_i \rho_j = \delta_{ij} + i\epsilon_{ijk} \rho_k \quad (92)$$

so that we can define an equivalent Hamiltonian H' via a simple unitary transformation

$$\begin{aligned} H' &= \exp\left(-i\frac{1}{2}\phi\rho_2\right) H \exp\left(i\frac{1}{2}\phi\rho_2\right) \\ &= \left(\cos\frac{\phi}{2} - i\rho_2 \sin\frac{\phi}{2}\right) H \left(\cos\frac{\phi}{2} + i\rho_2 \sin\frac{\phi}{2}\right) \\ &= \rho_1(\boldsymbol{\sigma} \cdot \mathbf{p} \cos\phi + m_R \sin\phi) + \rho_3(-\boldsymbol{\sigma} \cdot \mathbf{p} \sin\phi + m_R \cos\phi) \end{aligned} \quad (93)$$

(where we have assumed that $\boldsymbol{\sigma} \cdot \mathbf{p}$ commutes with $\cos(\phi/2)$, $\sin(\phi/2)$). Now pick

$$\tan\phi = -\frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{m_R} \quad (94)$$

to eliminate the off-diagonal term ρ_1 . Then

$$H' = \rho_3 \sqrt{(\boldsymbol{\sigma} \cdot \mathbf{p})^2 + m_R^2} = \rho_3 \sqrt{p^2 + m_R^2} \quad (95)$$

as expected.

Now try the same problem in the case that $T \neq 0$ but still assuming the absence of any external electromagnetic fields— $A_\mu = 0$. The Hamiltonian then becomes

$$\begin{aligned} H &= \boldsymbol{\alpha} \cdot \left(\mathbf{p} - \frac{\boldsymbol{\alpha}}{4\pi^2} \mathbf{I}\right) + \gamma_0 m_R + \frac{\boldsymbol{\alpha}}{4\pi^2} I_0 \\ &= \rho_1 \boldsymbol{\sigma} \cdot \left(\mathbf{p} - \frac{\boldsymbol{\alpha}}{4\pi^2} \mathbf{I}\right) + \rho_3 \left(m_R + \frac{\boldsymbol{\alpha}}{4\pi^2} I_0\right) + \frac{\boldsymbol{\alpha}}{4\pi^2} I_0 (1 - \rho_3). \end{aligned} \quad (96)$$

Now rotate this Hamiltonian as before but with

$$\tan\phi = \frac{\boldsymbol{\sigma} \cdot (\mathbf{p} - (\boldsymbol{\alpha}/4\pi^2) \mathbf{I})}{m_R + (\boldsymbol{\alpha}/4\pi^2) I_0}. \quad (97)$$

Here $\mathbf{I}$, I_0 are functions of $\mathbf{p}$ and ε_p , where ε_p is an eigenvalue of the full Hamiltonian. However, to the extent that we are working only to first order in the mass shift δm_β , we can write

$$I_\mu(\mathbf{p}, \varepsilon_p) = I_\mu(\mathbf{p}, E_p) + O(\delta m_\beta^2), \quad (98)$$

where

$$E_p = \sqrt{p^2 + m_R^2} \quad (99)$$

is the usual $T=0$ energy. We then have

$$\begin{aligned} H' &= \exp\left(-i\rho_2 \frac{\phi}{2}\right) H \exp\left(i\rho_2 \frac{\phi}{2}\right) \\ &\times \rho_3 \sqrt{\left(\boldsymbol{\sigma} \cdot \left(\mathbf{p} - \frac{\boldsymbol{\alpha}}{4\pi^2} \mathbf{I}\right)\right)^2 + \left(m_R + \frac{\boldsymbol{\alpha}}{4\pi^2} I_0\right)^2} + \frac{\boldsymbol{\alpha}}{4\pi^2} I_0 (1 - \rho_3 \cos\phi - \rho_1 \sin\phi). \end{aligned} \quad (100)$$

Dropping the off-diagonal term in ρ_1 and keeping only the dominant (upper) component of the wavefunction we find

$$H' \simeq \sqrt{\left(\boldsymbol{\sigma} \cdot \left(\mathbf{p} - \frac{\alpha}{4\pi^2} \mathbf{I}\right)\right)^2 + \left(m_{\mathbf{R}} + \frac{\alpha}{4\pi^2} I_0\right)^2} + \frac{\alpha}{4\pi^2} I_0 (1 - \cos \phi). \quad (101)$$

Then defining

$$F(x) = \frac{1}{x} \ln \frac{1+x}{1-x} \quad (102)$$

we can write

$$\begin{aligned} \frac{\alpha}{4\pi^2} \mathbf{I} &= \mathbf{p} \frac{m_{\mathbf{R}} \delta m_{\beta}}{2p^2} \left(F\left(\frac{p}{E_{\mathbf{p}}}\right) - 2 \right), \\ \frac{\alpha}{4\pi^2} I_0 &= \frac{m_{\mathbf{R}} \delta m_{\beta}}{2E_{\mathbf{p}}} F\left(\frac{p}{E_{\mathbf{p}}}\right). \end{aligned} \quad (103)$$

Keeping only terms linear in δm_{β} we have

$$\begin{aligned} H' &= \left[p^2 + m_{\mathbf{R}}^2 - \delta m_{\beta} m_{\mathbf{R}} \left(F\left(\frac{p}{E_{\mathbf{p}}}\right) - 2 \right) + m_{\mathbf{R}}^2 \frac{\delta m_{\beta}}{E_{\mathbf{p}}} F\left(\frac{p}{E_{\mathbf{p}}}\right) \right]^{1/2} \\ &\quad + \frac{m_{\mathbf{R}} \delta m_{\beta}}{2E_{\mathbf{p}}} F\left(\frac{p}{E_{\mathbf{p}}}\right) (1 - \cos \phi) \\ &\cong \left[p^2 + m_{\mathbf{R}}^2 + \delta m_{\beta} m_{\mathbf{R}} \left(F\left(\frac{p}{E_{\mathbf{p}}}\right) (1 - \cos \phi) + \frac{m_{\mathbf{R}}}{E_{\mathbf{p}}} F\left(\frac{p}{E_{\mathbf{p}}}\right) - \left(F\left(\frac{p}{E_{\mathbf{p}}}\right) - 2 \right) \right) \right]. \end{aligned} \quad (104)$$

But

$$\cos \phi = \frac{1}{\sqrt{1 + \tan^2 \phi}} = \left(1 + \left(\boldsymbol{\sigma} \cdot \left(\mathbf{p} - \frac{\alpha}{4\pi^2} \mathbf{I} \right) \right)^2 / \left(m_{\mathbf{R}} + \frac{\alpha}{4\pi^2} I_0 \right)^2 \right)^{-1/2} \quad (105)$$

so

$$\cos \phi = \frac{m_{\mathbf{R}}}{E_{\mathbf{p}}} + O(\delta m_{\beta}) \quad (106)$$

We find then

$$\begin{aligned} H' &= [p^2 + m_{\mathbf{R}}^2 + 2m_{\mathbf{R}} \delta m_{\beta}]^{1/2} \\ &\cong \sqrt{p^2 + m_{\mathbf{p}}^2(T)} \end{aligned} \quad (107)$$

so that the inertial mass is indeed $m_{\mathbf{p}}(T)$, as demonstrated earlier in the non-relativistic limit.

Now consider the modifications arising when an external electromagnetic field A_μ is present. As before, we first consider the $T=0$ case and, for simplicity, suppose that only a uniform magnetic field is involved, so we may take $A_0=0$. The Hamiltonian then becomes, including the $T=0$ radiative correction [12]

$$\begin{aligned} H &= \boldsymbol{\alpha} \cdot (\mathbf{p} - e\mathbf{A}) + \gamma_0 m_R - \frac{\alpha}{2\pi} \frac{e}{2m_R} \gamma_0 \boldsymbol{\sigma} \cdot \mathbf{B} \\ &= \rho_1 \boldsymbol{\sigma} \cdot (\mathbf{p} - e\mathbf{A}) + \rho_3 m_R - \frac{\alpha}{2\pi} \frac{e}{2m_R} \rho_3 \boldsymbol{\sigma} \cdot \mathbf{B}. \end{aligned} \quad (108)$$

Performing a rotation as before but with

$$\tan \phi = -\frac{\boldsymbol{\sigma} \cdot \boldsymbol{\pi}}{m_R}, \quad \text{where } \boldsymbol{\pi} = \mathbf{p} - e\mathbf{A} \quad (109)$$

we find

$$\begin{aligned} H' &= \exp(-i\frac{1}{2}\phi\rho_2) H \exp(i\frac{1}{2}\phi\rho_2) \\ &\cong \rho_3 \left[\sqrt{(\boldsymbol{\sigma} \cdot \boldsymbol{\pi})^2 + m_R^2} - \frac{\alpha}{2\pi} \frac{e}{2m_R} \left(\cos \frac{\phi}{2} \boldsymbol{\sigma} \cdot \mathbf{B} \cos \frac{\phi}{2} - \sin \frac{\phi}{2} \boldsymbol{\sigma} \cdot \mathbf{B} \sin \frac{\phi}{2} \right) \right] \\ &\quad + \text{off-diagonal terms.} \end{aligned} \quad (110)$$

Now $\cos \phi/2$, being even in ϕ , is a function only of $(\boldsymbol{\sigma} \cdot \boldsymbol{\pi})^2 = \pi^2$. On the other hand $\sin \phi/2$, being odd in ϕ , can be written as $\boldsymbol{\sigma} \cdot \boldsymbol{\pi}$ times a function of π^2 . Thus we have

$$\begin{aligned} \cos \frac{\phi}{2} \boldsymbol{\sigma} \cdot \mathbf{B} \cos \frac{\phi}{2} &= \boldsymbol{\sigma} \cdot \mathbf{B} \cos^2 \frac{\phi}{2} \\ \sin \frac{\phi}{2} \boldsymbol{\sigma} \cdot \mathbf{B} \sin \frac{\phi}{2} &= \boldsymbol{\sigma} \cdot \boldsymbol{\pi} \frac{1}{|\boldsymbol{\pi}|} \boldsymbol{\sigma} \cdot \mathbf{B} \boldsymbol{\sigma} \cdot \boldsymbol{\pi} \frac{1}{|\boldsymbol{\pi}|} \sin^2 \frac{\phi}{2} \\ &= \left(2\boldsymbol{\sigma} \cdot \boldsymbol{\pi} \boldsymbol{\pi} \cdot \mathbf{B} \frac{1}{|\boldsymbol{\pi}|^2} - \boldsymbol{\sigma} \cdot \mathbf{B} \right) \sin^2 \frac{\phi}{2} \end{aligned} \quad (111)$$

so that

$$\begin{aligned} H' &= \rho_3 \left[\sqrt{(\boldsymbol{\sigma} \cdot \boldsymbol{\pi})^2 + m_R^2} - \frac{\alpha}{2\pi} \frac{e}{2m_R} \left(\boldsymbol{\sigma} \cdot \mathbf{B} - \boldsymbol{\sigma} \cdot \boldsymbol{\pi} \boldsymbol{\pi} \cdot \mathbf{B} \frac{1}{|\boldsymbol{\pi}|^2} \left(1 - \frac{m_R}{E_p} \right) \right) \right] \\ &\quad + \text{off-diagonal terms.} \end{aligned} \quad (112)$$

Since

$$(\boldsymbol{\sigma} \cdot \boldsymbol{\pi})^2 = \pi^2 + i\boldsymbol{\sigma} \cdot \boldsymbol{\pi} \times \boldsymbol{\pi} = p^2 - 2\mathbf{p} \cdot \mathbf{A}e + e^2 A^2 - \boldsymbol{\sigma} \cdot \mathbf{B}e \quad (113)$$

and the potential is $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}$ this becomes

$$(\boldsymbol{\sigma} \cdot \boldsymbol{\pi})^2 \cong p^2 - \mathbf{L} \cdot \mathbf{B}e - \boldsymbol{\sigma} \cdot \mathbf{B}e \quad (114)$$

so that

$$H' = \rho_3 \left(\sqrt{p^2 + m_R^2} - \frac{e}{2E_p} (\mathbf{L} + \boldsymbol{\sigma}) \cdot \mathbf{B} - \frac{\alpha}{2\pi} \frac{e}{2m_R} \left(\boldsymbol{\sigma} \cdot \mathbf{B} - \boldsymbol{\sigma} \cdot \hat{v} \hat{v} \cdot \mathbf{B} \left(1 - \frac{m_R}{E_p} \right) \right) \right) \quad (115)$$

using $m\mathbf{v} = \boldsymbol{\pi}$. The equation of motion of the coordinate operator $\mathbf{r}$ is then

$$\frac{d}{dt} \mathbf{r} = i[H, \mathbf{r}] = \frac{1}{E_p} (\mathbf{p} - e\mathbf{A}) = \frac{1}{E_p} \left(\mathbf{p} + \frac{1}{2} e \mathbf{r} \times \mathbf{B} \right) \quad (116)$$

which describes the motion of a particle in a magnetic field rotating with the relativistic cyclotron frequency

$$\boldsymbol{\omega}_L = \frac{e}{2E_p} \mathbf{B}. \quad (117)$$

On the other hand for the spin operator we have

$$\frac{d}{dt} \boldsymbol{\sigma} = i[H, \boldsymbol{\sigma}] = + \frac{e}{2E_p} \boldsymbol{\sigma} \times \mathbf{B} + \frac{\alpha}{2\pi} \frac{e}{2m_R} \left(\boldsymbol{\sigma} \times \mathbf{B} - \boldsymbol{\sigma} \times \hat{v} \hat{v} \cdot \mathbf{B} \left(1 - \frac{m_R}{E_p} \right) \right). \quad (118)$$

Thus for motion with $\mathbf{v}$ respectively parallel and perpendicular to $\mathbf{B}$ we find

$$\begin{aligned} \mathbf{B} \parallel \mathbf{v} \quad \frac{d\boldsymbol{\sigma}}{dt} &= \left(1 + \frac{\alpha}{2\pi} \right) \boldsymbol{\sigma} \times \boldsymbol{\omega}_L, \\ \mathbf{B} \perp \mathbf{v} \quad \frac{d\boldsymbol{\sigma}}{dt} &= \left(1 + \frac{\alpha}{2\pi} \frac{E_p}{m_R} \right) \boldsymbol{\sigma} \times \boldsymbol{\omega}_L, \end{aligned} \quad (119)$$

in agreement with the results of Mendlowitz and Case [12].

Finally we examine an identical situation—motion of a relativistic charged particle in a uniform magnetic field—but at finite temperature. The original Hamiltonian is

$$\begin{aligned} H &= \boldsymbol{\alpha} \cdot \boldsymbol{\pi}_\beta + \gamma_0 m_R + \frac{\alpha}{4\pi^2} I_0 - \gamma_0 \frac{\alpha}{4\pi^2} e \mathbf{I} \cdot \mathbf{A} \frac{1}{m_R} \\ &\quad - e \frac{1}{2m} \gamma_0 \sigma_{\mu\nu} \left(\frac{\alpha}{2\pi} \partial^\nu A^\mu + \frac{\alpha}{4\pi^2} \partial^\nu A_\alpha I^{\mu\alpha}(p) \right), \end{aligned} \quad (120)$$

where

$$\boldsymbol{\pi}_\beta = \mathbf{p} - \frac{\alpha}{4\pi^2} \mathbf{I} - e\mathbf{A} \left(1 - \frac{1}{E_p} I_0 \frac{\alpha}{4\pi^2} \right) \quad (121)$$

and

$$I_{\alpha\mu}(p) = -\frac{\partial}{\partial p^\alpha} I_\mu(p). \quad (122)$$

We then rotate the Hamiltonian with

$$\tan \phi = -\frac{\boldsymbol{\sigma} \cdot \boldsymbol{\pi}_\beta}{m_R + (\alpha/4\pi^2) I_0} \quad (123)$$

which yields

$$\begin{aligned} H' &= \exp\left(-i\frac{\phi}{2}\rho_2\right) H \exp\left(i\frac{\phi}{2}\rho_2\right) \\ &= \rho_3 \sqrt{(\boldsymbol{\sigma} \cdot \boldsymbol{\pi}_\beta)^2 + \left(m_R + \frac{\alpha}{4\pi^2} I_0\right)^2} + \frac{\alpha}{4\pi^2} I_0 (1 - \rho_3 \cos \phi) \\ &\quad - \rho_3 e \frac{1}{m_R} \frac{\alpha}{4\pi^2} \left(\cos \frac{\phi}{2} \mathbf{I} \cdot \mathbf{A} \cos \frac{\phi}{2} - \sin \frac{\phi}{2} \mathbf{I} \cdot \mathbf{A} \sin \frac{\phi}{2} \right) \\ &\quad + \rho_3 \frac{e}{4m_R} \left(\cos \frac{\phi}{2} \left(\boldsymbol{\sigma} \cdot \mathbf{B} \left(\frac{\alpha}{4\pi^2} I_{kk} - \frac{\alpha}{\pi} \right) - \mathbf{B}_l \frac{\alpha}{4\pi^2} I_{lm} \sigma_m \right) \cos \frac{\phi}{2} \right. \\ &\quad \left. - \sin \frac{\phi}{2} \left(\boldsymbol{\sigma} \cdot \mathbf{B} \left(\frac{\alpha}{4\pi^2} I_{kk} - \frac{\alpha}{\pi} \right) - \mathbf{B}_l \frac{\alpha}{4\pi^2} I_{lm} \sigma_m \right) \sin \frac{\phi}{2} \right) \\ &\quad + i \frac{e}{4m_R} \left(\cos \frac{\phi}{2} \frac{\alpha}{4\pi^2} I_{0j} (\boldsymbol{\sigma} \times \mathbf{B})_j \sin \frac{\phi}{2} - \sin \frac{\phi}{2} \frac{\alpha}{4\pi^2} I_{0j} (\boldsymbol{\sigma} \times \mathbf{B})_j \cos \frac{\phi}{2} \right) \\ &\quad + \text{off-diagonal terms.} \end{aligned} \quad (124)$$

Since

$$\begin{aligned} \cos \frac{\phi}{2} \sigma_i \cos \frac{\phi}{2} - \sin \frac{\phi}{2} \sigma_i \sin \frac{\phi}{2} &= \sigma_i \cos^2 \frac{\phi}{2} - \boldsymbol{\sigma} \cdot \hat{\pi} \sigma_i \boldsymbol{\sigma} \cdot \hat{\pi} \sin^2 \frac{\phi}{2} \\ &= \sigma_i - \hat{\pi}_1 \boldsymbol{\sigma} \cdot \hat{\pi} \left(1 - \frac{m_R}{E_p} \right), \\ i \cos \frac{\phi}{2} \sigma_i \sin \frac{\phi}{2} - i \sin \frac{\phi}{2} \sigma_i \cos \frac{\phi}{2} &= (\boldsymbol{\sigma} \times \hat{\pi})_i \frac{p}{E_p}, \end{aligned} \quad (125)$$

we find, for the upper component after considerable algebra

$$\begin{aligned} H' &\cong \sqrt{(\boldsymbol{\sigma} \cdot \boldsymbol{\pi}_\beta)^2 + m_R^2} \left(1 + \frac{\delta m_\beta}{2E_p} F\left(\frac{p}{E_p}\right) \right)^2 - \frac{e}{2E_p} \mathbf{L} \cdot \mathbf{B} \frac{m_R}{2p^2} \delta m_\beta \left(F\left(\frac{p}{E_p}\right) - 2 \right) \\ &\quad + \frac{m_R}{2E_p} \delta m_\beta F\left(\frac{p}{E_p}\right) \left(1 - \frac{m_R}{E_p} - \frac{\alpha}{2\pi} \frac{e}{2m_R} \left(\boldsymbol{\sigma} \cdot \mathbf{B} - \boldsymbol{\sigma} \cdot \hat{\pi} \hat{\pi} \cdot \mathbf{B} \left(1 - \frac{m_R}{E_p} \right) \right) \right) \end{aligned}$$

$$\begin{aligned}
 & + e \frac{\delta m_\beta}{4} \left(\boldsymbol{\sigma} \cdot \mathbf{B} \left(\frac{1}{m_R^2} - \frac{1}{2p^2} \left(F \left(\frac{p}{E_p} \right) - 2 \right) \right) \right. \\
 & + \boldsymbol{\sigma} \cdot \hat{\pi} \hat{\pi} \cdot \mathbf{B} \left(\left(1 - \frac{m_R}{E_p} \right) \left(F \left(\frac{p}{e} \right) - 2 \frac{E^2}{m^2} \right) \right. \\
 & + \frac{m}{E_p} \left(3F \left(\frac{p}{E_p} \right) + 4 - 2 \frac{E_p^2}{m_R^2} \right) \left. \frac{1}{2p^2} \right) \\
 & + e \frac{\delta m_\beta}{4} (\boldsymbol{\sigma} \cdot \mathbf{B} - \boldsymbol{\sigma} \cdot \hat{\pi} \hat{\pi} \cdot \mathbf{B}) \left(\frac{1}{m_R^2} - \frac{1}{2E_p^2} F \left(\frac{p}{E_p} \right) \right) \\
 & + \text{off-diagonal terms.}
 \end{aligned} \tag{126}$$

Using

$$E_\beta = \sqrt{p^2 + m_p^2(T)} \tag{127}$$

we can write this result as

$$\begin{aligned}
 H' = & E_\beta - \frac{e}{2E_\beta} \mathbf{L} \cdot \mathbf{B} \left(1 + \frac{m_R \delta m_\beta}{2E_p^2} \left(2 - F \left(\frac{p}{E_p} \right) \right) \right) \\
 & - \frac{e}{2E_\beta} \boldsymbol{\sigma} \cdot \mathbf{B} \left(1 + \frac{\alpha}{2\pi} \frac{E_\beta}{m_R} + \frac{\delta m_\beta}{2E_p} \right. \\
 & \times \left[F \left(\frac{p}{E_p} \right) \left(\frac{1}{2} + \frac{E_p^2}{2p^2} - \frac{m_R E_p}{p^2} \right) - \frac{E_p^2}{p^2} - \frac{2E_p^2}{m_R^2} + 2 \frac{m_R E_p}{p^2} \right] \\
 & + \frac{e}{2E_\beta} \boldsymbol{\sigma} \cdot \hat{v} \hat{v} \cdot \mathbf{B} \left(\frac{\alpha}{2\pi} \frac{E_\beta}{m_R} \left(1 - \frac{m_R}{E_\beta} \right) + \frac{\delta m_\beta}{2E_p} \right. \\
 & \times \left[F \left(\frac{p}{E_p} \right) \left(1 + \frac{m_R^2}{2p^2} + \frac{m_R E}{p^2} \right) - 2 \frac{m_R E}{p^2} - \frac{E^4}{p^2 m_R^2} - \frac{E^2}{m_R^2} \right].
 \end{aligned} \tag{128}$$

We find then for the relativistic cyclotron frequency at finite temperature

$$\omega_L = \frac{eB}{2\varepsilon_p} \left(1 + \frac{\delta m_\beta m_R}{2E_p^2} \left(2 - F \left(\frac{p}{E_p} \right) \right) \right). \tag{129}$$

On the other hand for the spin rotation frequencies we have

$$\begin{aligned}
 \mathbf{v} \perp \mathbf{B} \quad \omega_s = & \omega_L \left(1 + \frac{\alpha}{2\pi} \frac{E_\beta}{m_R} + \frac{\delta m_\beta}{2E_p} \left[F \left(\frac{p}{E_p} \right) \left(\frac{1}{2} + \frac{m_R}{E_p} + \frac{E_p^2}{2p^2} - \frac{m_R E_p}{p^2} \right) \right. \right. \\
 & \left. \left. - \frac{E_p^2}{p^2} - \frac{2E_p^2}{m_R^2} + 2 \frac{m_R E_p}{p^2} - 2 \frac{m_R}{E_p} \right] \right)
 \end{aligned} \tag{130}$$

$$\mathbf{v} \parallel \mathbf{B} \quad \omega_s = \omega_L \left(1 + \frac{\alpha}{2\pi} + \frac{\delta m_\beta}{2E_p} \left[F\left(\frac{p}{E_p}\right) \left(-\frac{1}{2} + \frac{E_p^2}{2p^2} + \frac{m_R}{E_p} - 2 \frac{m_R E}{p^2} - \frac{m_R^2}{2p^2} \right) - \frac{E_p^2}{p^2} - \frac{E_p^2}{m_R^2} + 4 \frac{m_R E_p}{p^2} + \frac{E^4}{p^2 m_R^2} - 2 \frac{m_R}{E_p} \right] \right). \quad (131)$$

Then in the $T=0$ limit but at relativistic velocity or the nonrelativistic limit at finite temperature these reduce to their previously derived values. However, at finite T and relativistic speeds they possess their own distinctive (and quite nontrivial) forms.

V. THE VERTEX CORRECTION AT HIGH TEMPERATURE

One can study the general vertex correction at all temperatures through use of the Ward identity. The self-energy can be written in the general form of Eq. (32). Working at $p = p'$, one has

$$\text{Fig. 4d} = -\frac{d\Sigma}{dp_\mu} = \bar{u}(p) \left[-\gamma_\mu C - \frac{d}{dp_\mu} \left(B + \frac{1}{m} p \cdot D \right) + \frac{D_\mu}{m} \right] u(p), \quad (132)$$

$$\text{Figs. 4b, c} = 2\bar{u}(p) \gamma_\mu u(p) \left[C + \frac{m}{E} \frac{d}{dE} \left(B + \frac{1}{m} p \cdot D \right) \right],$$

and upon dividing by Z_2 , as given by Eq. (34), we find

$$M_\mu = \bar{u}_\beta(p) \left[\gamma_\mu \left(1 + \frac{m}{E} \frac{d}{dE} \left(B + \frac{1}{m} p \cdot D \right) - \frac{D_0}{E} \right) + \frac{D_\mu}{m} - \frac{d}{dp_\mu} \left(B + \frac{1}{m} p \cdot D \right) \right] u_\beta(p). \quad (133)$$

This simplifies significantly when one uses the finite T matrix elements

$$M_\mu = \frac{p_\mu}{E} + \frac{m}{E} \left(\frac{p_\mu}{E} \frac{d}{dE} - \frac{d}{dp_\mu} \right) \left(B + \frac{1}{m} p \cdot D \right). \quad (134)$$

One can immediately see that M_0 , the charge vertex, is always equal to unity, independent of *any* assumptions about the self energy [13]. This is confirmation that our procedure for identifying the wavefunction renormalization constant is indeed correct. If B and $p \cdot D$ are Lorentz covariant functions, then in addition the temperature-dependent modification in the 3-vector current is also seen to vanish. If we recall that [cf. Eq. (33)]

$$\delta m = B + \frac{1}{m} p \cdot D, \quad (135)$$

then one can see that the only modification in the 3-vector current comes from the Lorentz noninvariance of the mass of the particle. We recall that the explicit calculations above show that δm is Lorentz invariant for all momentum if $T \ll m$, and for momenta well above the Fermi level ($p \gg T$) in the case where $T > m$. It appears reasonable to interpret the physical origin of this modification as being due to the Pauli principle which restricts propagation in momentum modes already filled in the fermion heat bath. The actual form of this term in QED is

$$\begin{aligned} & \frac{m}{E} \left(\frac{p_\mu}{E} \frac{d}{dE} - \frac{d}{dp_\mu} \right) \left(B + \frac{1}{m} p \cdot D \right) \\ &= \frac{\alpha}{4\pi^2} \int \frac{d^3 l}{E_l} n_F(E_l) \frac{m^3}{E_p} \frac{(p^\mu E_l - E_p l^\mu)}{[(p \cdot l)^2 - m^4]^2}. \end{aligned} \quad (136)$$

In the case that $p \neq p'$ the analysis proceeds as before, but now with the contribution from the fermion integrals appended. Resulting is an expression more complex but similar to Eq. (65). Since there is no really new physics, we do not quote the detailed expression. However, in the limit $T \gg m$ the fermion integrals can be evaluated analytically to yield in the nonrelativistic approximation

$$\begin{aligned} i \frac{\partial u}{\partial t} &= \tilde{m}_p(T) + \frac{p^2}{2\tilde{m}_p(T)} - e_R \frac{\mathbf{A} \cdot \mathbf{p}}{\tilde{m}_p(T)} + e_R \phi \\ &\quad - \frac{e_R}{2\tilde{m}_p(T)} \boldsymbol{\sigma} \cdot \mathbf{B} \left(1 + \frac{\alpha}{2\pi} - \frac{\delta m_\beta}{m} \right), \end{aligned} \quad (137)$$

where

$$\tilde{m}_p(T) = m_R + \frac{3}{2} \delta m_\beta. \quad (138)$$

Thus we find that the inertial mass is increased to become

$$\tilde{m}_p(T) = m_R + \frac{\pi \alpha T^2}{2m_R}, \quad (139)$$

while the anomalous moment becomes

$$\mu_a = \frac{\alpha}{2\pi} - \frac{\delta m_\beta}{m_R} = \frac{\alpha}{2\pi} - \frac{\pi \alpha T^2}{3m_R^2} \quad (140)$$

as measured in $e/2\tilde{m}_p(T)$ magnetons. Again this appears to agree with the calculation of Fujimoto and Yee [7], although it is not completely clear which magnetons they are utilizing.

As mentioned before, there is no charge renormalization arising from the vertex renormalization no matter what the temperature is. However, in the situation that $T \gtrsim m$, one must also consider the renormalization of charge arising from vacuum polarization effects, and that is the subject of the succeeding section.

VI. CHARGE RENORMALIZATION AND THE RUNNING COUPLING CONSTANT

The vacuum polarization has been studied in detail by Weldon [14], with an emphasis on the many interesting plasma effects which can occur. We will adapt his results to discuss charge renormalizations. The reader is referred to Weldon's paper for a more complete discussion of plasma oscillations and screening phenomena.

The vacuum polarization can be decomposed into longitudinal and transverse components

$$\pi_{\mu\nu} = \pi_L(\omega, \mathbf{k}) \bar{P}_{\mu\nu} + [\pi_T(\omega, \mathbf{k}) - \pi_L(\omega, \mathbf{k})] P_{\mu\nu} \quad (141)$$

where

$$\begin{aligned} \bar{P}_{\mu\nu} &= g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \\ P_{\mu\nu} &= g_{\mu\nu} - u_\mu u_\nu + \frac{(k_\mu - \omega u_\mu)(k_\nu - \omega u_\nu)}{|\mathbf{k}|^2} \end{aligned} \quad (142)$$

with u_μ being a vector which has the form

$$u_\mu = (1, 0, 0, 0) \quad (143)$$

in the rest frame of the heat bath. In a high temperature expansion, Weldon finds the temperature dependent corrections to be (in these expressions $k \equiv |\mathbf{k}|$)

$$\begin{aligned} \text{Re } \pi_L^\beta(\omega, k) &= \frac{4\pi\alpha}{3} \left(1 - \frac{\omega^2}{k^2} \right) \left[\left\{ 1 - \frac{\omega}{2k} \ln \left| \frac{\omega + k}{\omega - k} \right| \right\} T^2 + \frac{k^2}{2\pi^2} \ln m_e/T \right. \\ &\quad \left. + \text{terms independent of } T + O(1/T^2) \right] \\ \text{Re } \pi_T^\beta(\omega, k) &= \frac{4\pi\alpha}{3} \left[\left\{ \frac{\omega^2}{k^2} + \left(1 - \frac{\omega^2}{k^2} \right) \frac{\omega}{2k} \ln \left| \frac{\omega + k}{\omega - k} \right| \right\} T^2/2 \right. \\ &\quad \left. + \frac{2k^2 - 2\omega^2}{4\pi^2} \ln m_e/T \right. \\ &\quad \left. + \text{terms independent of } T + O(1/T^2) \right] \end{aligned} \quad (144)$$

in the rest frame of the heat bath. The photon propagator is modified to become

$$\Delta_{\mu\nu}(k) = \frac{\bar{P}_{\mu\nu} - P_{\mu\nu}}{\omega^2 - k^2 - \pi_L} + \frac{P_{\mu\nu}}{\omega^2 - k^2 - \pi_T}. \quad (145)$$

Note that the $\omega \rightarrow 0$ limit of these expressions is particularly simple

$$\begin{aligned} \text{Re } \pi_L^\beta(0, k) &= \frac{4\pi\alpha}{3} \left\{ T^2 + \frac{k^2}{2\pi^2} \ln m_e/T + \cdots \right\}, \\ \text{Re } \pi_T^\beta(0, k) &= \frac{2\alpha}{3\pi} \{ k^2 \ln m_e/T + \cdots \}. \end{aligned} \quad (146)$$

It is possible to define a static charge at high temperature despite the fact that the electrons are relativistic. Consider, for example, the interaction of two protons at rest in some Lorentz frame at temperatures $m_e \ll T \ll m_p$. Because the temperature is assumed to be well below the proton mass, there will be no subtleties associated with its interpretation, and its vertex function is not modified by renormalization. The proton then serves as a static source which can polarize electron-positron pairs in the heat bath. The interaction in the rest frame of the proton is found by setting $\omega \rightarrow 0$, in which case the interaction is given by

$$V(k) = e_R \delta_{\mu 0} \left[\frac{u_\mu u_\nu}{k^2 - (4\pi\alpha/3) \{ T^2 + (k^2/2\pi^2) \ln m_e/T \}} + \frac{g_{\mu\nu} - u_\mu u_\nu}{k^2 - (2\alpha/3\pi) k^2 \ln m_e/T} \right] e_R \delta_{\mu 0}, \quad (147)$$

where e_R is the renormalized charge at $T=0$. Note that in the rest frame of the protons

$$u_\mu = \frac{1}{\sqrt{1-\beta^2}} (1, \boldsymbol{\beta}), \quad (148)$$

where $\boldsymbol{\beta}$ is the velocity of the heat bath in this frame. This means that the above formula describes the static interaction in any rest frame relative to the heat bath, not just in the rest frame of the heat bath itself. Neglecting terms of $O(\alpha^2)$, the interaction can be written as

$$V(k) = e_R^2 \left(1 + \frac{2\alpha}{3\pi} \ln T/m \right) \left[\frac{u_0^2}{k^2 + (4\pi\alpha T^2/3)} + \frac{1-u_0^2}{k^2} \right]. \quad (149)$$

The constant in the longitudinal propagator is the well known plasma screening mass [15]. The frame dependence of this part of the interaction has been studied by Weldon [14]. What is new here is the charge renormalization. We can see that the effective static charge is frame independent but is temperature dependent,

$$\alpha(T) = \begin{cases} \alpha(T=0), & T \ll m, \\ \alpha(T=0) \left(1 + \frac{2\alpha}{3\pi} \ln T/m \right), & T \gg m. \end{cases} \quad (150)$$

The constant in front of the logarithm is what one would expect from renormalization group considerations [16]. At high temperatures the effective charge becomes stronger.

We emphasize that the appearance of the term $\ln m_e$ in the running coupling does not constitute a mass singularity. It arises from the conventional procedure of defining the coupling constant at large distances $r \gg 1/m_e$. If one were to consider the $m_e \rightarrow 0$ limit, one would require a new procedure for defining the renormalized

coupling constant. The $\ln m_e$ singularity would be absorbed into the new definition of the coupling constant. For example, if one chose to renormalize at $q^2 = \mu^2$, the logarithm which appears in the running coupling would become $\ln T/\mu$. This behavior is in accord with the Kinoshita–Lee–Nauenberg theorem [17] which says that all mass singularities can be absorbed into the definition of the coupling constant.

The coupling constant defined above is appropriate for static heavy charges. If one wishes to consider a different physical situation, one would need a different definition of the charge, and in general the behavior of the effective charge would be expected to be different.

VII. THE ENERGY MOMENTUM TENSOR

As discussed earlier, it is of interest to study the gravitational mass in a finite temperature field theory. To do so, one needs to study the renormalization of the energy-momentum tensor, which determines the coupling to the gravitational field [19]. This has been studied in detail in a separate communication [11]. However, for completeness we summarize the results here as they provide a separate insight into the unusual energy and momentum relations in a heat bath.

The energy momentum tensor in QED is

$$\begin{aligned} \theta_{\mu\nu} = & \bar{\psi} \left[\frac{i}{2} (\gamma_\mu D_\nu + \gamma_\nu D_\mu) - g_{\mu\nu} (i \not{D} - m) \right] \psi \\ & + F_{\mu\lambda} F^\lambda{}_\nu + \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \end{aligned} \quad (151)$$

where D_μ is the usual gauge covariant derivative. Its matrix element in a one electron state at $T=0$ is

$$\langle p | \theta_{\mu\nu} | p \rangle = \bar{u}(p) \left(\frac{\gamma_\mu p_\nu + \gamma_\nu p_\mu}{2} \right) u(p) = \frac{p_\mu p_\nu}{E_p}. \quad (152)$$

The $T=0$ radiative corrections leave this unchanged if written in terms of the renormalized mass.

The diagrams which contribute to the renormalization are given in Fig. 5. Calculating these leads to a renormalized tensor ($T \ll m$) [11, 18],

$$\begin{aligned} \langle p | \theta_{\mu\nu} | p \rangle = & \bar{u}_\beta(p) \left[\left(\frac{\gamma_\mu p_\nu + \gamma_\nu p_\mu}{2} \right) \left[1 - \frac{\alpha}{4\pi^2} I_0/E_p \right] - \frac{2\alpha\pi T^2}{3} \delta_{\mu 0} \delta_{\nu 0} \right. \\ & \left. + \frac{\alpha}{4\pi^2} \frac{(p_\mu I_\nu + p_\nu I_\mu)}{2m} - g_{\mu\nu} \left(\not{p} - m_R - \frac{\alpha\pi T^2}{3m_R} \right) \right] u_\beta(p). \end{aligned} \quad (153)$$

FIG. 5. The radiative corrections to the energy-momentum tensor. Here the double wavy line represents the coupling to $T_{\mu\nu}$.

Using

$$E_\beta = \sqrt{p^2 + m_p^2(T)} \quad (154)$$

the matrix element simplifies when the spinors are evaluated

$$\langle p | \theta_{\mu\nu} | p \rangle = \frac{p_{\mu\nu} - (2\alpha\pi/3) T^2 d_{\mu 0} \delta_{\nu 0}}{E_\beta} \quad (155)$$

A noncovariant temperature-dependent component has been generated. However, aside from this constant the final result has a simple form. Note that the trace of $\theta_{\mu\nu}$ is proportional to the $T=0$ mass

$$\langle p | \theta_\mu^\mu | p \rangle = \frac{m_R^2}{E_\beta}. \quad (156)$$

In particular, as $m_R \rightarrow 0$ the trace remains zero even though the pole in the propagator is shifted away from zero by finite T corrections.

VII. SCALAR ELECTRODYNAMICS AT FINITE TEMPERATURE

Similar results to those found in the Dirac case obtain for a scalar field theory. Thus we have repeated many of the previous calculations for the case of charged,

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FIG. 6. The self-energy diagrams for a scalar particle.

spin-zero particles and quote the results here for comparison. We begin with the self-energy and vertex corrections at $T=0$. From the graphs of Fig. 6 we find that the mass shift and wavefunction renormalization are given by (cf. Sect. I)

$$m_R^2 = m_0^2 \left(1 - \frac{\alpha}{4\pi} \left(\frac{3}{\epsilon} - 7 \right) \right), \quad (157)$$

$$Z_2 = 1,$$

where m_R is the renormalized ($T=0$) mass. (The tadpole diagram does not contribute at $T=0$ in dimensional regularization.) We note that there is no wavefunction renormalization to first order in α in Feynman gauge when dimensional regularization is used. The vertex correction is obtained from the graphs of Fig. 7 and we find by explicit calculation that $Z_1 = 1$ as guaranteed by Ward identities.

FIG. 7. The diagrams used to renormalize the scalar electromagnetic vertex.

For $0 < T \ll m_s$, m_s the scalar mass, we consider the finite temperature corrections to the photon propagator only and find that the self-energy and vertex graphs now give

$$m_{\text{phys}}^2 = m_s^2 + \frac{\alpha}{2\pi^2} p \cdot I \simeq \left(m_s + \frac{\alpha\pi T^2}{3m_s} \right)^2 \quad (158)$$

and

$$Z_1 = Z_2 = 1 + \frac{\alpha I_A}{4\pi^2}$$

so that the finite T mass shift is identical to the corresponding fermion case as is the infrared divergent piece of the wavefunction and vertex renormalization constants.

Of course, for $T \gg m_s$ we must include the finite T correction to the scalar propagators

$$\Delta_\beta(q) = \frac{i}{q^2 - m_s^2} + 2\pi\delta(q^2 - m_s^2) n_B(E_q) \quad (159)$$

and we obtain the additional contribution to the mass and wavefunction renormalization constants

$$Z_1 = Z_2 \xrightarrow{T \gg m_s} 1 + \frac{\alpha I_A}{2\pi^2}, \quad m_{\text{phys}}^2 \xrightarrow{T \gg m_s} \left(m_s + \frac{\alpha\pi T^2}{2m_s} \right)^2. \quad (160)$$

Finally, using the methods of Section VI we can also derive the effective running coupling and photon mass (due to the vacuum polarization graphs of Fig. 3). We find that in the limit $\omega \rightarrow 0$ that the effective interaction between heavy static charges can be written as (in the limit $T \ll m_s$)

$$V(k) = e_R^2 \left(1 + \frac{\alpha}{6\pi} \ln \left(\frac{T}{m_s} \right) \right) \left[\frac{u_0^2}{k^2 + (8\pi\alpha T/3)} + \frac{1 - u_0^2}{k^2} \right]. \quad (161)$$

The contribution to the plasma screening mass is twice that for a fermion loop (due solely to the difference in the form of the thermal distribution functions for bosons versus fermions). The temperature dependence of the static charge is

$$\alpha(T) = \begin{cases} \alpha(T=0), & T \ll m_s, \\ \alpha(T=0) \left(1 + \frac{\alpha}{6\pi} \ln \left(\frac{T}{m_s} \right) \right), & T \gg m_s, \end{cases} \quad (162)$$

again as expected from renormalization group arguments.

IX. THE DIRAC QUANTIZATION CONDITION

The introduction of magnetic monopoles into quantum electrodynamics (or even classical electromagnetism) poses many theoretical questions, many of which can be

restated in the context of finite temperatures. In this section we address only one question, that of whether the Dirac quantization condition³

$$eg = n/2 \quad (163)$$

remains valid when finite temperature effects are included. At zero temperature the electron charge is renormalized (the running coupling varies with distance) and the question of whether g is renormalized in such a way that Eq. (163) remains true has been investigated several times [20], most recently by Goebel and Thomaz [21]. At finite temperature we have seen that the static electric charge (properly defined) also exhibits a temperature dependence and we are naturally led to ask whether g has a corresponding T dependence such as to keep (163) valid.

We follow Goebel and Thomaz and, for simplicity, we consider not the case of a magnetic monopole field but the vacuum polarization effects (one loop corrections due to spin-zero, charged particles only) on a uniform $\mathbf{B}$ field. We then include the effects of finite temperature and find that e and g (or rather μ and ε) do indeed change in the way required to keep $eg = \text{constant}$. (The connection between the uniform field case and that of a monopole is discussed by Goebel and Thomaz.) The strategy is to calculate the vacuum energy density, $E_V = \mathbf{B}^2/2\mu$, including one-charged scalar loop corrections at finite T , to extract the permeability, $\mu(B)$ in the weak field limit ($\mu(0)$), and to compare this to the one-loop calculation of $\varepsilon(0)$ which is essentially Z_3^{-1} , the photon wavefunction renormalization.

The energy density of a uniform $\mathbf{B}$ field including the vacuum polarization effects of charged scalars (at finite T) is

$$E_V(B) = \frac{B^2}{2} + \frac{eB}{2\pi^2} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dk E(k)(1 + 2n_B(E(k))) \quad (164)$$

where

$$E(k) = \sqrt{m^2 + k^2 + eB(2n+1)/2} \quad (165)$$

and the distribution n_B takes into account real scalars in the heat bath. (We assume that $T \gg m$; otherwise such contributions are exponentially suppressed.) The sum can be done by a Sommerfeld–Watson transform

$$\sum_{n=0}^{\infty} f(n) = \int_{-1/2}^{\infty} f(y) dy + 2 \int_0^{\infty} \frac{dx}{1 + e^{2\pi x}} \text{Im} \left[f\left(-\frac{1}{2} + ix\right) \right]. \quad (166)$$

The second integral is the one that will lead to a B^2 term in E_V so we are led to evaluate

$$\int_0^{\infty} \frac{dx}{1 + e^{2\pi x}} \text{Im} \left[\int_{-\infty}^{\infty} dk \sqrt{\mu^2 + k^2} (1 + 2n_B(\sqrt{\mu^2 + k^2}/2)) \right] \quad (167)$$

³ For an overview of magnetic monopole physics, see Ref. [20].

where $\mu^2 = m^2 + 2ieBx$. We do the k^2 integration first, using a cutoff Λ for the divergent $T=0$ piece,

$$\begin{aligned}
 & \int_{-\Lambda}^{\Lambda} dk \sqrt{\mu^2 + k^2} + 2 \int_{-\infty}^{\infty} dk \sqrt{\mu^2 + k^2} (\exp(\beta \sqrt{\mu^2 + k^2}/2) - 1)^{-1} \\
 &= \Lambda^2 + \mu^2 (\ln(\Lambda/\mu) + \text{constant}) + O(\mu^4/\Lambda^2) \\
 &+ \frac{16}{\beta^2} \left[\frac{a^2}{2} \int_0^{\infty} \frac{du}{\sqrt{a^2 - u^2} (\exp(\sqrt{a^2 + u^2}) - 1)} + \dots \right] \\
 &= \Lambda^2 + \mu^2 (\ln(\Lambda/\mu) + \text{constant}) \\
 &+ \dots + \frac{8a^2}{\beta^2} \left(\frac{\pi}{2a} + \frac{1}{2} \ln \left(\frac{a}{4\pi} \right) + \dots \right), \tag{168}
 \end{aligned}$$

where $a^2 = \beta^2 \mu^2/4$ and the integral has been expanded in powers of a [22]. Thus Eq. (176) reduces to

$$\Lambda^2 + \mu^2 \ln(\Lambda/\mu) + \dots + \frac{2\pi\mu}{\beta} + \mu^2 \ln(\beta\mu) + \dots \tag{169}$$

Taking the imaginary part, keeping only the terms of order B^2 , and doing the remaining integration over x yields

$$\begin{aligned}
 E_V(B) &= \frac{1}{2} B^2 \left[1 + \frac{\alpha}{6\pi} \left(\ln \left(\frac{\Lambda}{m} \right) + \ln \left(\frac{m}{T} \right) \right) \right] \\
 &= B^2/2\mu \tag{170}
 \end{aligned}$$

so that in the weak field limit

$$\mu(0) = \left(1 + \frac{\alpha}{6\pi} \ln \left(\frac{\Lambda}{T} \right) \right)^{-1} \tag{171}$$

which is simply the $T=0$ result with m replaced by T .

From Section VIII we have the temperature-dependent charge due to scalar loops (for heavy static charges)

$$\begin{aligned}
 e^2(T) &= e_R^2(T=0) \left(1 + \frac{\alpha}{6\pi} \ln \left(\frac{T}{m} \right) \right) \\
 &= e_0^2 \left(1 - \frac{\alpha}{6\pi} \ln \left(\frac{\Lambda}{m} \right) \right) \left(1 + \frac{\alpha}{6\pi} \ln \left(\frac{T}{m} \right) \right) \tag{172}
 \end{aligned}$$

where we express the renormalized charge e_R in terms of the bare charge e_0 and a cutoff. Thus

$$\begin{aligned}\varepsilon(0) &= Z_3^{-1} \\ &= \left(1 - \frac{\alpha}{6\pi} \ln\left(\frac{\Lambda}{T}\right)\right)^{-1} \quad \text{and} \quad \mu(0) \varepsilon(0) = 1.\end{aligned}\quad (173)$$

(This is consistent with the result of Tarrach [23] on the constancy of the velocity of light at finite temperature.) Thus the Dirac quantization condition obtains both at $T=0$ and at nonzero T .

X. THE LAMB SHIFT

We close our paper by examining the effects of finite temperature on two well-studied quantities calculable via quantum electrodynamics—the Lamb shift and the Casimir effect. First the former:

The degeneracy of the $2S_{1/2}$ and $2P_{1/2}$ energy levels in hydrogen is removed by radiative corrections. The standard treatment [24] of the Lamb shift involves matching Bethe's nonrelativistic calculation of the low energy virtual photon "bubble" with a relativistic treatment for high energy effects. In this section we will look at the dominant low temperature modifications to this energy splitting. For this topic we need only to apply Bethe's formalism, as there are no high energy photons in a low temperature heat bath. The subtle matching of low energy and high energy contributions is not needed.

In Bethe's treatment, the energy shift of a state A is given by

$$\Delta E_a = \frac{2\alpha}{3\pi m^2} \int_0^{E_a^{\max}} d\omega \sum_I \frac{|\langle \mathbf{P}_{IA} \rangle|^2 (E_A - E_I)}{E_A - E_I - \omega} \quad (174)$$

where $\langle \mathbf{P}_{IA} \rangle$ is the matrix element of the momentum operator between the state A and the intermediate state I . The calculation of decay rates and energy shifts at finite temperature is developed in atomic physics notation in Ref. [25] but it is also simple to obtain directly from the relativistic formalism. The static Green's function

$$\begin{aligned}G_{ij}(\mathbf{x}', t'; \mathbf{x}, t) &\equiv \langle 0 | A_i(x', t') A_j(x, t) | 0 \rangle \\ &\equiv \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega} \sum_{\lambda} \hat{\epsilon}_i(k, \lambda) \hat{\epsilon}_j(k, \lambda) e^{-i\omega(t'-t) + i\mathbf{k} \cdot (\mathbf{x}' - \mathbf{x})}\end{aligned}\quad (175)$$

is obtained from the full photon propagator and at finite temperature is modified to become

$$\begin{aligned}G_{ij}(\mathbf{x}', t'; \mathbf{x}, t) &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega} \sum_{\lambda} \hat{\epsilon}_i(k, \lambda) \hat{\epsilon}_j(k, \lambda) \\ &\times [(1 + n_B(\omega)) e^{-i\omega(t'-t) + i\mathbf{k} \cdot (\mathbf{x}' - \mathbf{x})} + n_B(\omega) e^{i\omega(t'-t) - i\mathbf{k} \cdot (\mathbf{x}' - \mathbf{x})}].\end{aligned}\quad (176)$$

In this expression the piece involving simply $e^{-i\omega t}$ corresponds to spontaneous emission, that with $n_B(\omega) e^{-i\omega t}$ is stimulated emission and $n_B(\omega) e^{+i\omega t}$ represents absorption. The effect of this Green's function is to lead to a modification of the energy denominator

$$\begin{aligned} \frac{1}{E_A - E_I - \omega} &\rightarrow \left[\frac{1}{E_A - E_I - \omega} (1 + n_B(\omega)) + \frac{1}{E_A - E_I + \omega} n_B(\omega) \right] \\ &= \left[\frac{1}{E_A - E_I - \omega} + \frac{2n_B(\omega)(E_A - E_I)}{(E_A - E_I)^2 - \omega^2} \right]. \end{aligned} \quad (177)$$

Thus the energy shift becomes

$$\Delta E_A = \frac{2\alpha}{3\pi m^2} \int d\omega \left(\sum_I |\langle \mathbf{P}_{IA} \rangle|^2 (E_A - E_I) \right) \left[\frac{1}{E_A - E_I - \omega} + \frac{2n_B(\omega)(E_A - E_I)}{(E_A - E_I)^2 - \omega^2} \right]. \quad (178)$$

At low temperatures ω will be much less than the energy differences in the denominator. Therefore we may expand the finite temperature contribution as follows:

$$\frac{(E_A - E_I)^2}{(E_A - E_I)^2 - \omega^2} = 1 + \frac{\omega^2}{(E_A - E_I)^2} + \dots \quad (179)$$

In addition, the sum over intermediate states may be calculated simply by use of sum rules

$$\sum_I |\langle \mathbf{P}_{IA} \rangle|^2 = \langle \mathbf{P}^2 \rangle_{AA} \quad (180)$$

and

$$\begin{aligned} \sum_I \frac{|\langle \mathbf{P}_{IA} \rangle|^2}{(E_A - E_I)^2} &= \sum_I m^2 \frac{|[H, \mathbf{x}]_{IA}|^2}{(E_A - E_I)^2} \\ &= m^2 \sum_I |\langle \mathbf{x}_{IA} \rangle|^2 \\ &= m^2 \langle x^2 \rangle_{AA}. \end{aligned} \quad (181)$$

This removes all reference to the intermediate states, and allows the matrix element to be calculated using the state A alone. The energy shift becomes

$$\Delta E_A = \Delta E_A(T=0) + \left[\frac{4\alpha}{3\pi m^2} \int d\omega n_B(\omega) \right] \langle \mathbf{P}^2 \rangle_{AA} + \left[\frac{4\alpha}{3\pi} \int d\omega \omega^2 n_B(\omega) \right] \langle x^2 \rangle_{AA}. \quad (182)$$

The first temperature-dependent correction is a common mass renormalization correction which applies to all states. Only the last term leads to energy differences between the levels. To calculate it we note

$$\int d\omega \omega^2 n_B(\omega) = 2\zeta(3) T^3$$

$$\langle x^2 \rangle = \frac{n^2}{2} [5n^2 + 1 - 3l(l+1)] a^2, \quad (183)$$

where $\zeta(x)$ is the Riemann zeta function ($\zeta(3) = 1.2$), n, l are the quantum numbers labeling the atomic state, and a is the Bohr radius

$$a = \frac{1}{\alpha m}. \quad (184)$$

Applied to the $2S - 2P$ splitting, one obtains

$$\Delta E_{2S-2P}^{(T)} = \frac{32\alpha}{\pi} \zeta(3) (aT)^2 T. \quad (185)$$

The high energy contributions will have an expansion parameter (T/m) rather than (aT) . Despite this relative enhancement of the low energy effects, the magnitude of the temperature dependent correction is too tiny to play a role at reasonable temperatures [26]

$$\Delta E_{2S-2P}^{(T)} = 6 \times 10^{-6} \text{ MHz} \quad (186)$$

compared to the full value of the Lamb shift

$$\Delta E_{\text{EXPT}} = 1057 \cdot 850(20) \text{ MHz}. \quad (187)$$

XI. THE CASIMIR EFFECT AT FINITE TEMPERATURE

The last of the classic QED tests that we will examine at finite temperature is the Casimir effect⁴—the existence of forces (attractive or repulsive) exerted on macroscopic conducting surfaces due to changes, caused by the imposition of boundary conditions, in the zero-point energy of the vacuum. While not a test of QED radiative corrections, the existence of these forces does not confirm the reality of vacuum quantum fluctuations of the electromagnetic field. At finite temperature they test the (classical) thermal fluctuations as well.

Casimir [28] first predicted an attractive force between two parallel conducting plates and found the energy per unit surface caused by the fluctuations

$$\varepsilon(a) = \frac{-\pi^2}{720a^3}. \quad (188)$$

⁴ For a review of the Casimir effect, including finite temperature corrections, see Ref. [27].

The resulting attractive force and its dependence on the plate separation, a , was first measured by Sparnaay [29]. Real photons in the heat bath will also contribute to this effect and their contribution to the energy must be weighted by their thermal distributions function and include a factor of two for polarization. In this way, Fierz [30] was able to calculate the Casimir energy at finite temperature and found

$$\varepsilon(a, T) = -\frac{1}{8\pi^2 a^3} f(x) \quad (189)$$

where

$$x = 2\pi a T$$

and

$$f(x) = \sum_{m=1}^{\infty} \frac{x^3 \cosh(mx)}{m \sinh^3(mx)}.$$

Note that this reproduces Casimir's result at $T=0$ and vanishes exponentially as $T \rightarrow \infty$ (i.e., there is no Casimir effect at high temperatures). In this case the temperature-dependent expansion parameter is $2\pi Ta$ rather than T/m which was relevant to the loop effects we have studied. In particular, at $T = 300^\circ\text{K}$ the distance at which $x = 1$ is $1 \mu\text{m}$ which is in the range of distances in which the experimental measurements were done. Since $f(1)/f(0) = 0.90$ (and $f(2)/f(0) = 0.56$), Sparnaay would in fact have seen $\sim 10\%$ deviations from the Casimir prediction if his precision had been sufficient. (In fact, this is the only finite temperature effect associated with field theory which is measurable in the lab of which we are aware.)

The Casimir effect can also be studied for geometries other than parallel plates [31, 32]. Lukosz [31], for example, has generalized that result by calculating the change in zero-point energy due to the presence of a rectangular conducting cavity (of dimensions L_1, L_2, L_3 .) The resulting energy density shift is

$$u(L_1, L_2, L_3) = -\frac{1}{16\pi^2} \left[\sum'_{m_1, m_2, m_3} (m_1^2 L_1^2 + m_2^2 L_2^2 + m_3^2 L_3^2)^{-2} - \frac{\pi^2}{3} \frac{1}{L_1 L_2 L_3} \left(\frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} \right) \right] \quad (190)$$

(where $\sum'$ means that $(0, 0, 0)$ is deleted from the sum). For $L_2, L_3 \gg L_1 = a$ we recover the parallel plate configuration and the Casimir result

$$\varepsilon(a) = au(a) = \frac{-\pi^2}{720a^3}. \quad (191)$$

For $L_1 = L_2 = L_3 = L$ (a cubical cavity) one finds

$$u(L) = -\frac{1}{16\pi^2 L^4} \left[\sum'_{m_1, m_2, m_3} (m_1^2 + m_2^2 + m_3^2)^{-2} - \pi^3 \right] \quad (192)$$

and, evaluating the generalized zeta-function, the change is given by

$$\Delta E = u(L) L^3 \simeq \frac{0.0916}{L} \quad (193)$$

which leads to a repulsive force for the cubical shell. Boyer [32] has calculated the energy shift for a thin spherical shell of radius r numerically and finds

$$\Delta E \simeq \frac{0.09}{2r} \quad (194)$$

which is approximately equal to that of a cubical shell with $L = 2r$.

We can now generalize the results for the rectangular cavity to finite temperatures by including the contributions from real photons to obtain

$$\begin{aligned} u(L_1, L_2, L_3, T) = & -\frac{\pi}{2} T^3 \sum_{\substack{m_1, m_2, m_3 \\ = -\infty}}^{\infty} \frac{1}{l} \frac{\cosh(2\pi T l)}{\sinh^3(2\pi T l)} \\ & + \frac{\pi}{2} \frac{T^2}{L_1 L_2 L_3} \sum_{j=1}^3 \sum_{m_j=1}^{\infty} \frac{L_j}{\sinh^2(2\pi T m_j L_j)}, \end{aligned} \quad (195)$$

where $l = \sqrt{m_1^2 L_1^2 + m_2^2 L_2^2 + m_3^2 L_3^2}$. This reproduces the Lukosz result when $T=0$ and for $L_2, L_3 \gg L_1 = a$, we obtain the Fierz result, Eq. (189). As before, temperature effects of 10% or more are expected when $T \gtrsim (2\pi \min(L_1, L_2, L_3))$ and the Casimir effect vanishes exponentially for high temperatures.

XII. CONCLUSIONS

We have looked at the finite temperature modifications to the self-energy, vertex function and vacuum polarization at one loop order. The results of the one loop calculations themselves look quite complicated (cf. Eq. (66)) because of the presence of many Lorentz noninvariant integrals. However, most of these modifications simply account for the change in the particle kinematics in the heat bath. This can be seen by taking the matrix elements with finite temperature spinors, in which case the result simplifies enormously (cf. Eq. (73)). In particular the vertex function at $p = p'$ is not affected by radiative corrections, exactly as occurs at $T=0$. We have presented several methods to take account of these kinematic changes in obtaining quantities which can, in principle, be measured.

There are temperature-dependent corrections to the basic parameters of the theory. We collect them here. The mass shift is

$$m_{\text{phys}}^2 = \begin{cases} m_R^2 + \frac{2\pi\alpha T^2}{3} & T \ll m \\ m_R^2 + \pi\alpha T^2 & T \gg m \end{cases} \quad (196)$$

the magnetic moment is (in the nonrelativistic limit)

$$|\mu_e| = \frac{e}{2m_{\text{phys}}} \begin{cases} 1 + \frac{\alpha}{2\pi} - \frac{2}{3} \frac{\delta m}{m_R} & T \ll m \\ 1 + \frac{\alpha}{2\pi} - \frac{\delta m}{m_R} & T \gg m \end{cases} \quad (197)$$

and the running coupling constant for static heavy charges becomes

$$\alpha(T) = \begin{cases} \alpha & T \ll m \\ \alpha \left(1 + \frac{2\alpha}{3\pi} \ln T/m_e \right) & T \gg m. \end{cases} \quad (198)$$

These effects are unfortunately too small to be measured at temperatures which are realistic in the lab, as

$$\begin{aligned} \frac{2\pi\alpha}{3} \left(\frac{3^\circ\text{K}}{m_e} \right)^2 &= 4 \times 10^{-21}, \\ \frac{2\pi\alpha}{3} \left(\frac{300^\circ\text{K}}{m_e} \right)^2 &= 4 \times 10^{-17}. \end{aligned} \quad (199)$$

The only finite temperature effect associated with field theory which appears measurable in the lab is the Casimir effect, as discussed in Section XI. The reason that temperature dependence is observable here is that the expansion parameter is $x = 2\pi Ta$. Loop effects such as we are studying have the expansion parameter being T/m , and would only be important in cosmological, or perhaps quark matter, applications.

Although various intermediate results appear to contain mass singularities, all physical quantities are free of them. (See Sect. VI for a discussion of the case of the running coupling constant.) This is in accord with the expectation that the Kinoshita–Lee–Nauenberg theorem should continue to remain valid at finite temperature.

The use of temperature as a tool for exploring the physical structure of a field theory has become increasingly common. Many such applications have been made for quantum chromodynamics. It is useful in this regard to first understand completely the finite temperature effects in a field theory which is well understood—QED. This paper has been devoted to this goal in the study of one loop corrections.

APPENDIX

We have relegated to this Appendix the mathematical details of many of the calculations reported in the main text. To perform the $T=0$ calculations, we need

the dimensionally regularized values of various integrals. The techniques of dimensional regularization is well known, so we shall give only the results. Thus we find (with d being the number of dimensions)

$$\begin{aligned}
 J(p, \lambda^2) &= \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2} \frac{1}{k^2 - 2p \cdot k + \lambda^2} = \frac{i}{(4\pi)^{d/2}} \Gamma\left(2 - \frac{d}{2}\right) \frac{1}{(p^2)^{2 - (d/2)}} \frac{1}{d - 3} \\
 &\quad \times \left(1 - \frac{\lambda^2}{2p^2} (d - 3) + \dots\right) \\
 &= -\frac{i}{16\pi^2} \left(\frac{1}{\varepsilon} - 2 - \frac{\lambda^2}{2p^2} \left(\frac{1}{\varepsilon} + \dots\right)\right) \\
 J_\mu(p, \lambda^2) &= \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2} \frac{1}{k^2 - 2p \cdot k + \lambda^2} = \frac{i}{(4\pi)^{d/2}} p_\mu \Gamma\left(2 - \frac{d}{2}\right) \frac{1}{(p^2)^{2 - d/2}} \frac{1}{d - 2} \\
 &\quad \times \left(1 - \frac{\lambda^2}{2p^2} \frac{(d - 4)(d - 2)}{d - 3} + \dots\right) \\
 &= -\frac{i}{16\pi^2} p_\mu \left(\frac{1}{2\varepsilon} - \frac{1}{2} - \frac{\lambda^2}{p^2}\right),
 \end{aligned} \tag{A1}$$

where $1/\varepsilon = (2/(d - 4)) + \gamma - \ln 4\pi + \ln p^2$. Since

$$\begin{aligned}
 \gamma_\mu A \gamma^\mu &= (2 - d) A \\
 \gamma_\mu \gamma^\mu &= d
 \end{aligned} \tag{A2}$$

we determine

$$\begin{aligned}
 \Sigma(p) &= -ie^2 [((2 - d) \not{p} + md) J(p, p^2 - m^2) - (2 - d) J_\mu(p, p^2 - m^2) \gamma^\mu] \\
 &= -\frac{\alpha}{4\pi} \left[m \left(\frac{3}{\varepsilon} - 4\right) - (\not{p} - m) \left(\frac{3}{\varepsilon} + 2\right) \right] \\
 &= \left(\text{expanding } \frac{1}{\varepsilon} \text{ about } p^2 = m^2 \right) - \frac{\alpha}{4\pi} \left(\frac{3}{\varepsilon} - 4\right) (m - (\not{p} - m)).
 \end{aligned} \tag{A3}$$

To evaluate the vertex diagram (Fig. 4) we need the integrals

$$\begin{aligned}
 K(p', p) &= \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2} \frac{1}{k^2 - 2p' \cdot k} \frac{1}{k^2 - 2p \cdot k} \\
 &= \int_0^1 dx \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2} \frac{1}{(k^2 - 2k \cdot s)} \quad \text{where } s = p'(1 - x) + px \\
 &= \int_0^1 dx \left[\frac{-i}{(4\pi)^{d/2}} \Gamma\left(3 - \frac{d}{2}\right) \frac{1}{(s^2)^{3 - (d/2)}} \frac{1}{d - 4} \right] \\
 &= \int_0^1 dx \cdot \frac{-i}{16\pi^2} \frac{1}{s^2} \frac{1}{2\varepsilon}
 \end{aligned} \tag{A4}$$

$$\begin{aligned}
 K_\mu(p', p) &= \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2} \frac{1}{k^2 - 2p' \cdot k} \frac{1}{k^2 - 2p \cdot k} k_\mu \\
 &= \int_0^1 dx \left[\frac{-i}{(4\pi)^{d/2}} s_\mu \Gamma\left(3 - \frac{d}{2}\right) \frac{1}{(s^2)^{3-(d/2)}} \frac{1}{d-3} \right] \\
 &= \int_0^1 dx \frac{-i}{16\pi^2 s^2} s_\mu
 \end{aligned}$$

$$\begin{aligned}
 K_{\mu\nu}(p', p) &= \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2} \frac{1}{k^2 - 2p' \cdot k} \frac{1}{k^2 - 2p \cdot k} k_\mu k_\nu \\
 &= \int_0^1 dx \left[\frac{i}{(4\pi)^{d/2}} \frac{1}{(s^2)^{(d/2)-2}} \left(g_{\mu\nu} \frac{1}{2} \Gamma\left(2 - \frac{d}{2}\right) \frac{1}{d-2} - \frac{s_\mu s_\nu}{s^2} \Gamma\left(3 - \frac{d}{2}\right) \frac{1}{d-2} \right) \right. \\
 &\quad \left. = \int_0^1 dx \frac{-i}{16\pi^2} \left[\left(\frac{1}{\varepsilon} - 1 \right) \frac{1}{4} g_{\mu\nu} + \frac{s_\mu s_\nu}{2s^2} \right] \right].
 \end{aligned}$$

Then since $s^2 = m^2 - q^2 x(1-x)$ we find

$$\begin{aligned}
 A_\mu(p', p) &= +ie^2 \int_0^1 dx \left[\gamma_\mu \left[(4m^2 - 2q^2) K(s) - 4(p + p')_\alpha K^\alpha(s) + 2g_{\mu\nu} K^{\mu\nu}(s)(1 + \varepsilon) \right] \right. \\
 &\quad \left. + 4(p + p')_\mu K_\alpha(s) \gamma^\alpha - 4mK_\mu(s) - 4K_{\mu\alpha}(s) \gamma^\alpha (1 + \varepsilon) \right] \\
 &= -\frac{\alpha}{4\pi} \int_0^1 dx \left(\gamma_\mu \left(-\left(\frac{1}{\varepsilon} + \ln \frac{s^2}{m^2} \right) \left(1 + \frac{4m^2 - 2q^2}{2s^2} \right) \right. \right. \\
 &\quad \left. \left. + \frac{4}{s^2} \left(2m^2 - \frac{1}{2} q^2 \right) - 2 \right) - 4m(p + p')_\mu \frac{1}{s^2} + 6m \frac{s_\mu}{s^2} \right) \quad (A5) \\
 &\cong (\text{dropping terms of } O\left(\frac{q^2}{m^2}\right) - \frac{\alpha}{4\pi} \left(\gamma_\mu \left(-\frac{3}{\varepsilon} + 6 \right) - 2 \frac{(p + p')_\mu}{2m} \right) \\
 &= (\text{using the Gordon decomposition}) \\
 &\quad -\frac{\alpha}{4\pi} \left[\gamma_\mu \left(-\frac{3}{\varepsilon} + 4 \right) + 2i\sigma_{\mu\nu} q^\nu \frac{1}{2m} \right].
 \end{aligned}$$

To deal with the vacuum polarization we need the regularized integrals

$$\begin{aligned}
 L(q) &= \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 - m^2)(k - q)^2 - m^2} \\
 &\approx \frac{i}{(4\pi)^{d/2}} \frac{1}{m^{4-d}} \left(\Gamma\left(2 - \frac{d}{2}\right) + \frac{1}{6} \frac{q^2}{m^2} \Gamma\left(3 - \frac{d}{2}\right) + \dots \right) \\
 &= -\frac{i}{16\pi^2} \left(\frac{1}{\varepsilon} - \frac{q^2}{6m^2} + \dots \right) \quad (A6)
 \end{aligned}$$

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$$\begin{aligned}
 L_\mu(q) &= \int \frac{d^4 k}{(2\pi)^4} \frac{k_\mu}{(k^2 - m^2)((k - q)^2 - m^2)} = \frac{q_\mu}{2} L(q), \\
 L_{\mu\nu}(q) &= \int \frac{d^4 k}{(2\pi)^4} \frac{k_\mu k_\nu}{(k^2 - m^2)((k - q)^2 - m^2)} \\
 &= \frac{-i}{(4\pi)^{d/2}} \left[\frac{1}{2} g_{\mu\nu} \frac{1}{m^{2-d}} \Gamma\left(2 - \frac{d}{2}\right) \left(\frac{1}{1 - d/2} + \frac{1}{6} \frac{q^2}{m^2} \right) \right. \\
 &\quad \left. - q_\mu q_\nu \frac{1}{3m^{4-d}} \Gamma\left(2 - \frac{d}{2}\right) \right] \\
 &= -\frac{i}{16\pi^2} \left[\frac{m^2}{2} g_{\mu\nu} \left(\left(\frac{1}{\varepsilon} - 1 \right) - \frac{1}{6} \frac{q^2}{m^2} \frac{1}{\varepsilon} \right) + q_\mu q_\nu \left(\frac{1}{3\varepsilon} \right) \right].
 \end{aligned}$$

We find then

$$\begin{aligned}
 e^2 \int \frac{d^4 k}{(2\pi)^4} \frac{\text{Tr } \gamma_\mu (\not{k} + m) \gamma_\nu (\not{k} - \not{q} + m)}{(k^2 - m^2)((k - q)^2 - m^2)} &= 4e^2 [g_{\mu\nu}(m^2 L(q) + q^\mu L_\mu(q) - L_{\alpha\alpha}(q)) \\
 &\quad - q_\nu L_\mu(q) - q_\mu L_\nu(q) + 2L_{\mu\nu}(q)] \\
 &= i \frac{\alpha}{\pi} (-q^2 g_{\mu\nu} + q_\mu q_\nu) \frac{1}{3\varepsilon} + \dots
 \end{aligned}$$

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