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Renormalization and radiative corrections at finite temperature

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We discuss the renormalization prescription for fermions at finite temperature and describe the procedure for calculating radiative corrections. Novel features arise, such as a lack of explicit Lorentz invariance, $1/k^2$ singularities, and the absorption of soft photons from the background heat bath. The methods are illustrated by explicit calculation of the electron renormalization and the radiative corrections to $H\rightarrow e^+e^-$ (with H being spin zero) in finite-temperature QED.

I. INTRODUCTION

In studying the applications of particle physics to the early universe, one needs to use field theory at nonzero temperature. The general formulation of finite-temperature field theory was developed by Weinberg,[1] Dolan and Jackiw,[2] and by Duncan[3] who obtained the tree-level propagators and Feynman rules, and calculated the effective potential in gauge theories. More recent applications concern how the temperature affects scattering processes and decay rates.[4,5] In these cases, the temperature enters through radiative corrections. However, the conventional procedures for renormalization and calculation of radiative corrections are not applicable. There are two main reasons for this. First, the processes do not take place in the usual vacuum but occur in the background of particles present at finite T, i.e., a heat bath. Additionally, finite temperature theories are not explicitly Lorentz invariant; the heat bath defines a preferred frame of reference. Standard renormalization prescriptions use Lorentz invariance heavily. It is the purpose of this paper to describe a renormalization procedure appropriate for finite temperature.

The outline of our paper is as follows. In Sec. II we briefly describe the real-time formulation of finite-temperature field theory. Then in Sec. III we discuss mass and wave-function renormalization for fermions by considering the fermion propagator. Weldon has studied the photon self-energy so we need not repeat his work.[6] We describe the procedure for calculating radiative corrections of decays or scatterings in Sec. IV. Here we present an explicit calculation of the decay of a neutral scalar boson (such as a Higgs particle) in order to demonstrate the cancellation of infrared divergences. We summarize the procedure and results in Sec. V. Finally in the appendices we give a particularly simple and transparent derivation of the finite-T propagator and the calculational details.

II. FINITE-TEMPERATURE FORMULATION

At nonzero temperature the presence of particles in the background heat bath modifies the propagators even at the tree level. For a rigorous derivation we refer the reader to the early literature.[7] However, in Appendix A, we present a simple derivation which demonstrates just how the modifications of the propagator come about.

In the "imaginary-time" formulation, the energy variable is treated as a discrete quantity and calculations involve integrals over momentum and sums over energy. We prefer to use the "real-time" formulation wherein energy is a continuous variable as in conventional field theory. The real-time formulation has the additional advantage of explicitly separating out the zero-temperature result from the finite-temperature corrections.

The tree-level fermion propagator in momentum space is

$$S_\beta(p)=\frac{i}{\not p-m+i\epsilon}-2\pi\delta(p^2-m^2)(\not p+m)n_F(E_p)\ , \tag{1}$$

where β is the inverse temperature

$$\beta=1/T \tag{2}$$

and

$$n_F(E_p)=\frac{1}{e^{\beta E_p}+1} \tag{3}$$

is the Fermi-Dirac distribution function. Note that we use units of temperature where Boltzmann's constant is set equal to unity. The tree-level photon propagator in the Landau gauge (which we adopt throughout this paper) is

$$D_\beta^{\mu\nu}(q)=-g^{\mu\nu}\left[\frac{i}{q^2+i\epsilon}+2\pi\delta(q^2)n_B(E_q)\right], \tag{4}$$

where

$$n_B(E_q)=\frac{1}{e^{\beta E_q}-1} \tag{5}$$

is the Bose-Einstein distribution function. Feynman diagrams are calculated in the usual fashion, except for the substitution of the above propagators in place of the usual ones.

The usual infinite ultraviolet renormalization of the masses and coupling constants are due to the zero-temperature portion of the propagators. The finite-T correction involves an exponential ultraviolet cutoff in the distribution functions. However, new infrared problems can (and do) arise because of the $1/E_q$ singularity in $n_B(E_q)$ as $E_q\rightarrow 0$. These require special treatment, as we shall demonstrate.

III. RENORMALIZING THE FERMION PROPAGATOR

At the one-loop level the fermion self-energy is given by the diagram in Fig. 1. Explicit calculation of this diagram yields

$$\Sigma_\beta(p)=\Sigma_{T=0}(p)+\frac{e^2}{4\pi^3}\int d^4k\,(2m-\not p+\not k)\left[\frac{n_F(E_{p-k})\delta((p-k)^2-m^2)}{k^2+i\epsilon}-\frac{n_B(k)\delta(k^2)}{(p-k)^2-m^2+i\epsilon}\right] \tag{6}$$

$$=\Sigma_{T=0}(p)+\frac{\alpha}{4\pi^2}[(\not p-m)I_A+\not I+(2m-\not p)J_A+\not J_B]\ , \tag{7}$$

where

$$I^A=8\pi\int\frac{dk}{k}n_B(k),\quad I^\mu=2\int\frac{d^3k}{k_0}n_B(k)\frac{(k_0,\vec{k})}{E_pk_0-\vec{p}\cdot\vec{k}}\ ,$$
$$J_A=\int\frac{d^3l}{E_l}n_F(E_l)\left[\frac{1}{E_pE_l+m^2-\vec{p}\cdot\vec{l}}-\frac{1}{E_pE_l-m^2+\vec{p}\cdot\vec{l}}\right]\ , \tag{8}$$
$$J_B^\mu=\int\frac{d^3l}{E_l}n_F(E_l)\left[\frac{(E_p+E_l,\vec{p}+\vec{l})}{E_pE_l+m^2-\vec{p}\cdot\vec{l}}-\frac{(E_p-E_l,\vec{p}+\vec{l})}{E_pE_l-m^2+\vec{p}\cdot\vec{l}}\right]\ .$$

At $T=0$, Lorentz invariance allows us to write the self-energy near the mass shell as

$$\Sigma_{T=0}(p)=-(Z_2{}^{-1}-1)(\not p-m)+\delta m\ . \tag{9}$$

The standard procedure identifies δm as the mass shift ($m^{\text{phys}}=m+\delta m$) and Z_2 as the wave-function renormalization constant. The lack of explicit Lorentz invariance obscures this identification when $T\neq 0$. We must then turn to the basic definitions of these renormalizations in order to define them unambiguously.

First we introduce some notation. We write the fermion self-energy as

$$\Sigma(p)\equiv A(p)E\gamma_0-B(p)\vec{p}\cdot\vec{\gamma}-C(p)\ , \tag{10}$$

where

$$A=\frac{\alpha}{4\pi^2}\left[I^A+\frac{1}{E}I^0-J_A+\frac{J_B^0}{E}\right]\ ,$$
$$B=\frac{\alpha}{4\pi^2}\left[I^A+\frac{1}{p^2}\vec{I}\cdot\vec{p}-J_A+\frac{1}{p^2}\vec{J}_B\cdot\vec{p}\right]\ ,$$
$$C=\frac{\alpha}{4\pi^2}m(I^A-2J^A)\ ,$$

and the coefficients may depend on E and $\vec{p}$ in a Lorentz-noninvariant fashion. The inverse propagator is then

$$S^{-1}(p)=(1-A)E\gamma_0-(1-B)\vec{p}\cdot\gamma-(m-C)$$
$$\equiv\tilde{\not p}-\tilde{m}\ , \tag{11}$$

where

$$\tilde{p}_\mu\equiv((1-A)E,(1-B)\vec{p})\ ,$$
$$\tilde{m}\equiv m-C\ . \tag{12}$$

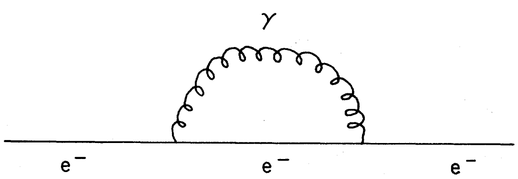

FIG. 1. The diagram for the fermion self-energy.

In order to find the physical mass of the particle we seek the pole in the propagator. This may be done fairly simply:

$$S(p)=\frac{i}{\tilde{\not p}-\tilde{m}+i\epsilon}=\frac{i(\tilde{\not p}+\tilde{m})}{\tilde{p}^2-\tilde{m}^2+i\epsilon}\ . \tag{13}$$

The pole occurs when

$$\tilde{p}^2-\tilde{m}^2=0 \tag{14}$$

or

$$(1-2A)E^2-(1-2B)\vec{p}^{\,2}-m_0{}^2+2mC=0\ ,$$
$$E^2-\vec{p}^{\,2}=m_0{}^2-2m_0C+2AE^2-2B\vec{p}^{\,2}$$
$$=m_0{}^2+\frac{\alpha}{4\pi^2}(2I\cdot p+2J_B\cdot p+2m^2J_A)$$
$$\equiv m_{\text{phys}}{}^2(\vec{p}^{\,2})\ . \tag{15}$$

On the right-hand side of this equation it is appropriate to this order in α to use $E=E_0=(\vec{p}^{\,2}+m_0{}^2)^{1/2}$. The physical mass thus found can be a function of the three-momentum $\vec{p}$.

For the one-loop calculation we find (see Appendix B)

$$m_{\text{phys}}{}^2=m_0{}^2+\tfrac{2}{3}\alpha\pi T^2+\frac{\alpha}{2\pi^2}m_0{}^2J_A(p)$$
$$+\frac{4\alpha}{\pi}\int_0^\infty\frac{l^2dl}{E_l}n_F(E_l)\ . \tag{16}$$

At low temperatures ($T\ll m_0$) the last two terms, due to fermions in the heat bath, are exponentially small and the physical mass is independent of momentum. However, at high temperatures it does depend on p through $J_A(p)$. As Weldon has pointed out, the finite-T mass is nonzero even if the $T=0$ mass, m_0, vanishes.[6]

The particles which propagate freely in the finite-T heat bath satisfy

$$(\tilde{\not p}-\tilde{m})u_\beta(p)=0\ . \tag{17}$$

If Σ were Lorentz invariant ($A=B$) this would reduce to the Dirac equation with the mass $\tilde{m}$ being the physical mass. However, in general, the finite-T spinors are

$$u_\beta(p)=\left[\frac{\tilde{E}+\tilde{m}}{2\tilde{E}}\right]^{1/2}\left[\begin{array}{c}\chi\\ \dfrac{\vec{\sigma}\cdot\vec{p}}{\tilde{E}+\tilde{m}}\chi\end{array}\right]=\left[\frac{E(1-A)+m-C}{2E(1-A)}\right]^{1/2}\left[\begin{array}{c}\chi\\ \dfrac{\vec{\sigma}\cdot\vec{p}(1-B)}{E(1-A)+m-C}\chi\end{array}\right] \tag{18}$$

with a corresponding expression for the case of antiparticles. The renormalized fermion field operator can be expanded in terms of these solutions:

$$\psi^R(x)=\int\frac{d^3p}{(2\pi)^3}\sum_s[u_{\beta,s}(p)b_s(p)e^{-ip\cdot x}+v_{\beta,s}(p)d_s^\dagger(p)e^{+ip\cdot x}]\ . \tag{19}$$

With the normalizations given, these fields satisfy the usual anticommutation rules

$$\begin{aligned}&\{\psi_\alpha^R(x),\bar{\psi}_\beta^R(x')\}_{x_0=x_0'}=\delta^3(x-x')\gamma_{0\alpha\beta}\ ,\\ &\{b_s(p),b_{s'}^\dagger(p')\}=\delta^3(p-p')\delta_{ss'}\ ,\\ &\{d_s(p),d_{s'}^\dagger(p')\}=\delta^3(p-p')\delta_{ss'}\ .\end{aligned} \tag{20}$$

It is these commutation rules which *define* the renormalized propagator. We write

$$\begin{aligned}S_\beta^R(x-y)&={}_\beta\langle 0\,|\,T(\psi^R(x)\bar{\psi}^R(y))\,|\,0\rangle_\beta\\ &=\int\frac{d^3p}{(2\pi)^3}\sum_s[\theta(x_0-y_0)u_\beta(p)\bar{u}_\beta(p)e^{-ip\cdot(x-y)}+\theta(y_0-x_0)v_\beta(p)\bar{v}_\beta(p)e^{ip\cdot(x-y)}]\ .\end{aligned} \tag{21}$$

If we use

$$\sum_s u_{\beta,s}(p)\bar{u}_{\beta,s}(p)=\frac{\tilde{\not p}+\tilde{m}}{2\tilde{E}},\quad \sum_s v_{\beta,s}(p)\bar{v}_{\beta,s}(p)=-\frac{\tilde{\not p}+\tilde{m}}{2\tilde{E}}\ , \tag{22}$$

this can be converted into a four-dimensional integral

$$S_\beta^R(x-y)=-i\int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}(1-A)\frac{(\tilde{\not p}+\tilde{m})}{\tilde{p}^{\,2}-\tilde{m}^{\,2}+i\epsilon}=-i\int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}\frac{(1-A)}{\tilde{\not p}-\tilde{m}+i\epsilon}\ . \tag{23}$$

The factor of $(1-A)$ is required to produce a correctly normalized propagator. Comparing this with the unrenormalized form permits us to identify the wave-function renormalization constant

$$Z_2{}^{-1}=(1-A)\ . \tag{24}$$

When the self-energy is Lorentz invariant this rule is identical to the conventional definition of wave-function renormalization

$$\frac{\partial\Sigma(p)}{\partial\not p}=1-Z_2{}^{-1}\ , \tag{25}$$

but the prescription is *not* the same in general. The explicit form of Z_2 is

$$\begin{aligned}Z_2{}^{-1}=&Z_2{}^{-1}(T=0)-\frac{2\alpha}{\pi}\int\frac{dk}{k}n_B(k)\\ &-\frac{\alpha\pi T^2}{6E^2}\frac{1}{v}\ln\left[\frac{1+v}{1-v}\right]+J_A-\frac{1}{E}J_B^0\end{aligned} \tag{26}$$

with J_A and J_B^0 given in Eq. (8).

IV. RADIATIVE CORRECTIONS

There are several novel aspects to the calculation of radiative corrections at finite temperature. We will discuss these in more detail below but here we list some of these features.

(a) One should use the finite-T spinors, Eq. (18), rather than the usual $T=0$ spinors.

(b) In addition to the inclusion of soft-photon emission, one must allow for processes involving the *absorption* of soft photons or fermions from the heat bath.

(c) Finite-T renormalization constants are used and the phase space is determined using the finite-T mass, m_{phys}.

(d) The density-of-final-states factor is modified by particles in the heat bath to

$$\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}[1-n_F(E_p)] \tag{27a}$$

for fermions, and

$$\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}[1+n_B(E_p)] \tag{27b}$$

for bosons. The existence of some of these features have been noted in previous studies.[4–6]

In addition there are two new types of divergences to deal with. The first of these comes because the usual $1/k$ bremsstrahlung singularities become intensified by the presence of bosons in the heat bath. We will see that the singularity is now of the form

$$\frac{1}{k}\frac{1}{e^{\beta k}-1}\underset{k\to 0}{\sim}\frac{1}{\beta k^2}\ .$$

It will turn out that these leading singularities will cancel between the vertex renormalization, soft emission, and soft absorption. Another potential divergence comes from

mass singularities. At $T=0$, the Kinoshita-Lee-Nauenberg theorem[8] assures that any singularities as $m \to 0$ can be absorbed into renormalization constants, leaving physical processes finite in this limit. However, at finite T, new mass singularities can in principle appear. A finite-T version of the KLN theorem has not yet been proven, although we feel that it must be a physical requirement that processes finite at $T=0$ should remain finite when $T \neq 0$. We have not completely answered this question, but the mass singularities in our explicit calculation do indeed disappear when all diagrams are included.

The need to utilize finite-T spinors is related to the method of mass counterterms at $T=0$. We recall that, since one wants to use the physical mass, the original Lagrangian is rewritten as

$$\begin{aligned}\mathcal{L}^{T=0} &= \bar{\psi}(i\nabla\!\!\!/ - m)\psi \\ &= \bar{\psi}(i\nabla\!\!\!/ - m_{\text{phys}} + \delta m)\psi \qquad (28)\end{aligned}$$

with $\delta m = m_{\text{phys}} - m_0$. One then employs spinors involving m_{phys} and uses δm as a perturbative counterterm. At finite T this can be generalized (in momentum space) to

$$\text{“}\mathcal{L}\text{”} \sim p\!\!\!/ - m_0 = [p\!\!\!/ - m_0 - \Sigma(p)]\,|_{p\!\!\!/=m_0} + \Sigma(p)\,|_{p\!\!\!/=m_0} = \tilde{p\!\!\!/} - \tilde{m} + \Sigma(p)\,|_{p\!\!\!/=m_0} \,. \qquad (29)$$

In this case we employ then the spinors associated with the “Lagrangian” $\tilde{p\!\!\!/} - \tilde{m}$, and use the self-energy as a “mass counterterm.” We will illustrate this in our explicit calculation.

The need to modify the density-of-states factor *and* to include absorption of particles from the heat bath has been previously noted by Cambier *et al.*[5] and Dicus *et al.*[6] That one must include the absorption diagrams in the total decay rate is clear. However, it is less obvious that this inclusion plays a crucial role in the calculation of radiative corrections. In our example below, however, the absorption of soft photons diverges like $(1/k)n_\beta(k)$ and is *required* in order to cancel similar divergences due to vertex and wave-function renormalization. The modification of the density-of-final-states factor is due to the stimulated emission caused by the particles in the heat bath. For bosons it adds to the total rate as in Eq. (27b). However, for fermions, the sign is changed and the “stimulated emission” term lowers the decay rate (due to the Pauli principle).

As an explicit example of the method of calculating radiative corrections at finite temperature we will consider the decay $H \to e^+e^-$, where H is a scalar particle (such as, for example, a Higgs boson). For simplicity, we will assume the H to be at rest with respect to the heat bath, and calculate at $T \ll m_e$. This latter condition means that we can neglect finite-temperature modifications to the tree-level *fermion* propagators; thus temperature effects enter only through the photons. The $T=0$ radiative corrections to this process have been discussed in detail by Braaten,[9] and we will only deal with the $T \neq 0$ corrections. The relevant diagrams to one-loop order are shown in Fig. 2, and we will discuss each in turn.

(i) The bare vertex is

$$M_0 = -ig\bar{u}(p')v(p) \qquad (30)$$

which leads to the lowest-order decay rate

$$\Gamma_0 = \frac{m_H}{8\pi} g^2 v^3 \,, \qquad (31)$$

where

$$v = \left[1 - \frac{4m_0^2}{m_H^2}\right]^{1/2} .$$

(ii) The vertex correction, Fig. 2(b), has a temperature-dependent part (see Appendix B)

$$M_{\text{vertex}} = -M_0 \frac{\alpha}{\pi} \frac{1+v^2}{v} \ln\left[\frac{1+v}{1-v}\right] \int_0^\infty \frac{dk}{k} n_B(k) \,. \qquad (32)$$

This term illustrates the two new possible divergences discussed earlier. As $k \to 0$ the integrand behaves as dk/k^2 instead of the usual dk/k infrared behavior. In addition M is singular when $m \to 0, v \to 1$. This divergence *cannot*

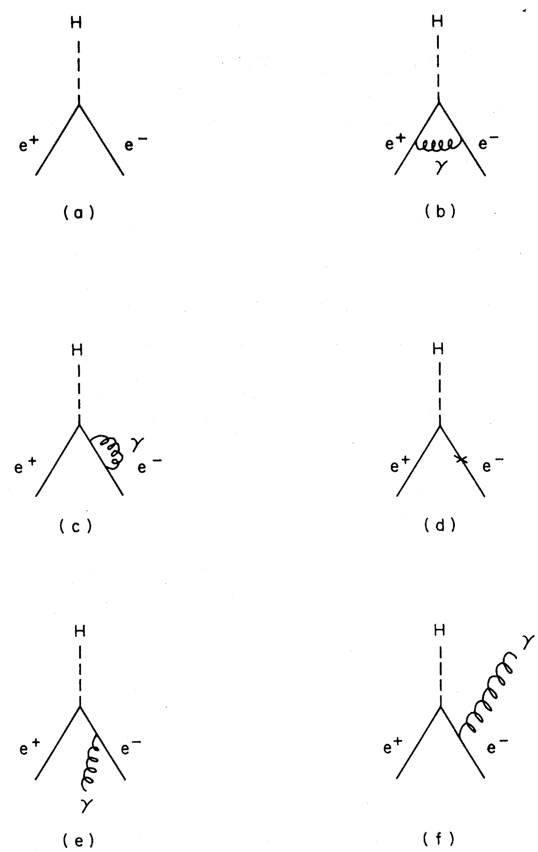

FIG. 2. Diagrams for $H \to e^+e^-$ to order α. Diagram (a) gives the bare coupling, (b) is the vertex correction, (c) and (d) are the self-energy and mass-counterterm contributions, respectively, while (e) and (f) correspond to real emission and absorption.

be absorbed into coupling-constant renormalization because the coupling constants can be renormalized at $T=0$. However, we will see that both of these singularities will disappear by the time we arrive at the final result.

(iii) In discussing the self-energy corrections let us recall the calculation of the previous section. At low T ($T \ll m_e$) the finite-T portion of the self-energy can be written as

$$\Sigma^T(p)=[(\not p-m)I_A+\not I(p)]\frac{\alpha}{4\pi^2}\,, \tag{33}$$

where (cf. Appendix B)

$$I_A=8\pi\int\frac{dk}{k}n_B(k)\,,$$
$$I_0(p)=\frac{2\pi^3T^2}{3E_p}\frac{1}{v}\ln\left[\frac{1+v}{1-v}\right]\,, \tag{34}$$
$$I_i(p)=\frac{2\pi^3T^2}{3E_p}\frac{1}{v^2}\left[\ln\left[\frac{1+v}{1-v}\right]-2v\right]\hat p_i\,.$$

The corresponding mass shift and renormalization constant are then

$$\delta m=\frac{\alpha\pi T^2}{3m}\,,$$
$$Z_2=\left[1+I_A\frac{\alpha}{4\pi^2}+\frac{I_0}{E}\frac{\alpha}{4\pi^2}\right]. \tag{35}$$

The self-energy contributions to the matrix element, Fig. 2(c), are then

$$M_{\rm SE}=-ig\bar u(p')\left[2I_A\frac{\alpha}{4\pi^2}+\frac{\delta m}{\not p-m}+\frac{\delta m}{\not p'+m}-\frac{\not I(p)}{2m}\frac{\alpha}{4\pi^2}-\frac{\not I(p')}{2m}\frac{\alpha}{4\pi^2}\right]v(p)\,. \tag{36}$$

(iv) In order to describe the mass counterterms, Fig. 2(d), we recall that we must employ the finite-T spinors in calculating the decay rate. This amounts to rewriting the momentum space "Lagrangian"

$$``\mathscr{L}"=\not p-m_0$$

as

$$``\mathscr{L}"=\not p-m+\not I(p)\frac{\alpha}{4\pi^2}\,. \tag{37}$$

The mass counterterm for Fig. 2(d) is then

$$``\delta\mathscr{L}"=+\not I(p)\frac{\alpha}{4\pi^2} \tag{38}$$

which leads to

$$M_{CT}=-ig\bar u(p')\left[-\frac{\delta m}{\not p-m}-\frac{\delta m}{\not p'+m}+\frac{\not I(p)}{2m}\frac{\alpha}{4\pi^2}+\frac{\not I(p')}{2m}\frac{\alpha}{4\pi^2}\right]v(p)\,. \tag{39}$$

(v) We multiply the matrix element by a factor of $Z_2^{-1/2}$ for each external fermion line.

Collecting all contributions to the two-body vertex (i)→(v) we have thus

$$M=-ig\bar u_\beta(p')\left[1-\frac{\alpha}{\pi}\frac{1+v^2}{v}\ln\left[\frac{1+v}{1-v}\right]\int\frac{dk}{k}n_B(k)-\frac{I_0}{E}\frac{\alpha}{4\pi^2}+I_A\frac{\alpha}{4\pi^2}\right]v_\beta(p)\,. \tag{40}$$

(vi) The matrix element for real emission is

$$M_{\rm RE}=-ieg\bar u(p')\left[\frac{p^\mu}{p\cdot k}-\frac{p'^\mu}{p'\cdot k}+\gamma^\mu\not k\left[\frac{1}{2p\cdot k}+\frac{1}{2p'\cdot k}\right]\right]v(p)\epsilon_\mu(k)\,. \tag{41}$$

When squared and summed over spins this leads to

$$\overline{|M|^2}=4e^2g^2\left\{(p\cdot p'-m^2)\left[\frac{2p\cdot p'}{p\cdot k\,p'\cdot k}-\frac{m^2}{(p\cdot k)^2}-\frac{m^2}{(p'\cdot k)^2}\right]+\left[2p\cdot p'-m^2\frac{p\cdot k}{p'\cdot k}-m^2\frac{p'\cdot k}{p\cdot k}\right]\left[\frac{1}{p\cdot k}+\frac{1}{p'\cdot k}\right]+\left[\frac{p'\cdot k}{p\cdot k}+\frac{p\cdot k}{p'\cdot k}+2\right]\right\}. \tag{42}$$

[We caution the reader that it is incorrect to set $m^2=0$ in the above result even if one is not interested in small terms such as $m^2/m_H{}^2$. This is because some of the denominators have $1/m^2$ singularities which renders the result of $O(1)$. These would be missed by setting $m^2=0$ too early in the calculation.[10]]

(vii) The matrix element for absorption of a photon is obtained from Eq. (42) by $k_\mu\to-k_\mu$.

We now evaluate all the corrections to the decay rate. The simplest correction is that due to the modification which takes place in "phase space" due to the temperature-dependent change in the mass of the electron. This yields (cf. Appendix B)

$$\Delta\Gamma=-\Gamma_0\frac{12m\delta m}{m_H{}^2v^2}+O(\delta m^2)$$
$$=-\frac{4\pi\alpha T^2}{m_H{}^2v^2}+O(T^4)\ . \tag{43}$$

The correction to the two-body decay rate is found by squaring Eq. (40) and summing over final states, giving (cf. Appendix B)

$$\Delta\Gamma=2\Gamma_0\left[\int\frac{dk}{k}n_B(k)\left[-\frac{\alpha}{\pi}\frac{1+v^2}{v}\ln\frac{1+v}{1-v}+\frac{2\alpha}{\pi}\right]-\frac{\alpha}{4\pi^2}\frac{\vec{\mathrm{I}}(p)\cdot\vec{\mathrm{p}}}{p^2}\right]\ . \tag{44}$$

The three-body rates (stimulated emission plus absorption) are more complicated because one must integrate over the whole of the Dalitz plot, weighted by the distribution function $n_B(k)$. The matrix element is a complicated function of k and the integrals of this with $n_B(k)$ cannot be evaluated analytically. What we do is to expand $d\Gamma/dk$ in powers of k [times $n_B(k)$] and then integrate term by term, converting this integral into a power series in T

$$\frac{d\Gamma}{dk}=n_B(k)\left[\frac{1}{k}R_{-1}+R_0+kR_1+O(k^2)\right]\ ,$$
$$\Gamma=\Gamma_{-1}+\Gamma_0+\Gamma_1+\cdots\ . \tag{45}$$

In doing the calculation one must expand not only the matrix element but also the size and shape of the Dalitz plot, which makes the calculation quite tedious. For the leading (most divergent) term we find [adding emission and absorption, which contribute equally (cf. Appendix B)]

$$\Gamma_{(-1)}=\frac{2\alpha\Gamma_0}{\pi}\left[\frac{1+v^2}{v}\ln\left[\frac{1+v}{1-v}\right]-2\right]\int\frac{dk}{k}n_B(k)\ . \tag{46}$$

As expected, this precisely cancels the leading soft-photon singularities in the two-body decay rate. The next term, $R_{(0)}$, would also lead to an infrared divergence because $n_B(k)\sim 1/\beta k$ as $k\to 0$. There is nothing in the two-body rate, Eq. (44), with which to ameliorate this singularity, and so $R_{(0)}$ must vanish. Indeed it does, due to a cancellation of the effect of emission and spontaneous absorption. Finally for the finite contribution of $\Gamma_{(1)}$ we find

$$\Gamma_{(1)}=\frac{8\alpha}{\pi}\frac{\Gamma_0}{m_H{}^2v^3}\left[\ln\frac{1+v}{1-v}+v\right]n_B(k)k\ . \tag{47}$$

Adding together the two-body and three-body rate corrections and integrating we find

$$\Delta\Gamma_{2,3}=\frac{4\pi\alpha T^2}{m_H{}^2v^2}\Gamma_0+O(T^4)\ . \tag{48}$$

Note that at this stage all the mass singularities have also disappeared, as we speculated must occur on physical grounds.[10]

The total correction, Eq. (43) plus Eq. (48), vanishes to the order in $T^2/M_H{}^2$ which we are working:

$$\Delta\Gamma_{\mathrm{TOT}}=\Delta\Gamma_{\mathrm{ps}}+\Delta\Gamma_{2,3}=0+O(T^4)\ . \tag{49}$$

This vanishing appears to be accidental, but we have also calculated the radiative corrections for the decay of a pseudoscalar H (instead of scalar) and found that to be zero also. The calculation, however, does serve to illustrate the finite-temperature renormalization techniques.

There exist calculations in the literature[4,5] evaluating radiative corrections at finite temperature, which have introduced some of the features described above. However, the appropriate procedure for wave-function renormalization appears to have been overlooked. For example, the authors of Ref. 5 miss the finite terms I_0/E in the wave-function renormalization because of their use of the conventional Eq. (25) instead of Eq. (24). The term I_0/E is significant because it is singular as $m\to 0$, and perhaps could remove the mass singularities found in Ref. 5.

V. SUMMARY

We have discussed mass and wave-function renormalization for a fermion at finite temperature. There exists a temperature-dependent and (generally) three-momentum-dependent mass shift which would be found by locating the pole in the propagator. The wave equation for a fermion in a heat bath is modified from the Dirac equation by one-loop corrections and we defined finite-T spinors [Eq. (18)] which solve the wave equation. By requiring that the canonical commutation rules remain satisfied we identify the renormalized propagator and thereby the wave-function renormalization constant Z_2. This identification is not the usual one, but does reduce to it if the self-energy is Lorentz invariant. The mass and wave-function renormalization for an electron were calculated in finite-temperature QED. We agree with previous results on the mass shift,[11] but feel that other workers have incorrectly calculated the wave-function renormalization Z_2.

Radiative corrections are calculated in the same way as at $T=0$, except for the additional features.

(a) Finite-T spinors should be used.

(b) Phase space has a temperature-dependent and (generally) momentum-dependent modification due to the mass shift.

(c) One must include diagrams involving the absorption of particles from the heat bath.

(d) One must include stimulated emission. This effectively modifies the density-of-final-states factor to

$$\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}[1+n_B(E_p)]$$

for bosons, and

$$\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}[1-n_F(E_p)]$$

for fermions.

(e) Care must be taken to correctly identify the "mass" counterterms as the Lorentz noninvariance of the calculation renders invalid the usual identification.

These points were illustrated by a sample calculation, $H\to e^+e^-$.

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APPENDIX A: FINITE-TEMPERATURE PROPAGATORS

In this appendix we present an elementary derivation of the tree-level propagator at finite T. For simplicity we treat a scalar field. The field operator is expanded in terms of creation and annihilation operators via

$$\phi(x)=\int \frac{d^3k}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega_k}}[e^{-ik\cdot x}a(k)+e^{+ik\cdot x}a^\dagger(k)] . \tag{A1}$$

In this normalization, we have

$$[a(k),a^\dagger(k)]=\delta^3(\vec{k}-\vec{k}\,') \tag{A2}$$

and

$$:H:=\int d^3k\,\omega_k a^\dagger(k)a(k) \tag{A3}$$

with H being normal ordered with respect to the $T=0$ vacuum. With this choice of normalization, we can read off the matrix element of the number operator at finite T

$$_\beta\langle 0\,|\,a^\dagger(k)a(k)\,|\,0\rangle_\beta=n_B(\omega_k)=\frac{1}{e^{\beta\omega_k}-1} . \tag{A4}$$

The finite-temperature propagator can now be calculated,

$$\begin{aligned}
D_\beta(x-y)&\equiv{}_\beta\langle 0\,|\,T[\phi(x)\phi(y)]\,|\,0\rangle_\beta\\
&=\int\frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\omega_k 2\omega_{k'})^{1/2}}\,{}_\beta\langle 0\,|\,\theta(x_0-y_0)[e^{-ik\cdot x}e^{+ik'\cdot y}a(k)a^\dagger(k')+e^{+ik\cdot x}e^{-ik'\cdot y}a^\dagger(k)a(k')]+x\leftrightarrow y\,|\,0\rangle_\beta\\
&=\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}[\theta(x_0-y_0)e^{-ik\cdot(x-y)}+\theta(y_0-x_0)e^{+ik\cdot(x-y)}]\\
&\quad+\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}n_B(\omega_k)(e^{ik\cdot(x-y)}+e^{-ik\cdot(x-y)}) .
\end{aligned}\tag{A5}$$

The $aa^\dagger$ terms generate the usual Feynman propagator while the $a^\dagger a$ terms count the particles in the heat bath. The propagator can be easily rewritten using contour-integral techniques

$$D_\beta(x-y)=i\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-y)}\left[\frac{1}{k^2-m^2+i\epsilon}-2\pi i\delta(k^2-m^2)n_\beta(\omega_k)\right] , \tag{A6}$$

which is the desired result. Similar derivations produce the spin-$\frac{1}{2}$ and spin-1 finite-temperature propagators. We observe then that the finite-T modification of the propagator is due to the existence of real particles in the heat bath.

APPENDIX B: CALCULATIONAL DETAILS

We calculate

$$I^\mu=2\int\frac{d^3k}{k_0}\frac{1}{e^{\beta k_0}-1}\frac{k^\mu}{E_pk^0-\vec{p}\cdot\vec{k}} . \tag{B1}$$

First note that

$$p_\mu I^\mu=2\int\frac{d^3k}{k_0}\frac{1}{e^{\beta k_0}-1}=8\pi\int_0^\infty\frac{k\,dk}{e^{\beta k}-1}=\frac{8\pi}{\beta^2}\int_0^\infty\frac{x\,dx}{e^x-1}=\frac{4\pi^3}{3\beta^2} . \tag{B2}$$

I^0 can be calculated directly

$$I^0=\frac{2}{E_p}\int k\,dk\,d\Omega_{\hat{k}}\frac{1}{e^{\beta k}-1}\frac{1}{1-\vec{v}\cdot\hat{k}}\left[\text{where }\vec{v}=\frac{\vec{p}}{E_p}\right]=\frac{4\pi}{vE_p\beta^2}\ln\frac{1+v}{1-v}\int_0^\infty\frac{x\,dx}{e^x-1}=\frac{2\pi^3}{3\beta^2}\frac{1}{vE_p}\ln\frac{1+v}{1-v} . \tag{B3}$$

Then

$$\vec{I}=\frac{\vec{p}}{p^2}\frac{4\pi}{\beta^2}\int_0^\infty\frac{x\,dx}{e^x-1}\left[\frac{1}{v}\ln\frac{1+v}{1-v}-2\right]=\frac{\vec{p}}{p^2}\frac{2\pi^3}{3\beta^2}\left[\frac{1}{v}\ln\frac{1+v}{1-v}-2\right] .$$

For the vertex renormalization diagram [Fig. 2(b)] we find

$$\delta M = g_0 e^2 \int \frac{d^4k}{(2\pi)^4} 2\pi\delta(k^2) n_B(k_0) \bar{u}(p_1)\gamma_\mu \frac{i}{\not p_1 - \not k - m} \frac{i}{\not p_2 - \not k - m} \gamma^\mu v(p_2)$$

$$= -M_0 \frac{\alpha}{2\pi^2} \int d^4k\, \delta(k^2) n_B(k_0) \frac{p_1 \cdot p_2}{p_1 \cdot k p_2 \cdot k}$$

$$= -M_0 \frac{\alpha}{2\pi^2} (1+v^2) \int \frac{d^3k}{k_0{}^3} \frac{1}{1-\vec{v}\cdot\hat{k}} n_B(k_0)$$

$$= -M_0 \frac{\alpha}{\pi} (1+v^2) \frac{1}{v} \ln\frac{1+v}{1-v} \int_0^\infty \frac{dk}{k} n_B(k) . \tag{B4}$$

We obtain the so-called "phase-space" correction by evaluating the effect of the change $m_0 \to m_{\text{phy}}$ on the zeroth-order decay rate. We find

$$\frac{\delta\Gamma}{\Gamma} = \frac{3v^2}{v^3} \frac{dv}{dm} \delta m = -12 \frac{m\,\delta m}{v^2 m_H{}^2} \tag{B5}$$

as given in Eq. (43).

In calculating the so-called two-body decay rate, we start with Eq. (40), square, and take traces using finite-temperature spinors. This yields

$$\Gamma^{(\text{two-body})} = \Gamma_0 \frac{\tilde{p}_2 \cdot \tilde{p}_1 - \tilde{m}^2}{2\tilde{E}^2 v^2} \left[1 - 2\frac{\alpha}{\pi} \frac{1+V^2}{V} \ln\left[\frac{1+v}{1-v}\right] \int \frac{dk}{k} n_B(k) - 2\frac{I_0}{E}\frac{\alpha}{4\pi^2} + 2I_A \frac{\alpha}{4\pi^2} \right]$$

$$= \Gamma_0 \frac{E^2 - m_{\text{phy}}{}^2}{E^2 - m_0{}^2} \left[1 - 2\frac{\alpha}{\pi} \frac{1+v^2}{v} \ln\frac{1+v}{1-v} \int \frac{dk}{k} n_B(k) + \frac{4\alpha}{\pi} \int \frac{dk}{k} n_B(k) - 2\frac{\alpha}{4\pi^2} \frac{\vec{\mathrm{p}}\cdot\vec{\mathrm{I}}(p)}{p^2} \right] . \tag{B6}$$

However, the factor $(E^2 - m_{\text{phy}}{}^2)/(E^2 - m_0{}^2)$ is already included in what we have called the phase-space correction. Thus we have

$$\Delta\Gamma^{\text{two-body}} = -2\Gamma_0 \frac{\alpha}{\pi} \left[\int \frac{dk}{k} n_B(k) \left[\frac{1+v^2}{v} \ln\frac{1+v}{1-v} - 2 \right] + \frac{1}{4\pi} \frac{\vec{\mathrm{p}}\cdot\vec{\mathrm{I}}(p)}{p^2} \right] \tag{B7}$$

as given in Eq. (44).

In order to calculate the rate for emission of real photons the matrix-element factor [Eq. (42)] must be multiplied by the phase-space factor

$$\int \frac{d^3k}{(2\pi)^3 2k} \frac{d^3p_1}{(2\pi)^3 2E_1} \frac{d^3p_2}{(2\pi)^3 2E_2} [1+n_B(k^0)](2\pi)^4 \delta^4(k - p_1 - p_2 - k) = \frac{1}{4} \frac{1}{(2\pi)^2} \int_0^\infty dk \int_{E_{\max}}^{E_{\max}} dE_1 [1+n_B(k_0)] \tag{B8}$$

and integrated over E_1, k. Here

$$E_{\max,\min} = \tfrac{1}{2} \left[m_H - k \pm k \left[1 - \frac{4m^2}{m_H(m_H - 2k)} \right]^{1/2} \right] \tag{B9}$$

represents the range in lepton energy allowed by kinematics. For the temperature-dependent piece we need only keep the terms in $n_B(k_0)$. Since by dimensional arguments

$$dk\, k^n n_B(k) \sim T^{n+1} \tag{B10}$$

as long as we are willing to work at low temperatures ($T \ll m_H$) it makes sense to expand the final integrand over dk as a power series in k/m_H. We find then for the rate of real-photon emission

$$\Gamma_\epsilon=\frac{g_0{}^2}{2m_H}\frac{\alpha}{8\pi^2}\int_0^\infty dk\, n_B(k_0)\left\{8k\ln\frac{\left[1+\left[1-\frac{4m^2}{m_H(m_H-2k)}\right]^{1/2}\right]^2}{\frac{4m^2}{m_H(m_H-2k)}}\left[1-\frac{m_H}{k}+\frac{1}{2}\frac{m_H{}^2}{k^2}-3\frac{m^2}{k^2}+4\frac{m^2}{m_Hk}\right.\right.$$

$$\left.\left.+\,4\frac{m^4}{m_H{}^2k^2}\right]-8(m_H-2k)\left[1-\frac{4m^2}{m_H(m_H-2k)}\right]^{1/2}\left[\frac{1}{2}\frac{m_H}{k}-2\frac{m^2}{km_H}\right]\right\}$$

$$=4g_0{}^2\frac{\alpha}{8\pi^2}\int_0^\infty dk\, n_B(k_0)\left\{\frac{m_H}{k}\left[(1+v^2)\frac{v^2}{4}\ln\frac{1+v}{1-v}-\frac{1}{2}v^3\right]+\left[-v^2\ln\frac{1+v}{1-v}\right]+\frac{k}{m_H}\left[v+\ln\frac{1+v}{1-v}\right]\right\}. \qquad \text{(B11)}$$

The rate for absorption is obtained from Eq. (B11) by changing $k\to-k$. Thus the combined absorption and emission rate is given by

$$\Gamma_\epsilon+\Gamma_a=g_0{}^2\frac{\alpha}{\pi^2}\int_0^\infty dk\, n_B(k_0)\left\{\frac{m_H}{k}\left[(1+v^2)\frac{v^2}{4}\ln\frac{1+v}{1-v}-\frac{1}{2}v^2\right]+\frac{k}{m_H}\left[v+\ln\frac{1+v}{1-v}\right]\right\}$$

$$=\Gamma_0\frac{8\alpha}{\pi}\left[\left[\frac{1+v^2}{4v}\ln\frac{1+v}{1-v}-\frac{1}{2}\right]\int_0^\infty\frac{dk}{k}n_B(k)+\left[\frac{1}{v^3}\ln\frac{1+v}{1-v}+\frac{1}{v^2}\right]\int_0^\infty dk\frac{k}{m_H{}^2}n_B(k)\right]. \qquad \text{(B12)}$$

Note here that the most singular term, $\int_0^\infty (dk/k)n_B(k)$, exactly cancels against the corresponding two-body infrared singularity. The term in $\int_0^\infty dk\, n_B(k)$, which is also singular, cancels between the emission and absorption diagrams. Finally, the remaining term, $\int_0^\infty dk\, kn_B(k)$, is nonsingular and is the three-body contribution to the radiative correction given in Eq. (47).

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