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It is shown that quarks in different situations are often related by a Bogoliubov transformation involving both quark and antiquark degrees of freedom. This is exploited, within the context of the MIT bag model, to provide a calculation of the πNN and ρNN coupling constants, with encouraging results.

I. INTRODUCTION

There exist in particle physics many different types of quark descriptions: current quarks, bag-model quarks, constituent quarks, etc. Each is useful in a particular context, but, although they carry the same charge, spin, isospin, etc., quarks in the various formulations are distinct. The choice of quark description is governed by the applications being considered.

It is, however, not generally appreciated just how subtle are the definitions of quark and antiquark degrees of freedom. Our intuition in composite models comes primarily from nonrelativistic physics, where antiparticles do not enter. However, in relativistic models, the labels of quark and antiquark are not invariant. It is the purpose of this paper to discuss this problem, and to show how it may be exploited to calculate coupling constants.

It will be shown that at least some of the different quark descriptions are related by a Bogoliubov transformation which relabels quark and antiquark degrees of freedom. More surprisingly, even within the same scheme, e.g., the bag model, fields in different particles may also be related in a similar manner. If fields in a given nucleon are designated as quarks, then the fields in a similar nucleon located in a different point, or moving with respect to the first, appear partially as antiquark holes rather than pure quarks. This, for example, allows the nonzero overlap of a QQQ nucleon and a $Q\bar{Q}$ pion onto a QQQ nucleon, which in turn can be related to the $NN\pi$ coupling constant.

II. BOGOLIUBOV TRANSFORMATIONS

Bogoliubov transformations can occur when there exist different bases in which the canonical

quantization procedure can be applied. For example, if in a given region of space we have a complete set of positive- and negative-energy solutions (quark and antiquark) to the equations of motion $\phi_{iq}(x)$ and $\phi_{i\bar{q}}(x)$, then the field operator can be expanded in this basis

$$\psi(x) = \sum_i [\phi_{iq}(x)b_i + \phi_{i\bar{q}}(x)d_i^\dagger]. \quad (1)$$

When the wave functions are normalized

$$\int d^3x \phi_i^\dagger(x)\phi_j(x) = \delta_{ij}, \quad (2)$$

the commutation relations are

$$\begin{aligned} \{b_i, b_j^\dagger\} &= \delta_{ij} = \{d_i, d_j^\dagger\}, \\ \{b_i, b_j\} &= \{d_i, d_j\} = \{b_i, d_j^\dagger\} = \{d_i, b_j^\dagger\} = 0. \end{aligned} \quad (3)$$

Of course, δ -function normalizations can also be treated in the usual manner.

There can also exist a second complete set of solutions, perhaps more complicated, which we can label $\tilde{\phi}_{iq}$ and $\tilde{\phi}_{i\bar{q}}$. The quantization procedure will go through in an identical fashion in this new basis:

$$\psi(x) = \sum_i [\tilde{\phi}_{iq}(x)\tilde{b}_i + \tilde{\phi}_{i\bar{q}}(x)\tilde{d}_i^\dagger]. \quad (4)$$

The new creation and annihilation operators are defined to satisfy the same commutation relations.

These two sets of operators may be related to each other by inverting

$$\begin{aligned} b_i &= \int d^3x \phi_{iq}^\dagger(x)\psi(x), \\ d_i^\dagger &= \int d^3x \phi_{i\bar{q}}^\dagger(x)\psi(x) \end{aligned} \quad (5)$$

and

$$\begin{aligned} \tilde{b}_i &= \int d^3x \tilde{\phi}_{iq}^\dagger(x)\psi(x), \\ \tilde{d}_i^\dagger &= \int d^3x \tilde{\phi}_{i\bar{q}}^\dagger(x)\psi(x). \end{aligned} \quad (6)$$

Inserting Eq. (4) into (5) yields

$$\begin{aligned} b_i &= \sum_j (\alpha_{ij} \tilde{b}_j + \beta_{ij} \tilde{d}_j^\dagger), \\ d_i^\dagger &= \sum_j (\gamma_{ij} \tilde{d}_j^\dagger + \epsilon_{ij} \tilde{b}_j), \end{aligned} \quad (7)$$

where

$$\begin{aligned} \alpha_{ij} &= \int d^3x \phi_{iq}^\dagger(x) \tilde{\phi}_{jq}(x), \\ \beta_{ij} &= \int d^3x \phi_{iq}^\dagger(x) \tilde{\phi}_{jq}(x), \\ \gamma_{ij} &= \int d^3x \phi_{iq}^\dagger(x) \tilde{\phi}_{jq}(x), \\ \epsilon_{ij} &= \int d^3x \phi_{iq}^\dagger(x) \tilde{\phi}_{jq}(x). \end{aligned} \quad (8)$$

Completeness requires

$$\begin{aligned} \sum_j (\alpha_{ij}^* \alpha_{jk} + \beta_{ij}^* \beta_{jk}) &= \delta_{ik}, \\ \sum_j (\alpha_{ij}^* \epsilon_{jk} + \beta_{ij}^* \gamma_{jk}) &= 0, \\ \sum_j (\gamma_{ij}^* \gamma_{jk} + \epsilon_{ij}^* \epsilon_{jk}) &= \delta_{ik}. \end{aligned} \quad (9)$$

These are precisely the conditions to leave the commutation relations form invariant under the transformation. The reverse transformation is

$$\begin{aligned} \tilde{b}_i &= \sum_j (\alpha_{ij}^* b_j + \beta_{ij}^* d_j^\dagger), \\ \tilde{d}_i^\dagger &= \sum_j (\alpha_{ij}^* d_j + \epsilon_{ij}^* b_j). \end{aligned} \quad (10)$$

These are the Bogoliubov transformations.¹ In this paper, we will primarily be concerned with the transformation where both basis wave functions satisfy the same equations of motion, at least locally. However, the transformations can be more general. For example, a common usage in condensed matter physics is to relate the free field to the interacting field (in some approximation) by such a transformation.²

An interesting property of these transformations is that the interpretation of states is mixed. For example, the vacuum of one representation, defined

$$\psi(x) = \int \frac{d^3p}{(2\omega_p)^{1/2}} \frac{1}{(2\pi)^{3/2}} [u(p)e^{-ip \cdot x} b(p) + v(p)e^{+ip \cdot x} d^\dagger(p)] . \quad (16)$$

The normalization is such that

$$\{b^\dagger(p'), b(p)\} = \{d^\dagger(p'), d(p)\} = \delta^3(p - p'), \quad u^\dagger(p)u(p) = \omega_p. \quad (17)$$

The ground-state bagged quark can be solved in terms of plane-wave quarks to be (m is a spin index which has heretofore been suppressed)

$$b_{om} = \int d^3p f(p) [b(\vec{p}, m) e^{-i(p - \omega)t} + \vec{\sigma}_{mm} \cdot \hat{p} d^\dagger(p, m') e^{+i(p + \omega)t}], \quad (18)$$

to have no particles by the requirement that

$$\begin{aligned} b_i |0\rangle &= 0, \\ d_i |0\rangle &= 0, \end{aligned} \quad (11)$$

for all i , contains particles of the other representation,

$$\begin{aligned} \langle 0 | \tilde{b}_i^\dagger \tilde{b}_i | 0 \rangle &= \sum_j |\beta_{ij}|^2, \\ \langle 0 | \tilde{d}_i^\dagger \tilde{d}_i | 0 \rangle &= \sum_j |\epsilon_{ij}|^2. \end{aligned} \quad (12)$$

In addition, the definitions of quark and antiquark are mixed. What is called quark in one basis has an antiquark hole component in the other. If a change of description is undertaken in a calculation, this relabeling must be taken into account.

Two examples will illustrate the procedure. In applications of the MIT bag model, one solves for the quark field operator inside a rigid cavity.^{3,4} Working with $m=0$, the wave functions are solutions to

$$i\not{\partial}\psi=0 \quad (13)$$

inside the cavity and

$$i\not{n}\psi=\psi \quad (14)$$

on the surface, with $n^\mu = (0, \hat{r})$. A complete set of solutions exists, with the lowest-energy wave functions being

$$\begin{aligned} \psi_{0q}(x) &= N \begin{bmatrix} ij_0(\omega r) \chi_m \\ -j_1(\omega r) \vec{\sigma} \cdot \hat{r} \chi_m \end{bmatrix} e^{-i\omega t}, \\ \psi_{0\bar{q}}(x) &= N \begin{bmatrix} -ij_1(\omega r) \vec{\sigma} \cdot \hat{r} \chi_m \\ j_0(\omega r) \chi_m \end{bmatrix} e^{+i\omega t}, \end{aligned} \quad (15)$$

with $\omega = 2.0428/R$ (R is the radius of the cavity) and $N = 2.27/(4\pi R^3)^{1/2}$. The former is determined by the boundary condition and the latter by the normalization. Inside the bag, the field operator can be expanded in these solutions. However, one can, if one wishes, also use a plane-wave expansion also satisfying the massless Dirac equation

where

$$f(p) = -iN \left\{ \frac{\omega - p}{\omega^2 p} \left[\frac{\sin(\omega - p)R}{\omega - p} - \frac{\sin(\omega + p)R}{\omega + p} \right] + \frac{2R^2}{\omega} j_0(\omega R) j_1(pR) \right\}. \quad (19)$$

The antiquark component is by no means small. If we had included a quark mass, the antiquark term would have vanished in the nonrelativistic limit ($mR \rightarrow \infty$), but for relativistic quarks it is important. The reverse transformation is not possible here as the bagged quarks are defined only over a limited region of space.

A second example is the relation between two sets of bagged quarks. Let the first set be those quarks in a nucleon, with $R_N = 5.5 \text{ GeV}^{-1}$, located at $\vec{x} = 0$, $t = 0$. Choose the second set to be those solutions appropriate to a smaller bag radius, e.g., a pion with $R_\pi \approx 3.6 \text{ GeV}^{-1}$, located not at the origin but at $\vec{x} = \vec{\delta}$, $t = 0$. We will assume that $\vec{\delta}$ is small enough that the entire smaller bag is enclosed in the larger bag. This is important as we do not know how to define the quark field outside of a bag. Within the pion we have two complete sets and the creation operators for quarks in the pion can be expressed in terms of those in the nucleon as in Eq. (8), where $\alpha, \beta, \gamma, \epsilon$ are functions of $\vec{\delta}$. It is the relationship of the ground-state solutions (subscript zero) which is of primary interest, and it is worthwhile to calculate these explicitly for $\vec{\delta}$ small:

$$\begin{aligned} \alpha_{00}(\vec{\delta}) &= NN' 4\pi \int_0^{R_\pi} dr r^2 [j_0(\omega r) j_0(\omega' r) + j_1(\omega r) j_1(\omega' r)] + O(\delta^2) \\ &= \gamma_{00}(\vec{\delta}), \\ \beta_{00}(\vec{\delta}) &= NN' \frac{\omega' \vec{\sigma} \cdot \vec{\delta}}{3} \int_0^{R_\pi} dr r^2 [j_0(\omega r) j_0(\omega' r) - j_1(\omega r) j_1(\omega' r)] + O(\delta^3) \\ &= \delta_{00}(\vec{\delta}) \equiv \bar{\beta} \vec{\sigma} \cdot \vec{\delta} / R', \end{aligned} \quad (20)$$

where primed quantities indicate quantities with radius R' . Numerically, for $R' = R_\pi = 3.6 \text{ GeV}^{-1}$ and $R = R_N = 5.5 \text{ GeV}^{-1}$,

$$\begin{aligned} \alpha_{00} &= \gamma_{00} = 0.58, \\ \beta_{00} &= \delta_{00} = 0.27 \frac{\vec{\sigma} \cdot \vec{\delta}}{R_\pi}, \end{aligned} \quad (21)$$

while for parameters appropriate to the ρ meson, $R' = R_\rho = 4.4 \text{ GeV}^{-1}$, $R = R_N = 5.5 \text{ GeV}^{-1}$,

$$\begin{aligned} \alpha_{00} &= \delta_{00} = 0.78, \\ \beta_{00} &= \delta_{00} = 0.31 \frac{\vec{\sigma} \cdot \vec{\delta}}{R_\rho}. \end{aligned} \quad (22)$$

These relations say that the antiquark creation operator used to create the antiquark in the pion has a piece which will destroy a quark in a nucleon. A nucleon plus pion state, with the pion at $\vec{\delta} = 0$, then has the usual $QQQQ\bar{Q}$ component (in terms of nucleon, $\vec{\delta} = 0$ quarks) plus a pure QQQ piece. Specifically the bag-model proton at $\vec{x} = 0$ has an overlap with a state consisting of a proton at $\vec{x} = 0$ and a pion at $\vec{x} = \vec{\delta}$, with $\vec{\delta}$ small, of

$$\begin{aligned} {}_B \langle p, \vec{0}; \pi^0, \vec{\delta} | p, \vec{0} \rangle_B &= \frac{5}{3\sqrt{12}} \alpha_{00} \bar{\beta} \frac{\vec{\sigma} \cdot \vec{\delta}}{R_\pi} \\ &= 0.08 \frac{\vec{\sigma} \cdot \vec{\delta}}{R_\pi}, \end{aligned} \quad (23)$$

where now $\vec{\sigma}$ is the nucleon to nucleon spin operator. Likewise for a ρ meson one obtains for the transition $p(\uparrow) \rightarrow p(\uparrow) + \rho^0(0)$

$$\begin{aligned} {}_B \langle p, \uparrow, \vec{0}; \rho^0, 0, \vec{\delta} | p, \uparrow, 0 \rangle_B &= \frac{1}{\sqrt{12}} \alpha_{00} \bar{\beta} \frac{\delta_z}{R_\rho} \\ &= 0.07 \frac{\delta_z}{R_\rho}. \end{aligned} \quad (24)$$

The relation between ρ and π overlaps is nearly that of $SU(6)$, as the ρ and π wave functions are nearly degenerate at this level, the only difference being in the radii. Spin-splitting corrections to the wave functions have not been taken into account, and as spin dependence is strong in the ρ , π system, it could be assumed that these overlaps could get modified substantially.

It is amusing to consider this result in light of the Chew-Low model. In the Chew-Low theory, the nucleon and the pion were the fundamental particles and through their interaction they generated the Δ resonance. In the quark model this appears unusual as one usually solves the model with no coupling to virtual or real channels. One finds three fundamental states, the nucleon, the pion, and the Δ in the assumed limit of no coupling. If one then "turns on" the coupling one would expect two Δ states, one the three-quark Δ and the other the Chew-Low Δ formed out of π

and p . The Bogoliubov transformation shows how, even in the naive quark model, these two can be the same. Via the transformation a three-quark proton and a $Q\bar{Q}\pi^+$, for example, have a three-quark component which in fact is identical to the three-quark Δ^{++} . In addition, the coupling cannot be turned off, as it arises due to the localization of confinement.

III. COUPLING CONSTANTS

It is clear that the overlaps found in the last section must have something to do with coupling constants. However, to make the correspondence precise one must know what the bag states $|N\rangle_B$ and $|\pi\rangle_B$ are. The coupling constants are defined in terms of plane-wave states $|N(p)\rangle$ and $|\pi(p)\rangle$. The bag states are not these, but in Ref. 4, Johnson and the author argued that they can be treated as superpositions of plane-wave eigenstates, with wave packet $\phi(p)$:

$$\begin{aligned} |\pi, x\rangle_B &= \int \frac{d^3p}{2\omega_p} \phi(p) e^{ip \cdot x} |\pi(p)\rangle, \\ |N, x\rangle_B &= \int d^3p \frac{m}{E} \chi(p) e^{ip \cdot x} |N(p)\rangle. \end{aligned} \quad (25)$$

For heavy states such as the nucleon, this prescrip-

tion reduces to the usual procedure for calculating within the quark model. For light states such as the pion it leads to new results.⁴ An explicit form for $\phi(p)$ was calculated to be

$$\phi(p) = N \int_{V_\pi} d^3x e^{ip \cdot x} [j_0^2(\omega'x) - j_1^2(\omega'x)], \quad (26)$$

where N is related to F_π and is determined by the normalized condition

$$\int \frac{d^3p}{2\omega} |\phi(p)|^2 (2\pi)^3 = 1. \quad (27)$$

A useful approximation to the wave packet is a Gaussian

$$\phi(p) = C e^{-\alpha^2 p^2}, \quad (28)$$

with

$$\alpha = \frac{(\ln 2)^{1/2} R}{\pi}. \quad (29)$$

This is accurate out to about $kR = 6$, and beyond that the wave packet is not large. We will follow Ref. 4 and use the same form for $\chi(p)$, as the precise form is not as important for nucleons.

The wave-packet formalism allows one to solve for $g_{\pi NN}$. We define

$$\langle P(p') | \phi_{\pi 0}(x) | P(p) \rangle = \frac{g_{\pi NN} \bar{u}(p') \gamma_5 u(p)}{(p-p')^2 - m_\pi^2} F((p-p')^2) e^{i(p-p') \cdot x}, \quad (30)$$

where $F(q^2)$ is a form factor for the transition, normalized to $F(0) = 1$, introduced as we will be probing momenta where $(p-p')^2$ is not small. For simplicity we will assume that $m_\pi^2 = 0$ and $\langle \vec{p}^2 \rangle \ll m_N^2$, so that $m/E = 1$, both of which are reasonable for the accuracy of the current calculation. Using the relation of $\phi(x)$ and $|\pi(p)\rangle$,

$$|\pi(p)\rangle = \int d^3x e^{-ip \cdot x} \vec{\partial}_0 \phi(x) |0\rangle, \quad (31)$$

and Eq. (25), one finds that

$$\begin{aligned} {}_B \langle p, \vec{0}; \pi, \vec{\delta} | p, \vec{0} \rangle_B &= \int \frac{d^3k}{2k} e^{i\vec{k} \cdot \vec{\delta}} \phi(k) \left[\int d^3p d^3p' \chi^\dagger(p') \chi(p) \right] \\ &\quad \times \int d^3x \omega_r \frac{g_{\pi NN}}{(p-p')^2} \frac{\vec{\sigma} \cdot (\vec{p} - \vec{p}')}{2M_N} e^{i(k-p+p') \cdot x} F(p-p')^2 \\ &= \frac{g_{\pi NN}}{2M_N} \int \frac{d^3k}{2k} \sigma \cdot \hat{k} e^{i\vec{k} \cdot \vec{\delta}} \phi(k) F(\vec{k}^2) \left[\int d^3p \chi^\dagger(p-k) \chi(p) (2\pi)^3 \right], \end{aligned} \quad (32)$$

which for small $\vec{\delta}$ becomes

$${}_B \langle p; \pi^0, \vec{\delta} | p \rangle_B = \frac{g_{\pi NN}}{2M_N} \frac{1}{6} \vec{\sigma} \cdot \vec{\delta} \int d^3k \phi(k) F(\vec{k}^2) \left[\int d^3p \chi^\dagger(p-k) \chi(p) (2\pi)^3 \right]. \quad (33)$$

What this result implies is that the bag overlap is an average over the wave packets of the momentum-space coupling. We need to determine a numerical value for this overlap.

One must specify $F(k^2)$. In general form factors fall with k^2 in a manner determined by the range of the interaction

$$F(k^2) \underset{k^2 \text{ small}}{\sim} 1 - \frac{1}{6} k^2 \langle r^2 \rangle. \quad (34)$$

If pion coupling to quarks were pointlike, and were of the form, e.g.,

$$f \bar{\psi} \gamma_\mu \gamma_5 \psi \partial^\mu \phi, \quad (35)$$

then $\langle r^2 \rangle$ would just be the axial-vector charge radius, which has been calculated in the bag model⁵ to be essentially equal to the vector charge radius $\langle r^2 \rangle_{ax} \approx \langle r^2 \rangle_{em}$. For nonpointlike pions we would expect the range to be somewhat larger, leading to a faster falling form factor (and ultimately to larger values of $g_{\pi NN}$). However, at the present level of unsophistication, we see no reason to go beyond the simpler estimate. In addition, we will choose the structure of the form factor in order to make the calculation simple. A dipole renders the integrals intractable while a Gaussian leads to considerable simplification. In summary we use

$$F(k^2) = e^{-\langle r^2 \rangle k^2 / 6} \quad (36)$$

with

$$\langle r^2 \rangle = \langle r^2 \rangle_{em} = 0.8 \text{ fm}^2.$$

The integrals may be done using Eqs. (28) and (36). I find

$$\int d^3 k \phi(k) F(k^2) \int d^3 p \chi(p-k) \chi(p) (2\pi)^3 = \frac{0.61}{R_N^2}. \quad (37)$$

This leads to a value for the pion coupling

$$g_{\pi NN} = 11.5 \quad (38)$$

which is quite close to the experimental value

$$(g_{\pi NN})_{\text{exp}} = 12.5. \quad (39)$$

Of course, the apparent accuracy of this result is probably illusory, as we should only expect to obtain the correct order of magnitude.

Despite this numerical success, the method has one unfortunate feature—it does not reproduce the Goldberger-Treiman relation. Perhaps one should not be surprised at this. The bag-model pion is at best an approximation to the chiral pion and does not possess the PCAC (partial conservation of axial-vector current) couplings. Despite some promising results of Ref. 4, the bag pion will not in general produce the form of the PCAC prediction. However, if the approximation is not too bad, the numbers should come out similar to PCAC. This appears to be the case here.

Finally, we may treat the ρ meson in the same way. We define

$$\langle P(p') | \phi_\rho^\mu(p) | P(p) \rangle = g_{\rho NN} \frac{\bar{u}(p') \gamma^\mu u(p)}{(p-p')^2 - m_\rho^2} F_\rho(p-p') e^{i(p-p') \cdot x}. \quad (40)$$

We now obtain

$${}_B \langle p; \rho^0, \vec{\delta} | p \rangle_B = \frac{g_{\rho NN}}{2M_N} \frac{1}{6} \delta_z \int d^3 k \phi(k) F(k^2) \frac{k}{(k^2 + m^2)^{1/2}} \left[\int d^3 p \chi^\dagger(p-k) \chi(p) (2\pi)^3 \right]. \quad (41)$$

In evaluating the integrals the awkward factor $k/(k^2 + m^2)^{1/2}$ was pulled outside of the integral, replacing k by its average value. We obtain

$$\frac{g_{\rho NN}}{g_{\pi NN}} = 0.67 \quad (42)$$

instead of the experimental value

$$\left. \frac{g_{\rho NN}}{g_{\pi NN}} \right|_{\text{exp}} = 0.46. \quad (43)$$

Again the agreement is fine considering the nature of the approximation.

These calculations may be objected to in that they neglect some difficult and very real physics involved in a fissioning of one particle into two. For example, the dynamics of the deformation of bags and their ultimate splitting is missing from the method. What has been done above is identify the overlap which produces the appropriate quarks in the final state. The estimates will only make

sense if fissioning is relatively easy and does not play a large role in the dynamics. However, the results are suggestive that coupling constants may be more tractable in quark models than previously

thought.

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¹N. N. Bogoliubov, J. Phys. (Moscow) 11, 23 (1947).

²See, for example, L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Addison-Wesley, Reading, Massachusetts, 1969).

³A. Chodos, R. L. Jaffe, K. Johnson, C. B. Thorn, and V. F. Weisskopf, Phys. Rev. D 9, 3471 (1974); A. Chodos, R. L. Jaffe, K. Johnson, and C. B. Thorn, *ibid.* 10, 2599 (1974); T. De Grand, R. L. Jaffe, K.

Johnson, and J. Kiskis, *ibid.* 12, 2060 (1975); J. F. Donoghue, E. Golowich, and B. R. Holstein, *ibid.* 12, 2875 (1975).

⁴J. F. Donoghue and K. Johnson, Phys. Rev. D 21, 1975 (1980).

⁵J. F. Donoghue and E. Golowich, Phys. Rev. D 14, 1386 (1976).