Comment on the proton decay mode
\( P \rightarrow e^+ \pi^0 \)

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On the proton decay mode $p \rightarrow \pi^0 e^+$

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We discuss the sources of uncertainty in the evaluation of the branching ratio for $p \rightarrow \pi^0 e^+$ within grand unified theories, and briefly comment on lifetime estimates.

One of the more dramatic predictions of grand unified theories is that the proton is unstable, with a lifetime not far beyond the present experimental limit.\(^1\) Several experiments are planned or underway in order to test this prediction.

There have been several attempts to calculate the proton lifetime and branching ratios.\(^2\)-\(^9\) The major uncertainty in the rate is the mass of heavy gauge bosons which must be inferred using an extrapolation from low energy. Beyond this, the calculation of the branching ratios involves low-energy quark dynamics, such as described by SU(6), the bag model, or the nonrelativistic quark model. At present, the origin of most of the variations in these models is understood. While further refinements could be made, it should be remembered that these details have little intrinsic interest beyond helping experimenters to observe proton decay.

There is, however, one area which needs clarification, i.e., the pion channels. Several of the proposed detectors are sensitive essentially only to the mode $p \rightarrow \pi^0 e^+$. Theory is divided into two camps. Two bag-model calculations\(^4,6\) have seen it appropriate to include form factors due to the sizable momentum transfer to the pion leading to a reduction of this rate. However, in harmonic-oscillator models,\(^5,8\) this form factor can be calculated and, with the parameters determined from the baryon spectrum, leads to almost no reduction. The bag-model branching ratios for $p \rightarrow \pi^0 e^+$ are near 10%, while the nonrelativistic models yield 30-40%. Below we discuss in more detail how this arises in the models and note that, if the oscillator parameter is determined from the nucleon's electromagnetic form factor, a more sizable suppression occurs, bringing the oscillator result closer to the bag-model result. We conclude that in both models there is an inherent uncertainty as to the suppression of the pionic mode and that a precise prediction is beyond the ability of either model at present. Experimenters should be aware of this problem when designing future equipment.

The arguments suggested in favor of the pionic suppression rely on an analogy between electromagnetic form factors, Figs. 1(a) and 1(b), and the $p \rightarrow e^+ \pi^0$ mode, Fig. 1(c). The electromagnetic form factors fall off with increasing $q^2$. In quark models this behavior can be calculated from the moments of the electromagnetic current. In particular the first term is given by the charge radius

FIG. 1. Diagrams representing (a) the proton's electromagnetic form factor, (b) the pion's electromagnetic form factor, and (c) the vertex for $p \rightarrow \pi^0 e^+$. 

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which can easily be calculated in a given model. Experimentally \(\langle r^2 \rangle = 0.81 \text{ fm}^2\). The pionic decay mode can in principle be similarly treated. One can imagine studying the \(p \rightarrow \pi\) part of the transition as a function of the momentum \(p\) transferred to the pion. As \(p\) increases, one could have a form-factor suppression, analogous to the electromagnetic form factors, and with some of the same parameters. In naive coordinate-space calculations the suppression would be quite large as the pion has \(v/c = 0.96\) in the frame where the proton is at rest, so that wave-function overlaps between the proton and the pion would be quite small. These considerations lead to the introduction of a form factor in Refs. 4 and 6. Using a form factor similar to that of the nucleon as an estimate leads to the predicted rate for \(p \rightarrow \pi^0 e^+\).

In a harmonic-oscillator quark model one can explicitly calculate these form factors. The electromagnetic case is given by

\[
F_{\text{EM}}(k^2) = e^{-k^2/\alpha^2},
\]

where \(\alpha^2\) is the harmonic-oscillator constant for baryons. For \(p \rightarrow \pi^0 e^+\) the corresponding form factor (for the amplitude) is

\[F_{\pi e}(k^2) = \text{constant} \times \exp[-3k^2/(4\alpha^2 + 3\beta^2)],\]

where \(\beta^2\) is the meson’s oscillator constant. As expected the forms are not too different. The actual suppression depends on the parameters in the model. Kane and Karl and Gavel et al. use \(\alpha^2 - \beta^2 = 0.17\ \text{GeV}^2\), with motivations to be reviewed below, leading to a rate suppression of only 15%.

Another estimate of the suppression can be obtained by comparison with the electromagnetic form factor. Expanding Eq. (2) and equating the first term with the corresponding expansion of

\[
F_{\text{EM}}(q^2) = \frac{1}{(1 + q^2/m_p^2)}
\]

yields \(\alpha^2 = 0.049\ \text{GeV}^2\). Now (with \(\alpha^2 = \beta^2\)) the pionic rate is suppressed by a factor of 2.6 leading to a branching ratio of 14–22%, not far from the bag-model estimates.

Clearly it is important to decide which of the estimates should be used. Reasonable cases can be made for both choices. The larger value of \(\alpha^2\) emerges from the fit to the baryon spectrum and is needed to obtain the correct excitation energies for baryon resonances. It is also the value used in the successful calculations of photon and pion couplings between nucleons and resonances. A test of it within the model is given by the single-pion photoproduction calculation of Copley et al. There a cancellation between spin and orbital excitations, depending sensitively on \(\alpha^2\), is needed to understand the data on the \(D_1\) resonance in the backward direction. All these favor \(\alpha^2 = 0.17\ \text{GeV}^2\).

This value of \(\alpha^2\) leads to a too small charge radius for the proton \(\langle r^2 \rangle = 0.26 \text{ fm}^2\). The smaller value of \(\alpha^2\) above is designed through Eq. (1) to give the ground-state baryons the right size. It is not so surprising that the harmonic-oscillator model has some trouble simultaneously fitting the excitation spectrum and the nucleon’s size. The approximation of nonrelativistic quarks and the steep potential at large distances would both tend to favor quarks which concentrate strongly near the center of the potential. In the photoproduction calculation, the nonrelativistic character would require a larger \(\alpha^2\) to get the required orbital excitation that would be needed for a relativistic quark.

We find it hard to give a compelling choice of one or the other parameters, and feel that it re-

TABLE I. Two-body proton-decay branching ratios in four models.

<table>
<thead>
<tr>
<th>Bag model Form factor included</th>
<th>Bag model No form factor</th>
<th>&quot;Relativistic&quot; model Harmonic oscillator Ref. 8: (\alpha^2 = 0.049\ \text{GeV}^2)</th>
<th>“Relativistic” model Harmonic oscillator Ref. 8: (\alpha^2 = 0.17\ \text{GeV}^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\omega^0 e^+)</td>
<td>50</td>
<td>25</td>
<td>39</td>
</tr>
<tr>
<td>(\rho^0 e^+)</td>
<td>19</td>
<td>10</td>
<td>16</td>
</tr>
<tr>
<td>(\pi^0 e^+)</td>
<td>9</td>
<td>41</td>
<td>22</td>
</tr>
<tr>
<td>(\eta^0 e^+)</td>
<td>3</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>(\rho^+ e^\mu)</td>
<td>8</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>(\pi^+ e^\mu)</td>
<td>3</td>
<td>14</td>
<td>9</td>
</tr>
<tr>
<td>(K^0 e^+)</td>
<td>7</td>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>(K^+ e^\mu)</td>
<td>1</td>
<td>0.5</td>
<td>1</td>
</tr>
</tbody>
</table>
presents a genuine uncertainty in the model. Likewise in the bag model, there is little hope of understanding the mechanism better in the near future. To give an idea of the plausible range of the proton decay modes, we give in Table I the bag-model estimates of Ref. 4 (adding in the kaonic modes from Ref. 6), both with and without form factors, and the "relativistic" estimate in the harmonic oscillator of Ref. 8 together with the result of using \( \alpha^2 = 0.049 \) in the latter calculation. It is interesting to note how well the two methods agree when there is no suppression of the pionic modes.

Finally, we would like to give a brief comment on the absolute magnitude of the proton lifetime. The calculated lifetimes in the two above models (for the same value of \( M_p = 6 \times 10^{14} \text{ GeV} \)) are \( 5 \times 10^{31} \text{ yr} \) for the bag-model estimates\(^5,6\) and \( 5.4 \times 10^{31} \text{ yr} \) for the model of Gavela \( et \ al.\)\(^5,12\). Despite this close agreement the calculations are somewhat different. The bag estimates use direct computation of the matrix elements involved, while Gavela \( et \ al.\) extract a \( |\psi(0)|^2 \) from \( \delta S = 1 \) hyperon-decay matrix elements. The latter approach is quite sensible; however, we wish to point out that there remains considerable uncertainty in the determination, depending on which information from hyperon decay is used.

Not all aspects of hyperon decay are understood. At present there are two ways in which parity-conserving (PC) matrix elements \( \langle B' \Gamma_{\text{PC}} | B \rangle \) (and hence \( |\psi(0)|^2 \)) enter the theory. In the parity-violating amplitudes (\( S \) wave), current algebra can be used to obtain the \( \langle B' \Gamma_{\text{PC}} | B \rangle \) amplitudes. The relation to the quark model \( |\psi(0)|^2 \) is some-

\[ \tau_p = 7.2 \times 10^{31} \text{ yr} \]

or

\[ \tau_p = 1.8 \times 10^{32} \text{ yr} \]

depending on whether one uses the \( F \) or \( D \) parameters to extract \( |\psi(0)|^2 \). Including the penguin diagram\(^13,15\) in hyperon decay could further reduce \( |\psi(0)|^2 \) and increase \( \tau_p \). However, again the bag-model and harmonic-oscillator estimates are not inconsistent within the uncertainties of the calculation.

Unfortunately we have found that calculations of proton decay are not clean in any model. There are considerable uncertainties in the rate and branching ratios. However, this should not obscure the common message of all calculations. If grand unified models are correct, proton decay is accessible to upcoming experiments, and we should soon be able to confirm or refute this daring prediction.

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12. Note that there is a factor of 6 in the lifetime between the report and published versions of Ref. 5 (W. Rolnick, private communication. We also thank Dr. G. Kane for discussions on this subject). We use the corrected version throughout.