1979

Tensor analysis of hadronic jets in quantum chromodynamics

John Donoghue, *University of Massachusetts - Amherst*
F.E. Low, *Center for Theoretical Physics*
So-Young Pi, *Center for Theoretical Physics*

Tensor analysis of hadronic jets in quantum chromodynamics

John F. Donoghue, F. E. Low, and So-Young Pi

Center for Theoretical Physics, Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

(Received 13 April 1979; revised manuscript received 11 June 1979)

We define tensors in momentum space for the events produced in e^+e^- annihilation. They are calculable in quantum-chromodynamics (QCD) perturbation theory and therefore can be used as a test of QCD. We have studied their distributions and average values which give characteristics of the global structure of hadronic jets in QCD.

Two-jet events have been observed in e^+e^- annihilation at center-of-mass energies greater than 6 GeV.¹ In quantum chromodynamics (QCD), as the center-of-mass energy increases, we expect that the effective transverse momentum relative to the jet axis will increase due to a hard-gluon bremsstrahlung process. Therefore, at very high energies three-jet events should be observed.

To analyze the jet structure experimentalists¹ have constructed a tensor in momentum space, which may be called a momentum ellipsoid:

$$M_{ij} \equiv \sum_a (\delta_{ij} |p^a|^2 - p_i^a p_j^a), \quad (1)$$

where a runs over the final hadrons and i, j indicate the components of the hadron momenta. Then the sphericity is defined as

$$S = \min \frac{3}{2} \frac{\sum_a |p_i^a|^2}{\sum_a |p^a|^2} \quad (2)$$

and can be used as a measure of jetlike properties. However, M_{ij} and S have mass singularities and are not calculable in QCD perturbation theory.² In addition to being calculable, it is desirable that QCD observables (1) vanish for two-jet events, and (2) not be obscured by nonperturbative effects, so that their measurement would give direct proof of the existence of three-jet events. In this note we define tensors in the momentum space of the final-state hadrons which are free of mass singularities and therefore can be computed perturbatively in QCD.³ The average values and distributions of these tensors give characteristics of the global structure of QCD jets and can provide useful tests of QCD.

Our tensors are defined as

$$Q_{ijk\dots n}^{(n)} \equiv \frac{1}{\sum_a |p^a|} \sum_a (p_i^a p_j^a p_k^a \dots p_n^a / |p^a|^{n-1}), \quad (3)$$

where a runs over final hadrons and $i, j, k, \dots, n$ indicate components of the vectors $\vec{p}_a$. We have studied the second- and third-rank tensors $Q_{ij}^{(2)}$

and $Q_{ijk}^{(3)}$ in the lowest nontrivial order in QCD for $e^+e^- \rightarrow$ hadrons. First we will discuss the second-rank tensor $Q_{ij}^{(2)}$. From Eq. (3) $Q_{ij}^{(2)}$ is given by

$$Q_{ij}^{(2)} = \left| \sum_a \frac{p_i^a p_j^a}{|p^a|} \right| / \sum_a |p^a|. \quad (4)$$

The geometrical meaning of the eigenvalues λ_i of $Q_{ij}^{(2)}$ is similar to the case of M_{ij} (the normalization chosen gives $\sum_{i=1}^3 \lambda_i = 1$): (a) For perfect two-jet events two of the eigenvalues are zero. (b) For three-jet events one of the eigenvalues is zero. (c) For events with spherical momentum distribution $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{3}$. (d) In QCD, the average of the eigenvalues is ordered:

$$\bar{\lambda}_1 \geq \bar{\lambda}_2 \geq \bar{\lambda}_3 \text{ with } \bar{\lambda}_1 = O(1), \bar{\lambda}_2 = O(\alpha_s), \bar{\lambda}_3 = O(\alpha_s^2).$$

In QCD up to order $\alpha_s = g_s^2/4\pi$, the processes contributing to $e^+e^- \rightarrow$ hadrons are

$$e^+(\vec{k}_1) + e^-(\vec{k}_2) \rightarrow q(\vec{p}_1) + \bar{q}(\vec{p}_2), \quad (5a)$$

$$e^+(\vec{k}_1) + e^-(\vec{k}_2) \rightarrow q(\vec{p}_1) + \bar{q}(\vec{p}_2) + G(\vec{p}_3), \quad (5b)$$

where $\vec{k}_1$ and $\vec{k}_2$ are beam momenta and $\vec{p}_1$, $\vec{p}_2$, and $\vec{p}_3$ are the momentum of the quark (q), antiquark ($\bar{q}$), and gluon (G), respectively. The process (5a) gives perfect two-jet events and the process (5b) gives coplanar events of a fundamental three-particle nature. The cross section of (5b) is

$$\frac{1}{\sigma_0} \frac{d\sigma}{dx_1 dx_2} = \frac{2}{3} \frac{\alpha_s}{\pi} \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)},$$

where $x_1 = 2p_1/\sqrt{s}$, subject to $0 \leq x_1 \leq 1$ and $x_1 + x_2 + x_3 = 2$.

Our tensors may be studied in various coordinate systems defined by the event. Two simple choices correspond to (i) the basis in which the tensor $Q^{(2)}$ is diagonal, with the jet direction $\hat{D}$ given by the diagonalization axis corresponding to the largest eigenvalue, or (ii) the thrust system $\hat{T}$, described below. For two-jet events $\hat{D}$ and $\hat{T}$ coincide, but they are in general different for three-jet processes. The physics in the two systems is similar, and we have chosen to give

our results in the thrust basis.

The thrust axis $\hat{T}$ is defined by the direction of the maximum directed momentum⁴

$$\max d(\hat{r}) = \max_a \sum \vec{p}_a \cdot \hat{r} \theta(\vec{p}_a \cdot \hat{r}). \quad (6)$$

We take the $\hat{T}$ axis to be in the $-z$ direction. The plane formed by the final-state hadrons is the x - z plane (Fig. 1). This plane can be defined operationally as being in the direction which maximizes the directed momentum perpendicular to the $\hat{T}$ axis, i.e., $\max d(\hat{r})|_{\hat{r} \perp \hat{T}}$. The sign of $\hat{r}$ may be chosen such that Q_{xz} is positive. Then, to order α_s , $Q_{ij}^{(2)}$ has the form

$$Q_{ij}^{(2)} = \begin{pmatrix} Q_{xx} & Q_{xz} & 0 \\ Q_{xz} & Q_{zz} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (7)$$

We have two independent components, Q_{xx} and Q_{zz} $= Q_{xx}$, which are of order α_s . $Q_{zz} = 1 - Q_{xx}$ is of order unity in α_s . It is easy to notice that Q_{zz} measures the difference between the thrust axis $\hat{T}$ and the largest axis of Q_{ij} when it is diagonalized ($\hat{D}$). The angle between the two frames is

$$\tan 2\theta_{DT} = \frac{2Q_{xz}}{1 - 2Q_{xx}}.$$

However, Q_{zz} is not a good variable for studying three-jet processes, since it becomes small for very pronounced three-jet events, vanishing at $|\vec{p}_1| = |\vec{p}_2| = |\vec{p}_3|$. A quantity which can more conveniently be used to measure jetlike properties is the component Q_{xx} . Q_{xx} plays a role analogous to sphericity:

$$Q_{xx} = \left[\sum_a \frac{(p_x^a)^2}{|\vec{p}^a|^2} \right] / \sum_a |\vec{p}^a|^2. \quad (8)$$

To first order in α_s , the maximum directed mo-

mentum is simply given by the momentum of the most energetic particle a . Therefore $\hat{T}$ will be the direction of $\vec{p}_a$ ($\hat{T} \equiv \hat{p}_a$) and Q_{xx} is given by

$$Q_{xx} = \frac{1}{\sqrt{S}} \left(\frac{p_{bx}^2}{p_b} + \frac{p_{cx}^2}{p_c} \right). \quad (9)$$

In terms of x_a , x_b , and x_c ,

$$Q_{xx} = 2 \frac{(1-x_a)}{x_a} \frac{(1-x_b)}{x_b} \frac{(1-x_c)}{x_c} \frac{(2-x_a)}{x_a}. \quad (10)$$

We have calculated the average value of Q_{xx} to order α_s and obtained

$$\begin{aligned} \langle Q_{xx} \rangle &= \int dx_1 dx_2 dx_3 \frac{1}{\sigma_0} \frac{d\sigma}{dx_1 dx_2} \\ &\quad \times \delta(x_1 + x_2 + x_3 - 2) Q_{xx} \\ &= \frac{8}{3} \frac{\alpha_s}{\pi} \left[\frac{29}{12} + \frac{\pi^2}{3} - 9 \ln^2 \frac{2}{3} - 2 (\ln^2 \frac{2}{3})^2 - 4 \sum_{k=1}^{\infty} \frac{(\frac{2}{3})^k}{k^2} \right] \\ &= 1.7 \alpha_s / \pi. \end{aligned} \quad (11)$$

As is the case for sphericity,⁵ however, $\langle Q_{xx} \rangle$ may not be a useful quantity at moderately high energies due to nonperturbative effects.

A better quantity for the study of QCD predictions is the distribution $d\sigma/dQ_{xx}$.⁶ In particular, for large values of Q_{xx} nonperturbative effects should be less pronounced than at low Q_{xx} . By expanding $d\sigma/dx_1 dx_2$ and Q_{xx} near the maximum value $Q_{xx} = \frac{1}{2}$, where the events have an extreme three-jet structure ($x_1 = x_2 = x_3 = \frac{2}{3}$), we have obtained an analytic expression

$$\frac{1}{\sigma_0} \frac{d\sigma}{dQ_{xx}} \bigg|_{Q_{xx}=1/2} = \frac{16\alpha_s}{\pi} \left(\frac{4}{3} \right)^3 \left[\frac{1}{2} - Q_{xx} + O\left(\left(\frac{1}{2} - Q_{xx}\right)\right) \right]. \quad (12)$$

(A numerical analysis of the behavior of $d\sigma/dQ_{xx}$ for $0 \leq Q_{xx} \leq \frac{1}{2}$ is shown in Fig. 2.)

Next we will discuss the third-rank tensor

Fig. 1. Definition of the $\hat{T}$ axis and the x - z plane: $|\vec{p}_a| \geq |\vec{p}_b|, |\vec{p}_c|$ and $\hat{T} \hat{p}_a$ is taken to be the $-\hat{z}$ direction. $\vec{p}_a, \vec{p}_b$, and $\vec{p}_c$ define the x - z plane. $\vec{k}$ is either beam direction.

FIG. 2. The distribution $(Q_{xx}/\sigma_0)(d\sigma/dQ_{xx})$ for $e^+e^- \rightarrow qqG$.

$$Q_{ijk}^{(3)} = \left[\sum_a \frac{p_i^a p_j^a p_k^a}{|p^a|^2} \right] / \sum_a |p^a|. \quad (13)$$

In the coordinate system we have defined all $Q_{yjk} = O(\alpha_s^2)$. From energy-momentum conservation $\delta^{ij} Q_{ijk}^{(3)} = 0$ and we have two independent components to $O(\alpha)$:

$$\begin{aligned} Q_{xxz} &= -Q_{zzx}, \\ Q_{xxx} &= -Q_{zzz}. \end{aligned} \quad (14)$$

It is important to notice that (a) *all* the elements of the third-rank tensor are at most of order α_s , whereas for the case of the second-rank tensor Q_{xx} is of order unity in α_s and (b) if the jet structure of an event is symmetric in x , $Q_{xxx} = 0$; if it is symmetric in z , $Q_{xxz} = 0$. Therefore, even two-jet events with nonvanishing transverse momenta coming from nonperturbative effects tend to cancel in both Q_{xxz} and Q_{xxx} . We have found that Q_{xxz} is a useful variable in studying the jet structure. In our coordinate system it is given by

$$Q_{xxz} = \frac{1}{\sqrt{s}} \left(\frac{p_{bz}}{p_b^2} + \frac{p_{cz}}{p_c^2} \right) p_{bx}^2. \quad (15)$$

In terms of energies of q , $\bar{q}$, and G ,

$$\begin{aligned} Q_{xxz} &= 2 \frac{(1-x_a)}{x_a^2} \frac{(1-x_b)}{x_b^2} \frac{(1-x_c)}{x_c^2} \\ &\times \left[x_a x_b x_c - 2 \frac{(1-x_a)}{x_a} (x_b - x_c)^2 \right]. \end{aligned} \quad (16)$$

Near the extreme configuration of three-jet events Q_{xxz} has its maximum value $\frac{1}{4}$ and the analytic expression for the distribution $d\sigma/dQ_{xxz}$ is given by

FIG. 3. The distribution $(Q_{xxz}/\sigma_0)(d\sigma/dQ_{xxz})$ for $e^+e^- \rightarrow qqG$.

FIG. 4. The distribution $(Q_{xx}/\Gamma)(d\Gamma/dQ_{xx})$ for $(Q\bar{Q})^{3S_1} \rightarrow 3G$.

$$\begin{aligned} \frac{1}{\sigma_0} \frac{d\sigma}{dQ_{xxz}} \Big|_{Q_{xxz}=1/4} &= \left(\frac{4}{3} \right)^4 \frac{2\alpha_s}{\pi} \left[a + b \left(\frac{1}{4} - Q_{xxz} \right) + O \left(\left(\frac{1}{4} - Q_{xxz} \right)^2 \right) \right], \end{aligned} \quad (17)$$

where

$$a = \frac{\pi}{4} + \sin^{-1} 1/\sqrt{5},$$

$$b = \frac{16}{3} \left(\frac{\pi}{4} + \sin^{-1} 1/\sqrt{5} - \frac{3}{20} \right).$$

The constant term in the cross section may lessen the sensitivity to nonperturbative effects. (In Fig. 3 we present the shape of $d\sigma/dQ_{xxz}$ for $0 \leq Q_{xxz} \leq \frac{1}{4}$ obtained from a numerical calculation. We also find $\langle Q_{xxz} \rangle = 1.1\alpha_s/\pi$.)

We also have studied the distributions and average values of the second- and third-rank tensors numerically for the three-gluon decay of a heavy-quark-antiquark vector S -wave bound state $[(Q\bar{Q})^{3S_1}]$ like $\Upsilon(9.4)$. For $(Q\bar{Q})^{3S_1} \rightarrow 3$ gluons in our coordinate system $Q_{ij, k, \dots, n}^{(n)} = 0$ for any $i, j, k, \dots, n = y$ and all other components $Q_{i, j, k, \dots, n}^{(n)}$, $i, j, k, \dots, n \neq y$, are of order α_s^3 . The distributions

$$\frac{1}{\Gamma} \frac{d\Gamma}{dQ_{ij, k, \dots, n}}$$

are of order unity in α_s . The differential cross section for $(Q\bar{Q})^{3S_1} \rightarrow 3$ gluons is given by^{7,8}

FIG. 5. The distribution $(Q_{xxz}/\Gamma)(d\Gamma/dQ_{xxz})$ for $(Q\bar{Q})^{3S_1} \rightarrow 3G$.

$$\frac{1}{\Gamma} \frac{d^2\Gamma}{dx_1 dx_2} = \frac{1}{\pi^2 - 9} \left[\frac{(1-x_1)^2}{x_2^2 x_3^2} + \frac{(1-x_2)^2}{x_3^2 x_1^2} + \frac{(1-x_3)^2}{x_1^2 x_2^2} \right], \quad (18)$$

where $x_i = 2|p_i|/\sqrt{s}$, subject to $0 \leq x_i \leq 1$ and $x_1 + x_2 + x_3 = 2$.

Notice that this distribution does not have the singularity near a two-jet configuration that the qqG cross section has. This leads to a more extreme three-jet structure, and yields typically higher values for our tensor elements.

The distributions $(Q_{xx}/\Gamma)(d\Gamma/dQ_{xx})$ and $(Q_{xxx}/\Gamma)(d\Gamma/dQ_{xxx})$ are shown in Fig. 4 and Fig. 5, respec-

tively.⁹ They are quite different from Figs. 1 and 2 which correspond to the continuum background when these decays are studied at e^+e^- facilities. These distributions yield average values $\langle Q_{xx} \rangle = 0.175$ and $\langle Q_{xxx} \rangle = 0.113$.

One can also carry out a similar analysis with the higher-rank tensors we have defined.

This work was supported in part through funds provided by the U. S. Department of Energy (DOE) under Contract No. EY-76-C-02-3069.

¹G. Hanson *et al.*, Phys. Rev. Lett. **35**, 1609 (1975).

²G. Sterman and S. Weinberg, Phys. Rev. Lett. **39**, 1436 (1977). In this connection, see also the demonstration of the lack of mass singularities to all orders in perturbation theory in G. Tiktopoulos, Nucl. Phys. **B147**, 371 (1979) and G. Sterman, Institute for Advanced Studies report (unpublished).

³A set of similar quantities which are rotationally invariant observables has been studied. G. C. Fox and S. Wolfram, Phys. Rev. Lett. **41**, 1581 (1978).

⁴E. Fahri, Phys. Rev. Lett. **39**, 1587 (1977).

⁵H. Georgi and M. Machacek, Phys. Rev. Lett. **39**, 1237 (1977). The average value of sphericity, is studied as a function of Q^2 ($Q=2E$). It is shown that the perturbative QCD effects become larger than the nonperturbative effects at Q^2 above 20 GeV (Ref. 2). Another re-

lated measure of jetlike properties is the energy correlation defined by C. L. Basham, L. S. Brown, S. D. Ellis, and S. T. Love, Phys. Rev. Lett. **41**, 1585 (1978); Phys. Rev. D **17**, 2298 (1978).

⁶The usefulness of the distribution of the second-rank tensor was also suggested by G. Parisi [Phys. Lett. **74B**, 65 (1978)] who called it a superinclusive cross section.

⁷A. Ore and J. Powell, Phys. Rev. **75**, 1696 (1949).

⁸K. Koller and T. F. Walsh, Phys. Lett. **72B**, 227; (1977) **73B**, 504(E) (1978).

⁹In order to compare our results with the experimental data for relatively-low-mass states such as $T(9.4)$ where each jet only carries about 3 GeV, one should perform a Monte Carlo calculation including nonperturbative effects.