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Cracks in conductors are detected through changes in the impedance of a coil that induces current in the material. In order to gain insight into the physics of the inspection, we have developed a theoretical and computational model that predicts the signals due to cracks in circular cylindrical holes using a boundary element calculation. In formulating the problem, the electromagnetic field is decomposed into transverse electric and transverse magnetic scalar modes. The effect of a planar crack in an electromagnetic field is represented by an electric current dipole layer orientated normal to the crack surface. The dipole density is determined by the integral equation whose dyadic kernel ensures that the tangential electric and magnetic fields are continuous at the surface of the hole. Instead of solving this equation, a numerical approximation is found in the form of a discrete system of linear algebraic equations formed using either boundary and volume elements depending respectively, on whether the crack opening is negligible or not. Because the kernel embodies the interface conditions at the surface of the hole, a discrete approximation of the field is only necessary in the flaw domain which means that relatively few unknowns are needed. The probe impedance variation has been computed for both ideal cracks, defined as having negligible opening but impenetrable to current, and open cracks/slots. Open crack model predictions of coil impedance variations with position relative to a semi-elliptical axial crack are in good agreement with measurements.

Index Terms—Bessel functions, borehole, boundary elements, cracks, eddy current, integral equation, nondestructive evaluation.

I. INTRODUCTION

CRACKS in metallic structures such as tubes and bolt-holes can be detected using eddy-current probes whose impedance varies due to the presence of flaws. Computational measurement models have been developed in an effort to refine inspection methods and as an aid to the interpretation of inspection data. Here, we describe one such model that uses boundary and volume elements for calculating eddy-current probe impedance variations due to cracks in circular cylindrical holes, such as fastener holes. Typically, when fastener holes are inspected with the fastener removed, a differential probe is used with a split ferrite core. However, here we consider an absolute rotary coil; one for which the coil axis is perpendicular to the borehole axis, Fig. 1, since one can achieve better control in validating the flaw model by using such probes.

In an early calculation of eddy-current coil signals due to defects, an attempt was made to solve a three dimensional inverse problem aimed at finding the geometry of flaws in tubes from eddy-current probe data [1]. The inversion was based on a forward model in which an electric field integral equation with a dyadic kernel determines the flaw field and a numerical approximation of a solution is sought via the volume element method. In this work, only one scalar function defined the kernel in each region but later [2], [3], an accurate kernel was derived using two transverse modes that are coupled at cylindrical interfaces. The use of dyadic kernels for modeling eddy-current tube inspection has been adopted by several authors [4]–[7]. These developments adhere to the conventions of electromagnetic wave scattering theory [8], [9] in that they retain displacement current and express the electromagnetic field in terms of Bessel functions of the first kind and Hankel functions. The construction of the kernel given here differs from approach taken in these studies, firstly, in that we use a scalar decomposition of the field that is valid in the source region. Secondly, quasi-static modal solutions are expressed in terms of associated (modified) Bessel functions and thirdly the dyadic kernel is constructed directly in terms of the transverse magnetic (TM) and transverse electric (TE) scalar kernels.

In a cylindrical system, the scalar modes are defined with respect to the direction of the axis of the structure. Usually, both TE and TM modes interact with a planar crack. However, in the case of a circumferential crack of negligible opening, only the TM mode is perturbed directly. This allows a simple TM potential formulation for the ideal planar transverse crack problem [10]. For open cracks and for a crack of arbitrary orientation, the calculation is more complicated since both modes are directly perturbed. In this article we describe the evaluation of coil impedance variation due to circumferential/transverse and
axial/longitudinal cracks, both ideal and open. Numerical results have been computed using a simple variant of the moment method.

II. FORMULATION

A. Scalar Decomposition

We consider conductors with linear material properties, having the permeability of free space and homogeneous apart from in the flaw region, \( \Omega \). At the flaw, the conductivity \( \sigma(\mathbf{r}) \) differs from that of the host conductivity \( \sigma_0 \). In the presence of an electromagnetic field, the inhomogeneity gives rise to a local induced electric dipole source \( \mathbf{P}(\mathbf{r}) = [\sigma(\mathbf{r}) - \sigma_0] \mathbf{E}(\mathbf{r}) \). In which case, one writes Ampère’s law as

\[
\nabla \times \mathbf{H} = \sigma_0 \mathbf{E} + \mathbf{P}
\]

and the quasi-static time-harmonic magnetic field, varying as the real part of \( \mathbf{H} \exp(-i \omega t) \), is a solution of

\[
\nabla \times \nabla \times \mathbf{H} - k^2 \mathbf{H} = \nabla \times \mathbf{P}(\mathbf{r})
\]

where \( k^2 = \omega \mu \sigma_0 \). The problem of finding the field due to the flaw thus reduces to one of finding the electric current dipole density, \( \mathbf{P} \).

We have determined the electric source density using a formulation based on a scalar decomposition of the field. Assuming the permeability of the material is that of free space, the magnetic field has zero divergence and can therefore be written in terms of TE and TM potentials, \( W_1 \) and \( W_2 \) respectively, as

\[
\mathbf{H} = \nabla \times \nabla \times (\hat{x} W_1) + k^2 \nabla \times (\hat{x} W_2)
\]

throughout the problem domain, \( \hat{x} \) being a unit vector in the axial direction. For the air-filled region of the borehole, we have a corresponding expression with \( k^2 \equiv 0 \). By substituting (3) into the magnetic field (2), one readily shows that the potentials satisfy

\[
(\nabla^2 + k^2) \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \begin{bmatrix} \hat{x} \cdot \nabla \times \mathbf{P} \\ \frac{k^2}{k^2} \hat{x} \cdot \nabla \times \nabla \times \mathbf{P} \end{bmatrix}
\]

where \( \nabla^2 \equiv \nabla^2 - \partial^2 / \partial z^2 \) is a transverse Laplace operator. The right-hand side of this relationship represents electromagnetic sources of TE and TM modes [11].

A method for finding the flaw field can be developed by constructing a dyadic kernel from scalar Green’s functions and using the kernel in an electric field integral equation to determine the source of the field. We start by expressing the scalar potentials in integral form using a set of scalar Green’s functions satisfying

\[
(\nabla^2 + k^2) \begin{bmatrix} G_{11} \\ G_{21} \\ G_{12} \\ G_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \delta(\mathbf{r} - \mathbf{r}').
\]

The subscripts denote the modal character of the field and the source respectively. A TE term is denoted by subscript 1 and a TM term by subscript 2. Mode coupling at the air-conductor interface is accounted for by including a kernel \( G_{12} \) to represent TE field migrating from the boundary originating at a TM source. Similarly \( G_{21} \) represents the TM field migration from the boundary of the conductive domain originating at a TE source.

A formal solution of (5) has been constructed using the fundamental solution of the scalar Helmholtz problem

\[
G_{0}(\mathbf{r} | \mathbf{r}') = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}
\]

and a related function \( U_{0}(\mathbf{r} | \mathbf{r}') \) defined such that

\[
G_{0}(\mathbf{r} | \mathbf{r}') = -\nabla^2 U_{0}(\mathbf{r} | \mathbf{r}').
\]

With these functions used to represent the singular fundamental solution, we can write a set of dedicated Green’s functions for the borehole problem as the sum of a singular part and a regular part as follows:

\[
G_{ij}(\mathbf{r} | \mathbf{r}') = \delta_{ij}G_{0}(\mathbf{r} | \mathbf{r}') + G_{ij}^{(T)}(\mathbf{r} | \mathbf{r}') \quad i, j = 1, 2
\]

where \( \delta_{ij} = 1 \) for \( i = j \) and zero otherwise. In addition one defines a set of functions \( U_{ij}(\mathbf{r} | \mathbf{r}') \) such that

\[
G_{ij}^{(T)}(\mathbf{r} | \mathbf{r}') = -\nabla^2 U_{ij}(\mathbf{r} | \mathbf{r}').
\]

Since (5), (7) and (9) reflect the form of (4) and the kernels are constructed to ensure the continuity of the tangential electric and magnetic field at the surface of the hole, one can identify \( U_{0}(\mathbf{r} | \mathbf{r}') \) as the unbounded domain TE/TM potential due to a TE/TM point source and \( U_{ij}(\mathbf{r} | \mathbf{r}') \) as representing the point source potentials associated with field migration from the borehole boundary.

From (4) and the principle of superposition, the TE and TM potentials due to a distributed electric source are given by

\[
\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \int_{\Omega} \begin{bmatrix} U_{0} + U_{11} \\ U_{21} \\ U_{0} + U_{22} \end{bmatrix} \begin{bmatrix} \hat{x} \cdot \nabla \times \mathbf{P} \\ \frac{1}{k^2} \hat{x} \cdot \nabla \times \nabla \times \mathbf{P} \end{bmatrix} \, d\mathbf{r}'.
\]

Integrating by parts gives

\[
W_1(\mathbf{r}) = \int_{\Omega} \left\{ \nabla \times [\hat{x}(U_{0} + U_{11})] + \frac{1}{k^2} \nabla \times \nabla \times [\hat{x} U_{21}] \right\} \cdot \mathbf{P} \, d\mathbf{r}'.
\]

and

\[
W_2(\mathbf{r}) = \int_{\Omega} \left\{ \nabla \times [\hat{x} U_{21}] + \frac{1}{k^2} \nabla \times \nabla \times [\hat{x} (U_{0} + U_{22})] \right\} \cdot \mathbf{P} \, d\mathbf{r}'.
\]

These expressions can be used to construct a dyadic kernel for an electric field integral equation as follows.

B. Electric Field Integral Equation

The induced source at a flaw is sought via the reduced equation for the quasi-static time-harmonic electric field in a conductor with linear material properties

\[
\nabla \times \nabla \times \mathbf{E} - k^2 \mathbf{E} = \omega \mu_0 \mathbf{P}(\mathbf{r})
\]

(13)
the formal solution of which can be written as

$$\mathbf{E}(r) = \mathbf{E}^{(0)}(r) + \omega \mu_0 \int_{\Omega} \mathbf{G}(r | r') \cdot \mathbf{P}(r') \, dr' \quad (14)$$

where $\mathbf{E}^{(0)}(r)$ is the coil electric field in absence of the flaw, the integral represents the flaw field, and $\mathbf{G}(r | r')$ is a dyadic Green’s function satisfying

$$\nabla \times \nabla \mathbf{G}(r | r') = k^2 \mathbf{G}(r | r') = \mathcal{I} \delta(r - r'). \quad (15)$$

For an unbounded domain, the dyadic kernel is [9], [12], [13]

$$\mathbf{G}^{(0)}(r | r') = \left( \mathcal{I} + \frac{1}{k^2} \nabla \nabla \right) \mathbf{G}_0(r | r') \quad (16)$$

with $\mathbf{G}_0(r | r')$ given by (6). Because the kernel is hyper-singular, (14) has a modified form when the field point is in the source region. In such cases, a local region enclosing the field point is excluded from the integral and a term added to the equation to compensate for the removal of the exclusion volume from the integral. The integral representation of the field in the source region for the limiting case of an infinitesimal exclusion volume has been reviewed by Yaghjian [14].

In devising numerical solutions, it is often necessary to express the field in an integral form involving a finite exclusion volume as in the analysis of Lee et al. [15]. In the numerical scheme used here, the exclusion volume is not needed because field testing functions are supported on the external boundary of the flaw region and not in the flaw region itself. However, for completeness, we shall express the electric field as a general scalar decomposition that is valid in the source region in order to show how (14) is derived starting with the scalar decomposition of the field.

In general, the dyadic kernel can be expressed as the sum of a singular unbounded domain term, plus a regular term that accounts for the interaction of the primary field with material boundaries. Here, as with the scalar Green’s functions in (8), the dyadic kernel for an infinite borehole is written as the sum of the unbounded domain kernel and a regular term representing field migration from the surface of the hole:

$$\mathbf{G}(r | r') = \mathbf{G}^{(0)}(r | r') + \mathbf{G}^{(I)}(r | r'). \quad (17)$$

In order to represent the dyadic kernels in terms of their scalar counterparts, we first express the electric field in terms of scalar potentials.

The scalar decomposition of the electric field can be found by taking the curl of the magnetic field in (3) and using the equation for the TE potential in (4) to transform result. In doing so it is helpful to expand $\nabla \times \mathbf{P}$ from the first line of (4) using the dyadic identity [12]

$$\nabla \times \nabla \mathbf{t} = -\nabla \times (\nabla \times \mathbf{z}) + (\nabla \times \nabla \times \mathbf{z})(\nabla \times \mathbf{z}). \quad (18)$$

As a result of these operations, one finds that

$$\nabla \times \mathbf{H} = k^2 \left[ \nabla \times \nabla \times (\mathbf{z} W_1) + \nabla \times (\mathbf{z} W_2) \right] + \frac{(\mathbf{z} \times \mathbf{z}) \cdot (\nabla \times \mathbf{z})}{\mathbf{v}_f^2} \cdot \mathbf{P}. \quad (19)$$

From Ampère’s law, (1), and the identity

$$\mathcal{I} = \mathbf{z} \times \nabla \left( \frac{\mathbf{z} \times \nabla \mathbf{v}_f}{\mathbf{v}_f^2} \right) + \frac{1}{\mathbf{v}_f} \left( \frac{\mathbf{z} \times \nabla \mathbf{v}_f}{\mathbf{v}_f^2} \right) \cdot \mathbf{P} \quad (20)$$

one gets

$$\mathbf{E} = \omega \mu_0 \left[ \nabla \times (\mathbf{z} W_1) + \nabla \times (\mathbf{z} W_2) \right] - \mathbf{v}_f \left( \frac{\mathbf{z} \times \nabla \mathbf{v}_f}{\mathbf{v}_f^2} \right) \cdot \mathbf{P} \quad (21)$$

at any point, including points in the source region. An integral expression of the form of the source integral of (14) is found by substituting (11) and (12) into (21), and by exchanging the order of the field derivatives and the integration over source coordinate (a step which generally implies the creation of an exclusion zone). These steps enable us to identify the kernel in two parts, as expressed in (17). Thus we find that

$$\mathbf{G}^{(0)}(r | r') = -\frac{1}{k^2} \left( \mathbf{z} \times \nabla \mathbf{v}_f \right) \delta(r - r')$$

$$+ \frac{1}{k^2} \left[ \nabla \times (\mathbf{z} W_1) \right] \mathbf{U}_0(r | r')$$

$$+ \frac{1}{k^2} \left[ \nabla \times (\mathbf{z} W_2) \right] \mathbf{U}_0(r | r') \quad (22)$$

and

$$\mathbf{G}^{(I)}(r | r') = \left( \nabla \times \mathbf{z} \right) \left( \nabla \times \mathbf{z} \right) U_{11}(r | r')$$

$$+ \frac{1}{k^2} \left[ \nabla \times (\nabla \times \mathbf{z}) \right] U_{12}(r | r')$$

$$+ \frac{1}{k^2} \left[ \nabla \times (\nabla \times \mathbf{z}) \right] U_{21}(r | r')$$

$$+ \frac{1}{k^2} \left[ \nabla \times (\nabla \times \mathbf{z}) \right] U_{22}(r | r'). \quad (23)$$

The integral equation with the unbounded domain kernel, (16), is typically derived without using scalar decomposition, for example, via an integral expression for the magnetic vector potential [13]. However, by using (5) and (7), one can express the right-hand side of (22) as a differential operator acting on $U_0(r | r')$. Then by using (20) and the identity

$$\nabla \times \nabla \mathbf{v}_f = \left( \mathbf{z} \times \nabla \mathbf{z} \right) \nabla \mathbf{v}_f - \left[ \nabla \times (\nabla \times \mathbf{z}) \right] [\nabla \times (\nabla \times \mathbf{z})] \quad (24)$$

one shows that this operator acting on $U_0(r | r')$ reduces to

$$- \left( \mathcal{I} + \frac{1}{k^2} \nabla \nabla \right) \nabla \mathbf{v}_f.$$
the excitation coil impedance due to the flaw can be determined from the relationship [17]

$$I^2 \Delta Z = - \int_{\Omega} E^{(0)}(r) \cdot P(r) \, dr$$  \hspace{1cm} (25)$$
deduced using a reciprocity theorem. In a discrete representation, the impedance integral over the flaw region $\Omega$ is approximated by a summation [17], as described in Section IV.

C. Kernels for Circularly Cylindrical Systems

The scalar kernels for circularly cylindrical systems can be expressed in cylindrical polar coordinates using the Fourier representation [18]. For a source in a conductive region we write

$$G_{ij}(r | r') = \frac{1}{4\pi^2} \sum_{m=\infty}^{\infty} e^{im(\varphi - \varphi')}$$

$$\times \int_{-\infty}^{\infty} \tilde{G}_{ij}(\rho, \rho', m, v) e^{im(z - z')} \, dv$$  \hspace{1cm} (26)$$
where $\tilde{G}_{ij}(\rho, \rho', m, v)$ satisfies

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \tilde{G}}{\partial \rho} \right) - \left( \xi^2 + \frac{m^2}{\rho^2} \right) \tilde{G} = - \frac{1}{\rho} \delta(\rho - \rho')$$  \hspace{1cm} (27)$$
with $\xi^2 = v^2 - i^2$. Solutions formulated in this way can be used to model fields in a number of basic engineering components, such as bolt holes, tubes, and rods.

The Fourier domain kernels for a singular source in the conductive region surrounding a circular hole takes the form

$$\tilde{G}_{ij}(\rho, \rho', m, v) = \delta_{ij} \tilde{G}_0(\rho, \rho', m, v) + \tilde{G}_{ij}^{(T)}(\rho, \rho', m, v)$$  \hspace{1cm} (28)$$
where $\tilde{G}_0(\rho, \rho', m, v)$ is a Fourier transformation of the singular, unbound domain kernel and $\tilde{G}_{ij}^{(T)}(\rho, \rho', m, v)$ represents field migration from the interface. The Green’s kernel for an unbound conductor, deduced by solving (27) or by adapting the method used in [18] is

$$\tilde{G}_0(\rho, \rho', m, v) = I_m(\xi \rho) K_m(\xi \rho')$$  \hspace{1cm} (29)$$
where $I_m(z)$ and $K_m(z)$ are associated Bessel functions, $\rho_\rho = \max(\rho, \rho')$ and $\rho_\rho = \min(\rho, \rho')$. The field migrating outwardly from the surface of the hole, radius $a$, is represented by

$$\tilde{G}_{ij}^{(T)}(\rho, \rho', m, v) = \frac{I_m(\xi \alpha)}{K_m(\xi \alpha)} \Gamma_{ij}(m, v) K_m(\xi \rho') K_m(\xi \rho)$$  \hspace{1cm} (30)$$
where $i, j = 1, 2$ and the $\Gamma_{ij}(m, v)$ are the boundary migration coefficients. We prefer to make explicit here an additional geometric factor $I_m(\xi \alpha)/K_m(\xi \alpha)$ which it seems best to separate from the term, $\Gamma_{ij}(m, v)$ characterizing the interface condition on a circularly cylindrical surface. A derivation of the boundary migration coefficients is given in Appendix A.

III. FLAW THEORY

A. Narrow Crack

The numerical calculation of the flaw signal can be carried out either by using volume elements or boundary elements. In general, all three components of the flaw polarization $P(r)$ are needed for a solution but for a parallel-sided narrow planar crack, the component normal to the crack faces dominates and the transverse components can be neglected. In the general volume element scheme [19] all three field components are computed but in the case of a narrow crack, this requires more unknowns than are necessary for reasonably accurate results. For a narrow crack it is more efficient to simply compute the component of the dipole density normal to the crack face thereby reducing the number of unknowns by a factor of three with a substantial reduction of the computational cost.

In the case of a parallel-sided narrow crack we write

$$P(r) \approx \hat{n} \cdot P(r)$$  \hspace{1cm} (31)$$
where $\hat{n}$ is normal to the crack face. Then multiplying the normal components of the terms in (14) by the host conductivity, and using (31), one finds that

$$J_n^{(0)}(r) + k^2 \int_{\Omega} G_{nn}(r | r') P(r') \, dr' = 0$$  \hspace{1cm} (32)$$
for a point on the outside surface of a crack that is impenetrable to eddy currents. Here $J_n^{(0)} = \sigma \hat{n} \cdot E^{(0)}$. The kernel is a component of the tensor Green’s function:

$$G_{nn}(r | r') = \hat{n} \cdot G(r | r') \cdot \hat{n}$$  \hspace{1cm} (33)$$
depending on the direction of the normal $\hat{n}$. In the case of an ideal crack, a surface dipole density is defined as the finite limit of

$$p = \delta_n P$$  \hspace{1cm} (34)$$
as the crack opening $\delta_n$ tends to zero. A boundary integral over the crack face then replaces the above volume integral. Here we will examine transverse/circumferential and longitudinal/axial cracks, both ideal and open.

B. Circumferential and Longitudinal Crack Kernel

For a circumferential crack, the tensor component of the kernel required for the calculation is $G_{zz}(r | r')$. This component, like the tensor it belongs to, can be subdivided into a singular, unbound domain part

$$G_{zz}^{(0)}(r | r') = \left( 1 + \frac{1}{k^2} \frac{\partial}{\partial \varphi} \right) G_0(r | r')$$  \hspace{1cm} (35)$$
and the regular part accounting for migration from the boundary. The latter is found in the last part of the expression for the boundary migration kernel in (23):

$$G_{zz}^{(T)}(r | r') = \frac{1}{k^2} \nabla_1^2 G_{22}^{(T)}(r | r').$$  \hspace{1cm} (36)$$
For an axial crack, the tensor component required for the calculation, $G_{22}(r | r')$, can likewise be subdivided into a singular unbounded domain part

$$G_{22}^{(0)}(r | r') = \left( 1 + \frac{1}{k^2 r^2} \right) G_0(r | r')$$  \hspace{1cm} (37)

and a regular part accounting for migration from the boundary and derived from (23):

$$G_{22}^{(T)}(r | r') = \frac{\partial^2 U_{11}(r | r')}{\partial \phi \partial \phi'} - \frac{1}{k^2 r^2} \frac{\partial^2 U_{12}(r | r')}{\partial \phi \partial \phi'} - \frac{1}{\rho} \frac{\partial U_{22}(r | r')}{\partial \phi} + \frac{1}{k^2 r^2} \frac{\partial^2 U_{22}(r | r')}{\partial \phi \partial \phi'}$$  \hspace{1cm} (38)

In forming a discrete approximation of the integral equations for the dipole density, matrix elements are calculated by numerical integration of these functions.

IV. NUMERICAL SOLUTION

A numerical approximation of the flaw polarization can be found by applying the moment method in which (32) is used to form the matrix equation

$$\mathbf{J} + \mathbf{M} \mathbf{P} = 0$$  \hspace{1cm} (39)

where $\mathbf{J}$ and $\mathbf{P}$ are column vectors. There is a vast literature on the moment method from which we shall draw on only to outline a basic application of a pulse-based point-matching scheme. Although lacking refinement, the approach has the virtues of being simple to apply and gives adequate results. In the point matching scheme we have adopted, the column vector $\mathbf{J}$ consists of values of the incident current density, $J_{in}^{(0)}(r)$, at points on the external crack face. The elements of the column vector $\mathbf{P}$, are coefficients in a series expansion of the dipole density using pulse functions defined on elemental cells. Consequently, the expansion approximates the dipole density as piecewise constant over volume elements, shown in Fig. 2, or boundary elements when the crack opening is negligible. The basic shape of the volume elements is similar for axial and circumferential cracks but the relative dimensions and appearance depends on the crack orientation. The direction of the dipole density is assumed to be that of the coordinate perpendicular to the cracks faces. Only one element is used in the direction of the crack opening but any number can be used to approximate the shape of the crack in the other two dimensions. The matrix $\mathbf{M}$ is found by integrating the appropriate component of the dyadic Green’s kernel over volume elements.

It is convenient to express the matrix in terms of the function $M(r | r_0)$ where $r_0$ is the coordinates of a reference point at the center of the cell occupying the region $\Omega_0$. This function is defined such that $M(r | r_j) = M_{ij}$ are the matrix elements of $\mathbf{M}$. Thus rows of $\mathbf{M}$ correspond to the matching points $r_j$ on the outside surface of the crack and $r_j$ are coordinates of the center of cell $j$.

For an ideal crack, one that is impenetrable to current yet has zero opening, the volume integral is replaced by an integral over a surface element, in which case one can extract matrix elements from the function

$$L(r | r_0) = k^2 \int_{S_0} G_{mn}(r | r') dS'.$$  \hspace{1cm} (41)

The integration is over the domain $S_0$ of a rectangular boundary element at the surface of the ideal crack. The equation to be solved is similar except that instead of (39) we write

$$\mathbf{J} + \mathbf{M} \mathbf{P} = 0$$  \hspace{1cm} (42)

where $\mathbf{P}$ is a column vector representing the strength of the surface dipole density associated with each boundary element.

For numerical calculations, the flaw-region integral in (25) is approximated numerically by a scalar product

$$I^2 \Delta Z = - \frac{1}{\sigma_0} \mathbf{J} \cdot \mathbf{P}.$$  \hspace{1cm} (43)

Next we shall describe the calculation of the matrix elements for both circumferential and axial cracks.

A. Singular Elements

Since the matching points are external to the domain of integration, the value of the singular element can be calculated using the following simple method. The singular term of the Green’s kernel, (16), gives rise to a contribution to the matrix that, for an ideal crack is expressed as a surface integral

$$L^{(0)}(r | r_0) = \int_{S_0} \left( k^2 + \frac{\partial^2}{\partial \phi' \partial \phi'} \right) G_{22}^{(0)}(r | r') dS'$$  \hspace{1cm} (44)

where $r'$ represents the coordinate normal to the crack face, $S_0$ is the domain of a boundary element and $r_0$ is the midpoint of the element. Because $G_{22}^{(0)}(r | r')$ is the fundamental solution of the scalar Helmholtz equation, one can transform the singular integrand using

$$\left( k^2 + \frac{\partial^2}{\partial \phi' \partial \phi'} \right) G_{22}^{(0)}(r | r') = - \nabla'_t \cdot \nabla_t G_{22}^{(0)}(r | r').$$

Here the subscript $t$ denotes components of the differential operator tangential to a surface of constant $\phi$. For a circumferential crack, for example, this surface is a plane in which $z$ is a constant whereas for an axial crack, the crack face is in a plane of constant $\phi$. Using the identity $\nabla'_t = \nabla'_\phi \cdot \nabla'_\phi$ to transform the 2-D Laplacian operator and applying Gauss divergence theorem in two dimensions, one can express the singular element as a path integral, written as

$$L^{(0)}(r | r_0) = - \oint_{C_0} \hat{u} \cdot \nabla'_t G_{22}^{(0)}(r | r') d\ell'$$  \hspace{1cm} (45)
where $C_0$ is the path bordering the face of the cell and $\hat{n}$ is a unit vector in the plane of the crack face outwardly normal to the cell boundary. For an open crack, one needs to include an integral over the crack width in addition to the integral in (45). For a circumferential crack calculation, this extra integral is with respect to the $z$ coordinate and ranges over the crack opening whereas for an axial crack, the additional integration is with respect to $\varphi$, Fig. 2. In both cases, the integration can be carried out numerically.

B. Circumferential Crack Reflection Matrix

From last term in (23), one identifies the reflection kernel for the circumferential crack, as given in (36). This has the Fourier domain representation

$$
\tilde{C}_{zz}^{(\Gamma)}(\rho, \varphi, m, v) = \frac{\xi_0}{k^2} \frac{I_m(\xi_0)}{K_m(\xi_0)} \Gamma_{22}^{(2)}(m, v) K_m(\xi \rho) K_m(\xi \rho')
$$

(46)

where $\Gamma_{22}^{(2)}(m, v)$ is a reflection coefficient given by (A31).

In view of the fact that $\tilde{C}_{zz}^{(\Gamma)}(\rho, \varphi, m, v)$ is invariant under a reversal of the sign of $m$ and $v$, the reflection kernel, previously expressed in the form of (26), can also be written as

$$
\tilde{C}_{zz}^{(\Gamma)}(r | r') = \frac{1}{2\pi^2} \sum_{m=0}^{\infty} \epsilon_m \cos[m(\varphi - \varphi' - \varphi_z)]
\times \int_{0}^{\infty} \tilde{C}_{zz}^{(\Gamma)}(\rho, \varphi, m, v) \cos[v(z - z')] dv
$$

(47)

where $\epsilon_0 = 1$ and $\epsilon_m = 2$ for $m = 1, 2, 3, \ldots$ For numerical calculations, the range of the integral can be truncated and approximated by a series.

In seeking numerical predictions for an ideal crack by approximating the dipole density as piecewise constant over a surface element, one integrates over the face of the source cell. For an open crack there is an integral over the opening to do as well, as in (40). In the latter case one finds that

$$
\tilde{M}_{zz}^{(\Gamma)}(\rho, \rho_0, m, v) = -\epsilon_0 \epsilon_2 \sin \left( \frac{\nu \delta_z}{2} \right) \sin \left( \frac{\nu \delta_z}{2} \right) 
\times \frac{I_m(\xi_0)}{K_m(\xi_0)} \Gamma_{22}^{(2)}(m, v) g_{m, 1}(\rho_1, \xi \rho_2) K_m(\xi \rho)
$$

(48)

where $\rho_1 = \rho_0 - \delta_\rho/2$ and $\rho_2 = \rho_0 + \delta_\rho/2$ are the radii of the annular sector at the face of a source cell, $\delta_\rho$ its azimuthal angle defining a cell sector. Also we define the function

$$
g_{\mu, \nu}(z) = \int_{0}^{z} r^\mu K_\nu(r) dr
$$

(49)

and for conciseness write $g_{\mu, \nu}(z_1 + z_2) = g_{\mu, \nu}(z_2) + g_{\mu, \nu}(z_2)$. The integral in (49) can be evaluated with the aid of a recursion relation found from a special case of (11.3.6) in [20]:

$$(m + \mu)g_{\mu, m+1}(z) = -2m z^\mu K_m(z) - (\mu + m)g_{\mu, m-1}(z)
$$

(50)

For the circumferential crack calculation, the case where $\mu = 1$ is needed whereas for the longitudinal crack we use $\mu = 0$ and with range of integration from $z$ to infinity, $\mu = -1$. The recursion relationship applies directly for $m \geq 2$ with $\mu = 1$. For lower orders

$$
g_{1, 0}(z) = -z K_1(z)
$$

(51)

$$
g_{1, 1}(z) = \int_{0}^{z} r K_1(r) dr
$$

(52)

and

$$
g_{1, 2}(z) = -z K_1(z) - 2K_0(z).
$$

(53)

The integral for $g_{1, 1}(z)$ can be evaluated analytically in terms of Struve and Bessel functions (6.561.4 of [21]).

In the case of an ideal closed transverse crack, the crack opening factor, $\delta_z$ in (48) is absorbed in the redefinition of the dipole density in accordance with limiting process associated with (34). In the ideal crack limit, the sinc function representing the crack opening in (48) tends to unity. Consequently, the ideal crack matrix elements for the boundary migration of the TM field can be deduced from

$$
\tilde{M}_{zz}^{(\Gamma)}(\rho, \rho_0, m, v) = \delta_z \sin \left( \frac{\nu \delta_z}{2} \right) \tilde{M}_{zz}^{(\Gamma)}(\rho, \rho_0, m, v)
$$

(54)

In general the matrix elements arising from a TM mode migrating from the interface having originated at a TM source are given by

$$
\tilde{M}_{zz}^{(\Gamma)}(r | r_0) = \frac{1}{2\pi^2} \sum_{m=0}^{\infty} \epsilon_m \cos[m(\varphi - \varphi_z)]
\times \int_{0}^{\infty} \tilde{M}_{zz}^{(\Gamma)}(\rho, \rho_0, m, v) \cos[v(z - z_0)] dv
$$

(55)

According to the prescription following (40), we index central points of the cell putting $\varphi - \varphi_0 = \varphi_i - \varphi_j = (i - j) \delta_\varphi$, $\rho = \rho_i$, $\rho_0 = \rho_j$ and with the field point on the crack face a distance $\delta_z/2$ from the cell center, put $z - z_0 = \delta_z/2$, to give the elements of that part of the system matrix representing the TM field migration from the surface of the hole.

C. Axial Crack Reflection Matrix

From (38), the Fourier transform of the kernel for the longitudinal crack is

$$
\tilde{G}_{zz}^{(\Gamma)}(\rho, \varphi, m, v) = -\frac{I_m(\xi_0)}{K_m(\xi_0)}
\times \left[ \Gamma_{11}^{(2)}(m, v) K_m'(\xi \rho) K_m(\xi \rho) 
+ \frac{1}{\xi_\rho} \Gamma_{12}^{(2)}(m, v) K_m'(\xi \rho') K_m(\xi \rho) 
+ \frac{m \nu v}{\xi_\rho} \Gamma_{22}^{(2)}(m, v) K_m'(\xi \rho) K_m(\xi \rho) 
+ \frac{m \nu v^2}{\xi_\rho} \Gamma_{22}^{(2)}(m, v) K_m'(\xi \rho') K_m(\xi \rho) \right]
$$

(56)

where the $\Gamma_{ij}^{(2)}(m, v)$ are given by (A28)–(A31). Assuming pulse basis functions for boundary elements on an idea crack,
one integrates over the surface of the source cell to get
\[
\hat{I}^{(1)}_{\text{source}}(\rho, \rho_0, m, v) = -\delta_1 \sin \left( \frac{v^2 \xi}{2} \right) \frac{I_m(\xi \alpha)}{K_m(\xi \alpha)} \times \left[ \frac{k_0^2}{\xi} \frac{\Gamma_{11}^{(2)}(m, v)M_{11}(\xi \rho_1, \xi \rho_2)}{K_m(\xi \rho_1, \xi \rho_2)}K_m(\xi \rho) \right.
\]
\[
+ \frac{mv}{\xi} \frac{\Gamma_{12}^{(2)}(m, v)M_{12}(\xi \rho_1, \xi \rho_2)}{K_m(\xi \rho_1, \xi \rho_2)}K_m(\xi \rho) \left. + \frac{k_0^2}{\xi^2} \frac{\Gamma_{21}^{(2)}(m, v)M_{21}(\xi \rho_1, \xi \rho_2)}{K_m(\xi \rho_1, \xi \rho_2)}K_m(\xi \rho) \right]
\]
(57)
where we have written \(K_m(\xi z_1, z_2) = K_m(z_2) - K_m(z_1)\), \(g_{-1,m}(z_2)\), \(g_{-1,m}(z_1)\) and defined
\[
g_{-1,m}(z) = \int_{z}^{\infty} \frac{1}{r} K_m(r) dr.
\]
(58)

The evaluation of (58) uses the recursion relation, (50), with \(g_{-1,0}(z)\) and \(g_{-1,1}(z)\) computed numerically.

For an open crack, the integration is prescribed by (40) and the matrix elements due to the effect of boundary migration are deduced from
\[
M_{\phi, \phi}^{(1)}(\rho, \rho_0, m, v) = -\delta_1 \sin \left( \frac{v^2 \xi}{2} \right) \sin \left( \frac{v^2 \xi}{2} \right) \frac{I_m(\xi \alpha)}{K_m(\xi \alpha)} \times \left[ \frac{k_0^2}{\xi} \frac{\Gamma_{11}^{(2)}(m, v)/h_m(\xi \rho_1, \xi \rho_2)}{K_m(\xi \rho_1, \xi \rho_2)}K_m(\xi \rho) \right.
\]
\[
+ \frac{mv}{\xi} \frac{\Gamma_{12}^{(2)}(m, v)/g_{0, m}(\xi \rho_1, \xi \rho_2)}{K_m(\xi \rho_1, \xi \rho_2)}K_m(\xi \rho) \left. + \frac{k_0^2}{\xi^2} \frac{\Gamma_{21}^{(2)}(m, v)/g_{0, m}(\xi \rho_1, \xi \rho_2)}{K_m(\xi \rho_1, \xi \rho_2)}K_m(\xi \rho) \right]
\]
(59)
where \(h_m(z_1, z_2) = h_m(z_2) - h_m(z_1)\),
\[
h_m(z) = \int_{z}^{\infty} rK_m(r) dr = zK_m(z) - g_{0, m}(z)
\]
(60)
and with \(g_{0, m}\) found from the recursion relationship (50). Thus higher orders can be found recursively from \(g_{0,1}(z) = -K_0(z)\) and \(g_{0,2}(z)\), the latter pair being computed numerically. Finally, we also need the analytical result for \(g_{0,0}(z)\), see (6.561.4) of [21], to complete the definition of boundary and volume elements for the circumferential and axial crack.

V. INCIDENT FIELD

A. General Form

In this section, general expressions are given for the current density in the conductor due to an arbitrary prescribed current source in the borehole. These are expressed in terms of a set of source coefficients, \(D_m^{(0)}\), which are used to define the field in an unbounded non-conductive region in the form
\[
W^{(0)}(r) = \sum_{m = -\infty}^{\infty} e^{\imath m \varphi} \int_{\infty}^{\infty} D_m^{(0)}(v)K_m(\xi \rho)e^{\imath v \xi z} dv
\]
(61)
at some radial coordinate \(\rho > \rho_s\) where \(\rho_s\) represents radial limit of the source defined with respect to the axis of a cylindrical polar coordinate system. The field in the conductor on the other hand, designated as region 2, is written as
\[
W^{(2)}(r) = \sum_{m = -\infty}^{\infty} e^{\imath m \varphi} \int_{\infty}^{\infty} D_m(v)K_m(\xi \rho)e^{\imath v \xi z} dv
\]
(62)
where \(W\) and \(D_m\) are two-component column vectors whose components correspond to TE and TM modes, denoted by subscripts 1 and 2, respectively. Thus
\[
W(r) = \begin{bmatrix} W_1(r) \\ W_2(r) \end{bmatrix}, \quad D_m(v) = \begin{bmatrix} D_{1m}(v) \\ D_{2m}(v) \end{bmatrix}
\]
(63)
The coefficients are linearly related
\[
D_m(v) = T_mD_m^{(0)}(v)
\]
(64)
where \(T_m\) is a 2 \times 2 matrix of transmission coefficients, for a source in the hole, see Appendix A. In the quasi-static limit, components of the transmission matrix associated with the TM source are zero, (A20). Hence the field is the conductor is determined by transmission coefficients associated with the TE field in the hole, (A18) and (A19).

The components of the induced current density, can be written in the form
\[
J(r) = \sum_{m = -\infty}^{\infty} e^{\imath m \varphi} \int_{\infty}^{\infty} J(\rho, \rho_0, m, v)e^{\imath v \xi z} dv
\]
(65)
Individual components, found from the source-free version of (21), are given by
\[
J_\rho = \frac{k_0^2}{\rho} [mD_{1m}(v) - nM_m(\xi \rho)D_{2m}(v)]K_m(\xi \rho)
\]
(66)
\[
J_\varphi = \frac{k_0^2}{\rho} [M_m(\xi \rho)D_{1m}(v) - mM_m(\xi \rho)D_{2m}(v)]K_m(\xi \rho)
\]
(67)
\[
J_z = -k_0^2 \xi^2 D_{2m}(v)K_m(\xi \rho)
\]
(68)
where \(M_m(\xi)\) is defined in (A10). In the quasi-static limit, the radial current at the surface of the conductor is zero and hence from (66)
\[
mD_{2m}(v) = nM_m(\xi \rho)D_{2m}(v)
\]
(69)
which means that one set of expansion coefficients can be eliminated. For example, by eliminating \(D_{1m}(v)\), the radial component of the current can be written as
\[
J_\rho = \frac{mv}{\rho} [M_m(\xi \rho) - M_m(\xi \rho)]D_{2m}(v)K_m(\xi \rho)
\]
(70)
It follows that the scalar modes representing the field in the conductor are interdependent, both being determined by the source
coefficients $D_{im}^{(0)}(v)$. Next we examine the derivation of the source coefficients for a rotary coil.

### B. Rotary Coil

Field theory for a circular coil that is designed to rotate in the circular hole with its axis normal to the cylindrical surface has been given in [16]. The coil field is derived from that of a filamentary current loop defined with respect to a cylindrical polar coordinate system whose axis is perpendicular to that of the loop and passes through its center. Next the expression is transformed by a change of variable using a coordinate system with an axis displaced a perpendicular distance $x_0$ from the plane of the loop, Fig. 3, the transformation between coordinate systems being carried out with the aid of an addition theorem. The source coefficients for a filament, radius $r_0$, centered at $z = 0$ and displaced from the coordinate axis by a perpendicular distance $x_0$, is thereby found to be [16]

$$\hat{D}_{lm}(v, r_0, x_0) = \frac{d}{\pi^2} \sum_{n=-\infty}^{\infty} \frac{n \sin(n\pi/2)}{v^3} I_{m-n}(|v|)F_m^{(1)}(v r_0)$$

(71)

where

$$F_m^{(1)}(r) = \int_0^{\infty} \frac{1}{\xi} I_m(\xi) \sin \left( \sqrt{r^2 - \xi^2} \right) d\xi$$

(72)

in which the prime denotes the derivative with respect to the argument.

The field of a coil with rectangular cross section, Fig. 4, is found by integrating over $x_0$ and over $r_0$. The integration with respect to $x_0$ is straightforward but finding $F_n(r)$ from (72), requires the evaluation of a double integral and is more challenging. In [16] the double integral was done numerically but here we express $F_n(r)$ in a convenient analytical form by first writing (72) as a series expansion that can be integrated term by term. First we change the variable of integration letting $u^2 = r^2 - \xi^2$ to give

$$F_m^{(1)}(r) = \int_0^{\infty} \frac{1}{u^2 - u^2} I_m \left( \sqrt{r^2 - u^2} \right) \sin(u) u du$$

(73)

This integrand is reducible to a suitable standard form by expanding the modified Bessel function according to

$$I_m(z) = \left( \frac{z}{2} \right)^m \sum_{k=0}^{\infty} \frac{(z^2/4)^k}{k!(m+k)!}$$

(74)

and substituting into (72). Noting from (3.771.10) of [21] that

$$\int_0^{\infty} (r^2 - x^2)^\nu \sin(x) x dx = 2^{-\nu+\nu} \pi^{\nu+1} u^{3+\nu} J_{3+\nu}(r)$$

(75)

enables one to write

$$F_m^{(1)}(r) = \sum_{k=0}^{\infty} \frac{\Gamma(m+k)}{2^k k!(m+k)!} r^{(m+k+\frac{1}{2})} J_{m+k+\frac{1}{2}}(r)$$

(76)

To determine the coil field, one can integrate this using (6.561.1) of [21]

$$\int_0^{\infty} x^\nu J_{\nu}(x) dx = 2^{-\nu-1} \pi^{\nu+1} \Gamma \left( \nu + \frac{1}{2} \right) \times \left[ I_{\nu-1}(\alpha) H_{\nu+1}(\alpha) - H_{\nu-1}(\alpha) I_{\nu+1}(\alpha) \right]$$

(77)

where $H_{\nu}(\alpha)$ is a Struve function. Thus

$$F_m(r) = \frac{\pi}{4} \sum_{k=0}^{\infty} \frac{\Gamma \left( \frac{m+k}{2} + k \right)}{k!(m+k)!} r^{(m+k+1)} \times \left[ J_{m+k+1}(r) H_{m+k}(r) - H_{m+k+1}(r) J_{m+k}(r) \right]$$

(78)

Having determined $F_m(r)$, the rotary coil field can be given in closed form.

The source expansion coefficients for a rotary coil are defined by

$$D_{im}^{(0)}(v) = \nu \int_{x_1}^{x_2} \int_{r_1}^{r_2} \hat{D}_{lm}(v, r_0, x_0) d\rho_0 dx_0$$

(79)

where $\nu$ is the wire turns density and the filament source coefficients in the integrand are given by (71). Integration with respect to $x_0$ is carried out using (6.511.11) of [21]

$$\Psi_{\mu}(s) = \int_0^{\infty} I_{\mu}(x) dx = 2 \sum_{j=0}^{\infty} (-1)^j I_{\mu+2j+1}(s)$$

(80)
for \( \Re \mu > -1 \) and the integral of \( F_n^1(r) \) in (71) has been given by (78). Thus we get

\[
D_{1m}^{(0)}(\psi) = \frac{nI}{\pi} \sum_{n=-\infty}^{\infty} \sum_{m=-n}^{n} \Psi_{m}(\psi_1, \psi_2) F_n^1(\psi_1, \psi_2)
\]

where

\[
\Psi_{m}(s_1, s_2) = \Psi_{m}(s_2) - \Psi_{m}(s_1)
\]

and

\[
F_n(\alpha_1, \alpha_2) = F_n(\alpha_2) - F_n(\alpha_1).
\]

In summary, three summations are used to evaluate the rotary coil source coefficients, each of which must be terminated appropriately to ensure accurate results. Equations (64), (62) and (21), then give the incident electric field on the crack.

VI. PREDICTIONS COMPARED WITH EXPERIMENT

Controlled experiments have been carried out to test predictions of coil impedance due to cracks in holes. Two aluminum alloy specimens were used each with a 25 mm diameter hole in which a notch has been cut. One of the test pieces has a semi-elliptical longitudinal notch cut into the wall of the hole as shown in Fig. 5. A second test piece has a circumferential notch whose subsurface edge forms a circular arc of diameter 5 mm centered at the surface of the hole. Fig. 6 shows both notches. These were cut using an electrical discharge machine with tools that produced openings of 130 micrometers. The conductivity of the samples was measured using an eddy-current conductivity meter and found to be 23.005 ± 0.02 S/m.

Coil dimensions were measured using a microscope to an accuracy of 0.02 mm, except for the lift-off value, \( a = x_2 \), Fig. 4. This is a critical parameter yet difficult to measure directly. Instead of attempting a direct measurement, the lift-off value was determined by parameter fitting to multi-frequency coil impedance measurements. Measurements are made with a coil in air and at the surface of a flaw-free part of the hole, the difference being the impedance due to induced current. The predicted impedances due to induced current are calculated for a range of lift offs and compared with the multi-frequency impedance measurements to obtained the value that gives optimal agreement between predictions and measurements at a flaw free location.

Coil impedance measurements and their variation with position are made with an Agilent 4294A impedance analyzer and a four axis (\( x \), \( y \), \( z \) and \( \phi \)) computer controlled precision scanner. A comparison of impedance predictions with measurements at 40 kHz on a circumferential notch formed by electrical discharge machining is shown in Fig. 7. A similar comparison at the same frequency for an axial notch is shown in Fig. 8. At the test frequency, the notches are approximately 10 skin-depths deep and hence the measurements are carried out in a regime where the notch depth is much larger that the skin depth. For both notches, prediction were made using 50 cells in the radial direction and 150 in the direction tangential to the surface of the hole.

VII. DISCUSSION AND CONCLUSION

Eddy-current coil impedance variations due to planar cracks in holes have been evaluated using a scalar decomposition of the quasi-static electromagnetic field leading to an integral equation for the flaw field. The attraction of the integral approach is that it can be formulated for simple structures such that a discrete approximation of the field is needed only in the flaw region. Usually the flaw is relatively small and therefore few unknowns are needed for a numerical solution. Furthermore, because the evaluation of the matrix elements is done by computing summations of Fourier series in two dimensions, the computational cost is modest.
and are two component column vectors representing both TE and TM modes, denoted by subscripts 1 and 2, respectively. For example

$$W(r) = \sum_{m=-\infty}^{\infty} e^{im\varphi} \int_{-\infty}^{\infty} \hat{W}_m(v, \rho) e^{imz} dv$$

(A1)

where \( W \) and \( \hat{W}_m \) are two component column vectors representing both TE and TM modes, denoted by subscripts 1 and 2, respectively. For example

$$W(r) = \begin{bmatrix} W_1(r) \\ W_2(r) \end{bmatrix} \quad \hat{W}_m(v, \rho) = \begin{bmatrix} \hat{W}_{1m}(v, \rho) \\ \hat{W}_{2m}(v, \rho) \end{bmatrix}.$$  

(A2)

The vector function \( \hat{W}_m \) satisfies the Bessel equation

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \hat{W}_m}{\partial \rho} \right) - \left( \xi^2 + \frac{m^2}{\rho^2} \right) \hat{W}_m = 0$$

(A3)

with \( \xi = \sqrt{\nu^2 - \frac{\rho^2}{\rho^2}} \). The transverse potentials are coupled at the air-conductor interface as exhibited by the reflection and transmission coefficients characterizing the field interaction at the surface of the hole.

In this Appendix, the coefficients are derived using the quasi-static limit of Maxwell’s equations by considering firstly a source in the hole and secondly a source in the conductor.

### TRANSMISSION AND REFLECTION COEFFICIENTS

Solutions of the modified Helmholtz equation, (4), for the TE and TM potentials can be found in the form of a Fourier representation

$$W(r) = \sum_{m=-\infty}^{\infty} e^{im\varphi} \int_{-\infty}^{\infty} \hat{W}_m(v, \rho) e^{imz} dv$$

(A1)

where \( W \) and \( \hat{W}_m \) are two component column vectors representing both TE and TM modes, denoted by subscripts 1 and 2, respectively. For example

$$W(r) = \begin{bmatrix} W_1(r) \\ W_2(r) \end{bmatrix} \quad \hat{W}_m(v, \rho) = \begin{bmatrix} \hat{W}_{1m}(v, \rho) \\ \hat{W}_{2m}(v, \rho) \end{bmatrix}.$$  

(A2)

The vector function \( \hat{W}_m \) satisfies the Bessel equation

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \hat{W}_m}{\partial \rho} \right) - \left( \xi^2 + \frac{m^2}{\rho^2} \right) \hat{W}_m = 0$$

(A3)

with \( \xi = \sqrt{\nu^2 - \frac{\rho^2}{\rho^2}} \). The transverse potentials are coupled at the air-conductor interface as exhibited by the reflection and transmission coefficients characterizing the field interaction at the surface of the hole.

In this Appendix, the coefficients are derived using the quasi-static limit of Maxwell’s equations by considering firstly a source in the hole and secondly a source in the conductor.
region. The coefficients are determined, as usual, from the continuity conditions on the field at the air-conductor interface that are specified using superscript (1) to denote the field in the hole and superscript (2) to denote the field in the conductor. From the continuity of the $z$-component of the electric and magnetic fields, one deduces respectively that

$$
\nu^2 \left[ \tilde{W}^{(1)}_{2m} \right]_{\rho = a_+} = \zeta^2 \left[ \tilde{W}^{(2)}_{2m} \right]_{\rho = a_+}
$$

(A4)

and by imposing the continuity of the azimuthal magnetic and electric fields, we get a relationship which couples the TE and TM modes:

$$
\begin{bmatrix}
\frac{m_v}{a} \frac{a}{\partial \rho} & 0 \\
\frac{m_v}{a} \frac{a}{\partial \rho} & k^2 \frac{a}{\partial \rho} - \frac{m_v}{a} \frac{a}{\partial \rho} - \mu_\nu \frac{m_v}{a} \\
\end{bmatrix}
\begin{bmatrix}
\tilde{W}^{(1)}_{2m} \\
\tilde{W}^{(2)}_{2m}
\end{bmatrix}
_{\rho = a_+} = 0.
$$

(A5)

where the derivatives are performed prior to evaluation of the expressions at $\rho = a_+$ corresponding to the limits as the bore surface is approached from within the conductor or within the bore, respectively.

A. Source in the Hole

For the potential in the hole at a point whose radial coordinate satisfies $\rho > \rho_h$, where $\rho_h$ is the radial distance from the axis to the point on the source closest to the bore surface, we write

$$
\tilde{W}^{(1)}_m = \frac{K_\nu(\rho a)}{K_\nu(\rho a)} \mathcal{C}^{(0)}_m + \frac{I_\nu(\rho a)}{I_\nu(\rho a)} \mathcal{C}^{(1)}_m
$$

(A6)

where $\mathcal{C}^{(0)}_m$ are prescribed source coefficients and

$$
\mathcal{C}^{(1)}_m = \mathbf{T}^{(1)}_m \mathcal{C}^{(0)}_m
$$

(A7)

where $\mathbf{T}^{(1)}_m$ is a $2 \times 2$ matrix of reflection coefficients allowing for the coupling between TE and TM modes, with the superscript (1) in this case denoting the region of the source. For region 2, outside the hole, the potential is written

$$
\tilde{W}^{(2)}_m = \frac{K_\nu(\xi a)}{K_\nu(\xi a)} \mathcal{C}^{(2)}_m
$$

(A8)

and since the source coefficients in the conductor are linearly related to those of the source

$$
\mathcal{C}^{(2)}_m = \mathbf{T}^{(2)}_m \mathcal{C}^{(0)}_m
$$

(A9)

where $\mathbf{T}^{(2)}_m$ is a $2 \times 2$ matrix of transmission coefficients. Again superscript (1) indicates the region of the source. In seeking the reflection and transmission matrices, it is convenient to use the following notation

$$
\begin{align*}
L_m(x) &= \frac{xI_\nu'(x)}{I_\nu(x)} \\
M_m(x) &= \frac{xK_\nu'(x)}{K_\nu(x)}
\end{align*}
$$

(A10)

and

$$
L_m(x) + M_m(x) = \frac{1}{I_\nu(x) K_\nu(x)}.
$$

(A11)

The last relationship here is derived using a Wronskian for the associated Bessel functions:

$$
K_m(x) \frac{dI_m(x)}{dx} - I_m(x) \frac{dK_m(x)}{dx} = \frac{1}{x}.
$$

(A12)

Next, allow the derivatives on the right of (A5) to act on the vector, $\mathbf{W}^{(2)}_m$ and substitute for this vector using (A4) to give

$$
\begin{bmatrix}
\mu_\nu m \\
-\nu^2 M_m(\xi a) - \nu M_m(\xi a) \\
\end{bmatrix}
\mathbf{T}^{(1)}_m \mathcal{C}^{(0)}_m = 0.
$$

(A13)

One can show that

$$
\begin{bmatrix}
\mu_\nu m \\
-\nu^2 M_m(\xi a) - \nu M_m(\xi a) \\
\end{bmatrix}
\mathbf{T}^{(2)}_m \mathcal{C}^{(1)}_m = 0.
$$

(A14)

by using (A6). This linear system is solved to yield the reflection matrix in the quasi-static limit whose components are as follows:

$$
\Gamma^{(1)}_{11} = -\frac{k^2 m^2 - M_m(\xi a)}{k^2 m^2 - M_m(\xi a)} \left[ I_m(\xi a) + (\xi^2/\mu_\nu)M_m(\xi a) \right]
$$

(A15)

$$
\Gamma^{(1)}_{21} = -\frac{L_m(\xi a)}{k^2 m^2 - M_m(\xi a)} \left[ I_m(\xi a) + (\xi^2/\mu_\nu)M_m(\xi a) \right]
$$

(A16)

$$
\Gamma^{(1)}_{12} = 0, \quad \Gamma^{(1)}_{22} = -1.
$$

(A17)

Note that the TM mode exhibits perfect reflection in the quasi-static limit. The components of the transmission matrix are

$$
T^{(1)}_{11} = \frac{L_m(\xi a)}{k^2 m^2 - M_m(\xi a)} \left[ I_m(\xi a) + (\xi^2/\mu_\nu)M_m(\xi a) \right]
$$

(A18)

$$
T^{(1)}_{21} = \frac{L_m(\xi a)}{k^2 m^2 - M_m(\xi a)} \left[ I_m(\xi a) + (\xi^2/\mu_\nu)M_m(\xi a) \right]
$$

(A19)

$$
T^{(1)}_{12} = 0, \quad T^{(1)}_{22} = 0
$$

(A20)

showing that no transmission from a TM source occurs and that transmissions coefficients for the field with a TE source are related by

$$
\nu T^{(1)}_{11} = \nu M_m(\xi a) T^{(1)}_{21}
$$

(A21)

as in (69). As before this condition that ensures that the radial electric field is zero at $\rho = a_+$.

B. Source in the Conductor Surrounding the Hole

Inside a source-free hole, the potential can be written

$$
\tilde{W}^{(3)}_m = \frac{I_m(\nu a)}{I_m(\nu a)} \mathcal{C}^{(1)}_m
$$

(A22)
where the coefficient vector $C_m^{(1)}$ is linearly related to a prescribed source coefficient vector $C_m^{(0)}$ by a $2 \times 2$ transmission matrix:

$$ C_m^{(1)} = T_m^{(2)} C_m^{(0)} . $$

The superscript on the matrix indicates the region of the source where the transverse potentials are given by

$$ \tilde{W}_m^{(2)} = \frac{I_m(\xi a)}{I_m(\xi a)} C_m^{(0)} + \frac{K_m(\xi a)}{K_m(\xi a)} C_m^{(2)} . $$

Once again, the source coefficients $C_m^{(0)}$ are linearly related to those representing the amplitudes of the scalar fields migrating from the boundary:

$$ C_m^{(2)} = \Gamma_m^{(2)} C_m^{(0)} $$

and $\Gamma_m^{(2)}$ is a $2 \times 2$ reflection matrix for a source in region 2.

Proceeding as with the internal source, we allow the derivatives on the right of (A5) to act on the vector, $\tilde{W}_m^{(1)}$, and substitute for this vector using (A4) to give

$$ \begin{bmatrix} a q \partial_\nu + m & \frac{a v}{\mu_r k^2 \nu m} \\ \mu_r k^2 \nu m & \frac{a v}{\mu_r k^2 \nu m} \end{bmatrix} \begin{bmatrix} \tilde{W}_m^{(2)} \\ \tilde{W}_m^{(2)} \end{bmatrix} = 0. $$

This can be expanded to give

$$ \begin{bmatrix} m & \frac{v a}{\mu_r k^2 \nu m} \\ \mu_r k^2 \nu m & \frac{v a}{\mu_r k^2 \nu m} \end{bmatrix} \Gamma_m^{(2)} = 0 $$

by using (A24) and (A25). Then the linear system is used to deduce the components of the reflection matrix:

$$ \Gamma_m^{(2)} = \frac{k^2 m^2 + M_m(\xi a)}{k^2 m^2 - M_m(\xi a)} \frac{[\partial_\nu M_m(\xi a) - (\xi^2 / \mu_r) L_m(\nu)]}{[\partial_\nu M_m(\xi a) + (\xi^2 / \mu_r) L_m(\nu)]} $$

(A28)

$$ \Gamma_m^{(2)} = \frac{-[L_m(\xi a) + M_m(\xi a)] V_{\nu m}}{k^2 m^2 - M_m(\xi a)} \frac{[\partial_\nu M_m(\xi a) - (\xi^2 / \mu_r) L_m(\nu)]}{[\partial_\nu M_m(\xi a) + (\xi^2 / \mu_r) L_m(\nu)]} $$

(A29)

$$ \Gamma_m^{(2)} = \frac{-[L_m(\xi a) + M_m(\xi a)] V_{\nu m}}{k^2 m^2 - M_m(\xi a)} \frac{[\partial_\nu M_m(\xi a) - (\xi^2 / \mu_r) L_m(\nu)]}{[\partial_\nu M_m(\xi a) + (\xi^2 / \mu_r) L_m(\nu)]} $$

(A30)

and

$$ \Gamma_m^{(2)} = \frac{k^2 m^2 + M_m(\xi a)}{k^2 m^2 - M_m(\xi a)} \frac{[\partial_\nu M_m(\xi a) + (\xi^2 / \mu_r) L_m(\nu)]}{[\partial_\nu M_m(\xi a) + (\xi^2 / \mu_r) L_m(\nu)]}. $$

(A31)

These relationships are used in the Green’s kernel for a source in the conductor. Although they can be easily determined, we shall not state the corresponding transmissions coefficients since they are not needed for the present calculation.

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REFERENCES


