A Numerical Model of Optimal Differentiated Indirect Taxation

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Summary

This study examines the structure of optimal commodity tax rates in a many-person many-goods static computational model using segmented LES utility. One of the major findings is that with non-linear Engel curves and linear income tax, optimal commodity tax rates will be progressive and highly dispersed under logarithmic utility specifications. The dispersion of tax rates is reduced if the inequality aversion rate of society is low. With exogenously given non-optimal and non-linear income tax schedules, usually there is still a need for differentiated and progressive indirect taxation. These findings are in marked contrast to the continuing preoccupation of much of the literature with uniform indirect taxation for redistributive purposes. The results also indicate that if tax evasion incurs substantial deadweight costs, it usually reduces optimal tax rates by over a half of the evasion/revenue ratio of the product, with the reduction being larger for necessities and smaller for luxuries. Private compliance costs and government administration costs reduce optimal tax rates by a similar amount to the share of these costs from taxes. In a model with linear income tax, the effect of externalities on optimal tax rates substantially exceeds the corresponding Pigouvian tax rates or subsidies. The main benefit of higher taxes on leisure complements than leisure substitutes appears to be in boosting tax revenue for redistribution, rather than in improving the utility position of those paying the taxes. The effect of complexities such as tax evasion, administrative costs, externalities and leisure complements/substitutes on redistribution is not neutral. Generally, these factors tend to increase the progressivity of optimal commodity tax rates.

Keywords: Optimal taxation, commodity taxation, indirect taxation, tax evasion, externalities, merit goods.

JEL Classification: H21, H23, H24, H26, C63.

1. Introduction

The purpose of this paper is to examine the pattern of optimal commodity tax rates using a computational model. Much of the discussion is concerned with the debate on progressive tax rates.
versus uniform taxation \(^1\). It also covers other issues, such as the impact on optimal tax rates of administrative and compliance costs, tax evasion, externalities, paternalistic concerns as well as leisure substitution or complementarity. A major aim of this study is to contribute to the current policy debate about the partial replacement of income tax by consumption taxes and to highlight possible reforms in the consumption tax area.

The numerical results from this static model suggest that given non-linear Engel curves and logarithmic utility, commodity tax rates tend to be progressive and widely dispersed. Adding “real life” complexities to the model increases further the progressivity and dispersion of tax rates. These findings are in marked contrast to the continuing preoccupation of much of the literature with uniform commodity taxation for redistributive purposes, which in my view, applies only under highly abstract and unrealistic conditions. The numerical examples presented in this paper do not prove that under realistic conditions optimal indirect taxation should be differentiated and progressive. They only illustrate and substantiate analytical arguments to that effect that appeared in the theoretical literature decades ago. Unfortunately, these analytical arguments have been often ignored in recent policy discussions. Hopefully, the present study will help to restore some common sense in this important policy-related area.

The paper is structured as follows:

Section 2 reviews briefly the history of the controversy on whether for redistributive purposes optimal commodity tax rates should be uniform or otherwise, and the current state-of-affairs in this debate. This is followed by a review of previous computational studies on optimal indirect taxation. The review suggests that this numerical study is more comprehensive than anything that was done on this subject before. The third sub-section examines some recent policy oriented studies on taxation and their relevance to the discussion on differentiated commodity taxation. It also outlines the “growth friendly taxation” approach.

Section 3 describes the main mathematical features of the segmented Linear Expenditure System (LES) model. Segmented LES means two LES functions with different parameters for two population groups. This arrangement is needed in order to obtain non-linear Engel curves with LES. Attention is given to the fact that while many functional forms could be used to investigate optimal commodity tax rates, segmented LES, which has explicit demand formulas, presents probably one of the easiest options to attack the problem. For the purpose of analysing the numerical results, we develop in appendix 2 an approximate formula for optimal tax rates called the modified inverse elasticity rule. It is used to explain and analyse the numerical results. It is not applied to calculate optimal tax rates, which are found using iterative calculations. Further details about the structure of the model are outlined in appendix 3.

Section 4 examines the role of indirect taxation for redistributive purposes. The model can incorporate various inequality aversion rates to represent political value judgments concerning income distribution. The numerical results indicate that given non-linear Engel curves and linear income tax, generally luxuries should be taxed at much higher rates than necessities. However, if the inequality aversion is low then the dispersion of optimal tax
rates is considerably reduced. The second sub-section examines the structure of optimal commodity tax rates in combination with some exogenously given non-linear and non-optimal income tax schedules. The third sub-section deals with a two composite goods extension of the model, which in some respects resembles the tax structure of current VAT systems. The fourth sub-section reviews some analytical arguments in favour of progressive indirect taxation, mainly on grounds of better labour incentives and the likely sub-optimality of direct support payments. The fifth sub-section comments on a proposition in favour of uniform taxation presented by Laroque (2005) and Kaplow (2006). The final sub-section examines how government subsidisation of education and health can fit into the picture.

Section 5 deals with a number of factors that can justify differentiated indirect taxation and are not directly related to distributional objectives. These include:

- tax compliance costs
- public administration costs
- indirect tax evasion
- externalities
- paternalistic concerns
- non-separable utility between commodities and leisure

The modified inverse elasticity rule is extended to accommodate these factors and yields some fairly simple approximate formulas for their impact on optimal tax rates. In some cases these formulas provide reasonably good approximations to the outcomes from iterative calculations. In other cases systematic differences emerge that will be noted and where possible analysed. The modified inverse elasticity formulas highlight the fact that the impact on optimal tax rates of factors that are seemingly unrelated to distributional objectives is actually influenced by distributional considerations. Generally, these factors tend to reinforce the progressivity of optimal indirect taxation.

Section 6 presents summary and qualifications. The first sub-section summarises the principal findings. The second examines possible weaknesses in the present model. The third sub-section highlights some policy implications.

The source code for this computational model is available on the Internet. Appendix A explains where to find it and how to use it. The interested reader can use this program for numerical experimentations by putting in parameters of his/her choice.

2. Background

2.1. Review of the tax uniformity debate

The theory of optimal taxation can be dated back to Ramsey’s work in the 1920’s, but only since the early 1970s were mathematical models developed that focus on the redistrib-
utive aspects of taxation. This theory considers both equity and efficiency. In these models individuals differ in term of labour productivity (ability) and redistributive taxation is used in line with the inequality aversion of society. The pioneering model in this field is Mirrlees (1971), who examined how to optimise a non-linear income tax function combined with a uniform lump-sum grant (called the demogrant), taking into account that more income redistribution will cause a reduction in labour supply.

Some of the early studies on commodity taxation in redistributive models favoured the idea of progressive indirect taxation. Feldstein (1972) showed that the pricing of goods supplied by public agencies should be progressive. Diamond (1975) indicated that in the presence of a redistributive demogrant, commodity tax rates will tend to be progressive. Neither of these studies included explicitly labour supply.

Shortly after the publication of Diamond’s article, a growing emphasis emerged in the literature on conditions that lead to optimally uniform commodity tax rates for distributional purposes. These conditions include weakly separable utility between commodities and leisure, perfect competition, no differentiation in support payments, no administrative-compliance costs, no tax evasion and consumers with identical characteristics apart from the wage rate. Atkinson and Stiglitz (1976) extended the Mirrlees (1971) income tax model and showed that provided income tax is the optimal control theoretic solution, then optimal commodity tax rates will be zero or uniform. Christiansen (1984) extended the Atkinson-Stiglitz result to a simpler model where commodity tax rates are proportional. Deaton (1979) showed that with linear Engel curves and linear income tax, optimal commodity tax rates will be uniform. Laroque (2005) and Kaplow (2006) suggested that optimal commodity tax rates should be uniform, even if the income tax function is not optimal to start with. Ruiz del Portal (2012) extended the validity of the Atkinson-Stiglitz theorem, to cover bunching as well as non-continuous and non-differentiable tax schedules. The only significant qualification to uniformity presented in these articles is that utility should be weakly separable between commodities and leisure. Otherwise, leisure-substitutes should be taxed more lightly and leisure complements more heavily, in line with the analysis of Corlett and Hague (1953-1954).

An early qualification to the Atkinson-Stiglitz theorem was presented by Mirrlees (1976), who found that when the population is defined not by a single characteristic (ability) but by multiple characteristics, the Atkinson-Stiglitz result will not necessarily apply. In context of the analysis on mixed proportional and non-linear taxes, Mirrlees (1976) found that optimal proportional taxes should bear more heavily on commodities that high ability individuals have relatively stronger tastes for. In the 1980s, the main objection to the uniform tax proposition was concerned with lump-sum support grants. Deaton and Stern (1986) indicated that if differentiated support grants are not set at the optimal level then non-uniform commodity taxes may be justified. Their proposition was tested and confirmed by Ebrahimi and Heady (1988) through a computational study, using child benefits as the reason for differentiating lump-sum grants. Another objection to commodity tax uniformity is concerned with the infeasibility of providing optimal income subsidies in
certain situations. This is particularly relevant in many developing countries where direct support payments are absent. In this situation, only progressive indirect taxation can be used to provide some support to the needy, as limited as it may be. Computational studies by Ray (1986), Murty and Ray (1987) and Srinivasan (1989), showed that given inequality aversion, in the absence of a demogrant the optimal commodity tax structure will be progressive.

In the 1990s the controversy about uniform commodity taxation has developed further. A number of mathematical studies pointed out the restricted validity of the uniform tax proposition, besides leisure substitution or complementarity. Cremer and Gahvari (1993) showed that optimal commodity taxes will not be uniform in the presence of commodity tax evasion. Under certain conditions, the optimal official tax rate will increase on evasion affected goods, while the effective post-evasion tax rate will decrease. Boadway, Marchand and Pestieau (1994) looked at the effect of income tax evasion and concluded that in the presence of such evasion optimal commodity tax rates will not be uniform, unless commodity preferences are quasi-homothetic. Later other factors were noted that may invalidate the uniform tax proposition. These include Cremer and Gahvari (1995) on uncertainty about future incomes and home purchases, Myles (1995) on imperfect competition, Naito (1999) on non-linear technologies and imperfect mobility between skills, Cremer, Pestieau and Rochet (2001) on differences in wealth endowments and Saez (2002) on heterogeneous preferences between households. In a critical review of earlier studies, Alm (1996) questioned the practical relevance of the uniform tax propositions derived from highly simplistic models. He noted that the administrative and compliance costs of taxes are often comparable or even larger than the economic efficiency costs measured in terms of excess burden. Boadway and Pestieau (2003) pointed out that the Atkinson-Stiglitz (1976) uniformity theorem is only valid if the income tax function is the Mirrlees (1971) type optimal solution. Thus given actual income tax schedules, the conditions for commodity tax uniformity may not apply. They also examined other sources of violations due to different needs and endowments, multiple forms of labour supply and home production. Kessing and Koldert (2013) showed that the Atkinson-Stiglitz result will be violated in the presence of cross-border shopping and exogenously given taxes on non-transportable goods. Bastani, Blomquist and Pirttila (2013) examined the need for child care subsidies and found that provided child care is not fully paid by the government then commodity tax rates should be progressive. Jacobs and Boadway (2014) examine analytically departures from the Atkinson-Stiglitz result due to non-separable utility between commodities and leisure. They note that econometric evidence does not support the assumption of weakly separable utility.

Despite numerous objections and qualifications, the proposition about uniform commodity taxation for distributional purposes seems to be alive and well. A number of recent policy oriented studies came out in favour of uniform indirect taxation by recommending to abolish existing tax exemptions and reduced rates in VAT systems. These studies include Mirrlees et al. (2011), Arnold et al. (2011), European Commission (2013) and IMF (2014). We shall review these reports briefly in section 2.3.
2.2. Previous computational studies

The continuing preoccupation with commodity tax uniformity in distributional models may partly explain the paucity of computational research on optimal commodity taxation. A literature review on this subject is presented by Nygard (2008). Most of the computational studies on optimal indirect taxation deal with the single-person Ramsey model and are largely irrelevant to redistributive taxation.

The few studies that have been published on many-person models, deal mainly with the situation in developing countries where direct support payments are absent (see Ray (1986), Murty and Ray (1987) and Srinivasan (1989)). They found that in the presence of egalitarian objectives and in the absence of other means of support (such as a demogrant), progressive commodity taxation will increase social welfare. This modelling approach was extended by Ray (1989) and Asano et al. (2004), who examined situations where the demogrant is sub-optimal but is above zero. They found that in these situations as well, progressive commodity taxation is justified. We shall take a critical look at the zero or sub-optimal demogrant approach in section 4.4.

Ebrahimi and Heady (1988) paper on the influence of demographic variables on optimal commodity tax rates is perhaps the most sophisticated and comprehensive computational model among early studies. Ebrahimi and Heady (1988) examined the effect of providing differentiated support payments depending on the number and age of children in the household. They also examined differences in male and female labour supply. They found that non-uniform commodity tax rates are justified when:

I. commodities and leisure are not weakly separable,
II. Engel curves are not parallel across households,
III. the demogrant per household is not linked to the age and number of children.

Their model is based on econometric estimates. It is restricted to only four commodities (energy, food, clothing and other goods), as well as female and male leisure. They did not examine non-linear Engel curves, but examined non-parallel Engel curves across households. The heterogeneity of Engel curves led to some departure from strict linearity. Even that slight non-linearity of Engel curves caused a perceptible differentiation in optimal tax rates. They also found that if child benefits are set at a sub-optimal level then optimal commodity tax rates will be differentiated.

Revesz (1997) describes a fairly extensive computational model of optimal commodity taxation, but unfortunately that paper has been largely ignored in the literature. Revesz (1997) provides the foundation for the model discussed in the present study.

A recent computational study by Bastani, Blomquist and Pirttila (2013) examines the effect of strong leisure substitutes, such as child-care and aged-care services on optimal commodity tax rates. They examine the role of tagging, optimal taxes-subsidies and changes in
labour supply using a Stiglitz (1982) type self-selection model, involving two composite goods plus child care and leisure and four population groups – low and high wage earners, parents and non-parents. They find that provided child care is not fully paid by the government, progressive taxation of commodities is justified.

It should be noted that in contrast to commodity taxation, there is no shortage of computational research on the optimal income tax. The Mirrlees (1971) non-linear income tax model was published together with numerical results, and various extensions of the model have been subject later to further computational studies. The linear income tax model was examined through numerical studies by Stern (1976) and others. Needless to say, the extensive numerical research on optimal income taxation and the paucity of commensurate research on commodity taxation has created a rather unbalanced situation in the theory of optimal taxation.

2.3. Some recent policy related studies

A number of recent policy studies examined possible reforms to improve the operation of tax systems. Here we shall review briefly five such studies: Mirrlees et al. (2011), Arnold et al. (2011), OECD (2012) European Commission (2013) and IMF (2014). The emphasis will be on what these studies say in relation to differentiated commodity taxation. With the exception of Mirrlees et al. (2011), this is not a central issue in these reports, nonetheless, they have something to say on this subject.

We start the discussion with the Mirrlees et al. (2011) review of the UK tax system. Among other things, this report recommended to abolish all VAT exemptions and reduced rates that were introduced because of distributional considerations and replace them by the standard rate. The rationale is that the revenue raised by increasing tax rates on necessities could be used more effectively to provide support to the needy through reduced income taxes or by increasing welfare benefits.

It is not easy to pass a judgment on this recommendation. To start with, let us note that reducing income tax rates is not a strong option to assist low income households in the UK, because most of them would be on zero or low marginal income tax rates. For that reason, we shall focus here mainly on compensation through welfare payments, whereas Mirrlees et al. (2011) put more emphasis on compensation to the needy through lower income tax. Welfare payments include pensions, child benefits, unemployment benefits, disability benefits and in-work credits for low wage earners. By using these selective support payments, it is indeed conceivable, although far from certain, that more effective support could be provided to the needy through zero or reduced VAT rates. Much depends on the administrative effectiveness of expanding these income support schemes. With the possible exception of old age pensions and child benefits, all these support schemes are fairly demanding in terms of monitoring and administration and are vulnerable to false reporting and other misuse. In addition, empirical studies and theoretical considerations (see Revesz (1989)) suggest
that unemployment and disability benefits can cause significant work disincentives. Hence the substitution of lower VAT rates by selective welfare payments is not without its problems. Moreover, given price increases that will affect mainly higher income groups, the overall effect on aggregate welfare is not clear. Nonetheless, there is no denying that such a reform has the potential to be welfare improving. That does not contradict the arguments presented in this paper in favour of progressive indirect taxation. VAT in the UK (as well as in other countries) is far from being a well-developed progressive indirect taxation system (see section 4.3). Even in VAT using countries where some progressivity has been retained (such as the UK), items subject to zero or reduced VAT rates cover only a minor share of the value of goods and services sold on the market (see European Commission (2003)). To eliminate the minor distributional component of such a system and replace it by targeted income support to the needy, may lead to welfare improvement under specific circumstances. Yet, this takes a rather narrow view on possible reforms. Had the Mirrlees Review posed the question whether it would not be advisable to replace high marginal income tax rates by higher taxes on luxury goods and housing, then the Review would have covered more comprehensively possible reforms in the indirect taxation area. But no such question was raised.

The four other policy studies are all concerned with possible changes in the tax mix in order to promote “growth-friendly taxation”. All of them recommend replacing at least part of capital and labour taxes by higher taxes on consumption and real estate. European Commission (2013) and IMF (2014) also mention inheritance and gift duties as possible sources for increased revenue, although the importance of these taxes has diminished markedly in recent decades. These studies also recommend to rely more on environmental taxes. The rationale for these recommendations is based on empirical observations that such changes in the tax mix are beneficial for economic development. In particular, Arnold et al. (2011) presents econometric evidence, using a longitudinal database of OECD countries, to show that when tax revenue is held constant, consumption and recurrent property taxes have a positive effect on long-term economic growth, whereas the effect of corporate and personal income taxes is negative. In a world of mobile capital, corporate taxes seem to have the worst effects on growth. Generally, internationally mobile tax bases (capital and skilled labour), as well as tax havens, have become an increasingly serious problem in recent decades. In order not to get too far away from the main theme, we shall not enter into the various reasons for the empirically observed effects on growth, with one exception.

All the studies agree that as much as possible, labour taxes should be replaced by consumption taxes. The rationale for these tax mix changes is partly based on the disincentive effects of higher marginal income tax rates. No similar disincentives are mentioned in regard to consumption taxes. By implication, these studies perceive income taxation as more detrimental to work incentives than indirect taxation. With the exception of my earlier publications (Revesz (1986, 1997)), I did not see a similar argument appearing in the optimal taxation literature. The less detrimental effect of progressive indirect taxation on labour supply is the main theoretical reason for its advantage over direct taxation, as will be discussed in section 4.4. Only Arnold et al. (2011) provide some explanation why consumption taxes have a less negative effect on labour supply than income taxes. They reason that since VAT
is nearly a flat rate tax, whereas marginal income tax rates are rising with income, income taxation causes more work disincentives. While this may be a valid partial explanation, as we shall see in section 4.4 there are also other, perhaps stronger, explanations for the different incentive effects.

With the exception of OECD (2012) all the “growth friendly” taxation studies recommend the replacement of zero or reduced VAT rates by the standard rate (so called “base broadening”). The reasoning is similar to that in Mirrlees et al. (2011) and we shall not discuss it any further. It is interesting to note that both Mirrlees et al. (2011) and IMF (2014) accept the proposition that there is scope for using progressive indirect taxation in developing countries, where administrative capacity is weak and direct transfer payments are insignificant or non-existent. This idea was raised in the optimal taxation literature in studies dealing with zero or sub-optimal demogранts.

3. The mathematical framework of the segmented LES model

The basic structure of the computational study discussed in this paper follows the specifications used in the commodity taxation combined with linear income tax models of Deaton (1979) and Atkinson and Stiglitz (1980). The uniform tax solution is used as the benchmark for investigating departures from uniformity under non-linear Engel curves demand conditions, or as a result of the inclusion of other factors into the model.

The government’s problem is to maximise the social welfare function:

\[ U = \sum_h a_h u_h (q_h) = \sum_h a_h u_h (p, W_h, y_h) \] (1)

where \( u_h \) is the direct or indirect utility function of taxpayer \( h \) and \( a_h \) represents politically determined utility weights. \( W_h \) is the gross wage rate (or ability level) of taxpayer \( h \) and \( y_h \) is his/her lump-sum income. Maximisation is carried out subject to the revenue constraint

\[ \sum_h \sum_i t_i \hat{p}_i q_{ih} - Hb - R_0 = 0 \] (2a)

and the production possibilities constraint

\[ \sum_h W_h \ell_h - \sum_h \sum_i \hat{p}_i q_{ih} - R_0 = 0 \] (2b)

where \( q_i \) are commodities, \( \hat{p}_i \) producer prices, \( t_i \) commodity tax rates, \( R_0 \) is fixed public goods expenditure requirement, \( H \) the total number of taxpayers and \( b \) is a uniform lump-sum grant per taxpayer (called the demogrant). Producer prices are fixed. Setting \( \hat{p}_i \) to the numeraire value one, consumer prices are given as: \( p_i = 1 + t_i \).

At the start the model is confined only to commodity taxation - income taxation will be introduced later. Demographic issues, such as household composition are ignored in the
model. The demogrant is the same for all taxpayers. Further, it is assumed that taxpayers differ in their earning capacity, represented by the gross wage rate $W$. Assuming all income comes from wages, income ($m$) is the same as output and is given by $W\ell$, where $\ell$ is labour supply. Gross income in the post-tax situation after receiving the demogrant ($b$) is: $\hat{m} = W\ell + b$. For the time being, lump-sum income ‘$y$’ equals ‘$b$’. Various additions to ‘$y$’ will be introduced later. Preferences are assumed to be weakly separable, which means that utility $u = u(v(q), L)$, where $q$ is the vector of commodities, $L$ is leisure and $v$ is a sub-utility function of commodities.

The model employs Linear Expenditure System (LES) utility. The LES utility function is defined as:\(^7\)

$$ u = \sum \beta_i \log (q_i - \alpha_i) $$

The corresponding demand functions for commodities are:

$$ q_i = \alpha_i + \frac{\beta_i}{p_i}(Zw + y - \sum_j p_j \alpha_j) $$

Labour supply is given as:

$$ \ell = Z - q_L = Z(1 - \beta_L) - \alpha_L - \frac{\beta_L}{w}(y - \sum_j p_j \alpha_j) $$

where $w$ is net after-income-tax wage rate and $y$ is net after-income-tax lump-sum income. $Z$ represents the total time available and $q_L$ is leisure. Given that income tax is not yet present, $w = W$ and $y = b$.

The only constraint on the parameters is that $\sum_i \beta_i = 1$. The $\alpha_i$ can be positive or negative, but with negative $\alpha_i$ care must be taken to ensure that $q_i$ will not turn out to be negative at low income levels. By virtue of being an additive utility function, LES satisfies weak separability between commodities and leisure. Additivity also implies that LES is globally quasi-concave\(^8\).

A unique feature of the model is the incorporation of a segmented utility function in order to obtain non-linear Engel curves. The segmentation is defined as follows. There are 15 taxpayers in the model. It is assumed that the eight lower $W$ taxpayers consume only 9 goods (the necessities). The higher $W$ seven taxpayers consume 18 goods, including the 9 necessities plus 9 luxuries. In order to obtain non-linear Engel curves, the $\beta_i$ parameters of the two groups must be different. Obviously, we are dealing here with non-identical preferences\(^9\). Appendix 3 explains how the utility parameters of the two groups are set. It also explains the definition of the inequality aversion rate and its incorporation into the social welfare function weights. The inequality aversion rate ranges from 0 to 1. One represents a high level of inequality aversion corresponding to logarithmic utility, while zero represents the absence of distributional concerns. A few other technical details are also relegated to appendix 3.
In regard to the choice of LES as the utility function in the model, let me note that I am not aware of any direct utility function apart from LES, CES and the homogeneous Cobb-Douglas function that yields explicit demand formulas. Neither CES nor Cobb-Douglas seems suitable for a multi-product flexible utility model, which leaves LES as the best choice. Needless to explain, the availability of explicit demand formulas makes it much easier to find the numerical optima. Segmented LES is well suited to examine a wide range of non-linear Engel curves configurations 10.

In the present model the search for the optimum is based exclusively on a computational approach, without any formulas apart from the utility and demand functions. The value of the utility function described in (3) is evaluated at each step, using the demand functions defined in (4) and (5). At each step (k) of the calculations a new demogrant (b) is found using constraint (2a). Successive approximations to optimal tax rates are carried out using the gradient equation:

\[ t_i(k) = t_i(k-1) + \frac{c}{q_i} \frac{\Delta U_i(k)}{\Delta t_i} \]  \hspace{1cm} (6)

where \( U \) is total utility and \( C \) is a scaling factor determined at the first iteration, \( \Delta U_i(k) = U(t_i(k-1) + 0.03) - U(t_i(k-1)) \) and \( \Delta t_i = 0.03 \). Notice that the gradient represents a total differentiation with respect to \( t_i \), including its effect on the demogrant ‘b’. In the uniform tax calculations the program carries out 60 iterations. In the non-uniform calculations 25 separate iterations are carried out for 18 goods. The number of iterations are fixed - no convergence condition has been set when to stop the calculations. In practice, by the 25th iteration the difference between \( t_i(25) \) and \( t_i(24) \) is almost always below 0.01. Mathematical reasoning indicates that given quasi-concave preferences and convex budget constraint, the gradient-based iterations must converge to a unique optimum 11.

Given that one of the main objectives of the study is to compare results of uniform versus differentiated taxation, the program calculates total utility and total output under uniform commodity taxation and under conditions when commodity tax rates are individually optimised, and reports the differences in these totals. The difference in total output is:

\[ \Delta M = \sum_h W_h \Delta \ell_h \]  \hspace{1cm} (7)

The difference in aggregate welfare is converted from utility to monetary values using the Lagrangian multiplier:

\[ \Delta U = \sum_h \Delta u / \lambda. \]  \hspace{1cm} (8)

The difference is reported as a percentage of total output. The Lagrangian is evaluated using the definition:

\[ \lambda = \frac{U(R_0 + \Delta R_0) - U(R_0)}{\Delta R_0} \]
where $R_0$ is expenditure on public goods. Changes in $R_0$ can be used to estimate the marginal utility of public expenditure.

In order to explain better the numerical results, I developed an approximate formula for optimal commodity tax rates titled the modified inverse elasticity rule. While this formula was not used in the iterative calculations based on (6), it provides a convenient analytical tool to interpret some of the numerical results. It can be extended further to accommodate factors such as administration, compliance, evasion, externalities and leisure complements and substitutes. The mathematical development of the modified inverse elasticity rule is described in appendix 2.

The relevant formula for the model without income tax is (A.11 in appendix 2):

$$
t_i = \frac{1 + t_i - \bar{u}_{mi}(1 + t_i + \bar{c}_i (t_i - t_{iA}) + \bar{e}_{it} \bar{T}_i')}{\bar{e}_i(u_0)}
$$

(9)

The term

$$
\bar{u}_{mi} = \frac{\partial u}{\partial b} = \frac{\partial u_{mi}}{\partial \bar{a}_i} / \partial y
$$

(10)

is called the marginal utility ratio of product $i$. $\partial u / \partial y$ is the average marginal utility of product $i$, that is, the average marginal utility of income of the consumers of product $i$, weighted according to their consumption shares (see A.13 in appendix 2). $\partial u / \partial b$ represents the utility value of the demogrant. Given the same demogrant for all taxpayers, it is the simple average of the marginal utilities of income (see A.2 in appendix 2). $t_{iA}$ is the average commodity tax rate on products other than $i$. $\bar{c}_i$ is the consumption weighted average price elasticity of good $i$, and $\bar{e}_i(u_0)$ is the average compensated elasticity of demand. $\bar{e}_{it}$ is the labour supply elasticity term (see A.5a and A.10 in appendix 2) and $\bar{T}_i'$ is the average marginal indirect tax rate of the consumers of product $i$ (defined in A.5b).

While the modified inverse elasticity rule is not particularly accurate (as illustrated in Table A.1 in appendix 2), it has some advantages over traditional first order conditions. It provides an explicit formula for optimal tax rates. In addition, the unspecified Lagrangian has been replaced by a ratio of marginal utilities $\bar{u}_{mi}$ in (9), which will be of considerable help in some explanations. A similar approximation for optimal tax rates has been worked out by Sandmo (1975), using an entirely different mathematical approach. Sandmo derived his approximation under the assumption of demand separability, i.e.: $\partial q_j / \partial p_k = 0$ for $j \neq k$. No such restrictive assumption was used in developing the modified inverse elasticity rule and the two approximations are not the same (see A.14 in appendix 2).

When income tax is included, the modified inverse elasticity rule is changed. It will be (A.15 in appendix 2):

$$
t_i = \frac{1 + t_i - \bar{u}_{mi}(1 + t_i + \bar{c}_i (t_i - t_{iA}) + \bar{e}_{it} (\bar{T}_m + \bar{T}_i'))}{\bar{e}_i(u_0)} - \bar{T}_i (1 - \gamma)
$$

(11)
where $T_{mi}$ is the average marginal income tax rate of the consumers of product i, $\bar{T}_i$ is the corresponding average income tax rate and $\gamma$ is a general adjustment factor that ensures the equalisation of actual and predicted total tax revenue. $\gamma$ is usually below plus or minus 0.15, indicating that the optimal combined tax rate ($t_i + \bar{T}_i$) in the presence of income tax is not much different from the optimal tax rate without it.

4. The redistributive model

4.1. The model without income tax

In this section we shall examine the basic redistributive model. Various “real life” complexities will be introduced later. The parameters used in all the scenarios reported in this paper are arbitrary rather than empirical. This approach is perhaps acceptable given that the paper deals with the indirect tax uniformity debate and other broad issues, rather than with the determination of actual tax rates. Moreover, to a large extent the conclusions in this paper are based on analytical arguments and are unrelated to the parameters used in the numerical calculations. The numerical examples are used to illustrate and substantiate the analytical points.

Table 1 shows numerical results from the model where income tax is absent. In the light of the discussion on zero homogenous utility and demand (see section C.3 in appendix 3), this represents a particular form of linear income tax combined with commodity taxation. All the calculations were done subject to a fixed public goods expenditure requirement of 10% of total output in the pre-tax situation. In the first scenario utilities are defined by LES without any transformation. Utility transformation, defined by eq. (C.2) in appendix 3, occurs in the second scenario, where the inequality aversion rate is set to 0.3. Looking at the tax rates, the most striking feature is that optimal tax rates are highly differentiated and progressive. This is even true when the inequality aversion rate is reduced from 1 to 0.3. It is not difficult to see from the figures that optimal tax rates are positively related to the marginal utility ratios of goods. Given the definition of these ratios in (10), that means that optimal tax rates are negatively related to the average social marginal utilities of products. Notice that apart from commodity 3, all other marginal utility ratios are above one. The reason is that the mean of marginal utilities, where each marginal utility of income is counted the same (that is, $\partial u / \partial b$), will usually be larger than the average marginal utility of goods, $\partial u / \partial y_i$, because in the later the low marginal utilities of incomes associated with higher consumption have a larger weight. That is true even for necessities, where higher income earners tend to consume more, unless the Engel curve is backward sloping (inferior good). With commodity 3 there is a drop in demand at the border between the two consumer groups, which results in low consumption at high income levels (inferior good), leading to the marginal utility ratio of the product to fall below one. The finding from the simulations that apart from inferior goods the marginal utility ratio is above one, will assume some importance in later discussion.

It is not difficult to see from the figures that optimal tax rates are negatively related to compensated demand elasticities, particularly among luxuries, in line with the modified in-
verse elasticity rule. The nearly inverse relationship between tax rates and compensated demand elasticities introduces another source of dispersion in the results besides inequality aversion. In the 0.3 inequality aversion rate scenario the dispersion of optimal tax rates is reduced, because generally tax rates are much smaller. The change in welfare terms compared with the uniform tax solution is defined in (8) and the change in output in (7). Notice that the gains in welfare and output over the uniform solution are much larger in the high inequality aversion scenario. It appears that these gains are positively correlated with the dispersion of optimal tax rates. Gains in welfare and/or output of differentiated taxation over the uniform solution in excess of 3% of total output appear also in a number of more complex scenarios that will be discussed later. Another point to note is that the average tax rate tends to be slightly lower under the non-uniform than under the uniform solution.

### Table 1

**Optimal Commodity Tax Rates Without Income Tax**

<table>
<thead>
<tr>
<th>Inequality aversion = 1</th>
<th>Inequality aversion = 0.3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>marginal utility Ratios</td>
</tr>
<tr>
<td>1</td>
<td>1.16</td>
</tr>
<tr>
<td>2</td>
<td>1.29</td>
</tr>
<tr>
<td>3</td>
<td>0.92</td>
</tr>
<tr>
<td>4</td>
<td>1.43</td>
</tr>
<tr>
<td>5</td>
<td>1.13</td>
</tr>
<tr>
<td>6</td>
<td>1.39</td>
</tr>
<tr>
<td>7</td>
<td>1.09</td>
</tr>
<tr>
<td>8</td>
<td>1.23</td>
</tr>
<tr>
<td>9</td>
<td>1.08</td>
</tr>
<tr>
<td>10</td>
<td>2.49</td>
</tr>
<tr>
<td>11</td>
<td>2.28</td>
</tr>
<tr>
<td>12</td>
<td>2.44</td>
</tr>
<tr>
<td>13</td>
<td>2.46</td>
</tr>
<tr>
<td>14</td>
<td>2.57</td>
</tr>
<tr>
<td>15</td>
<td>2.34</td>
</tr>
<tr>
<td>16</td>
<td>2.32</td>
</tr>
<tr>
<td>17</td>
<td>2.58</td>
</tr>
<tr>
<td>18</td>
<td>2.42</td>
</tr>
</tbody>
</table>

- **Average tax rate on the 9 necessities**: 0.70 | 0.27
- **Average tax rate on the 9 luxuries**: 1.83 | 0.41
- **% average tax from expenditure – non-uniform**: 48.4 | 23.8
- **% average tax from expenditure – uniform**: 50.8 | 25.7
- **% demogrant to average labour income**: 69 | 17
- **% change in welfare terms compared to uniform solution**: 3.5 | 0.2
- **% change in total output compared to uniform solution**: 2.1 | 1.9

* Fixed expenditure on public goods = 10% of total income under zero tax.
While this numerical study breaks some new grounds, not all the findings presented here are entirely original. Diamond (1975) analytical study, using a production possibilities frontier model rather than variable labour supply, indicated that in the absence of income tax but in the presence of a demogrant optimal commodity taxes will tend to be differentiated and progressive. As explained in section C.3 in appendix 3, in a basic redistributive model (no administration and evasion) zero income tax yields the same demand and utility outcomes as a range of linear income tax functions combined with proportionally adjusted commodity tax rates. Therefore, the conclusion of Diamond (1975) effectively applies to all linear income tax functions.

On a similar vein, Atkinson and Stiglitz (1980) concluded that an optimal commodity tax system, when the income tax is linear progressive, will not generally be uniform under weak separability. Among earlier computational studies, I think only Ebrahimi and Heady (1988) contains all the essential ingredients of a fully-fledged redistributive model, including the presence of lump-sum support payments and variable labour supply. The results presented by Ebrahimi and Heady (1988) indicate that under weakly separable utility optimal commodity tax rates will be differentiated, provided Engel curves are not parallel across households. The heterogeneity of Engel curves leads to some departure from strict linearity. Even that slight non-linearity of Engel curves caused a perceptible differentiation in optimal tax rates.

4.2. The model with non-linear income tax

So far we discussed the structure of optimal commodity tax rates under a linear income tax. At this stage the question arises to what extent the pattern of results would be different under non-linear and non-optimal income taxes. We already know from the Atkinson-Stiglitz (1976) theorem that when utility is weakly separable then the optimal non-linear income tax [Mirrles (1971)] should be combined with zero or uniform commodity tax rates. To examine what happens if income tax is not optimal, the computational model contains a piecewise linear income tax schedule, defined by five marginal income tax rates over five income intervals. The five income intervals cover about three taxpayers each, but this number can vary in each bracket, depending on variations in taxable income as a result of changing labour supply. Obviously the piecewise linear arrangement can approximate a wide range of non-linear income tax schedules.

The mathematical framework is changed after the introduction of income tax. Following the “virtual budget” framework outlined in Revesz (1986, 1997), Roberts (2000) and Saez (2001), the earning parameters used in (4) and (5) have to be redefined.

The post-income-tax wage rate is: \[ w = W (1 - T_m) \] (12)
where \( T_m \) is the marginal income tax rate and \( W \) is the pre-income-tax wage rate (or ability level). Post-income-tax ‘virtual’ lump-sum income (\( y \)) is defined so that \( w \ell + y \) add up to disposable income (\( m \)):

\[ m = \ell W - T + b = \ell w + y \] (13)
From (12) and (13) it follows that post-income-tax “virtual” lump-sum income is:

\[ y = T_m W \ell - T + b \]  

(14)

In (14) pre-tax lump-sum income \((Y_0)\) assumed to be zero. Note, in the presence of income tax, the linearised earning parameters \(w\) and \(y\) are the dual counterparts to income and leisure in the indirect utility space. The “virtual” lump-sum income component \((y)\) depends on the curvature of the income tax function \((T)\). It will equal ‘b’ with a linear income tax. It will be above ‘b’ if marginal income tax rates are increasing and vice versa if they are decreasing.

Table 2 shows results with two arbitrarily defined income tax schedules, one progressive the other regressive. The progressive schedule is made up of five tax brackets with marginal income tax rates increasing from zero at the bottom to 40% at the top bracket, each bracket being 10% higher than the preceding one. In the regressive schedule there is a rapid climb from zero marginal tax rate at the lowest bracket to 40% at the second bracket followed by 5% decrease in each of the following three brackets. The shape of the regressive schedule resembles the shape of some of the solutions to the Mirrlees (1971) model reported in Tuomala (1984). In the following discussion, when we investigate scenarios where income tax is present, it will be always the progressive schedule described above. The regressive schedule is mainly of theoretical interest.

In three out of the four scenarios in Table 2 commodity tax rates are differentiated and progressive. The exception is progressive income taxation combined with 0.3 inequality aversion, where the solution turns out to be negative and nearly uniform commodity taxes (see Table D.1 in appendix 4). When the inequality aversion rate is one, both progressive and regressive income tax schedules yield strongly differentiated and progressive commodity tax rates and there are significant gains over the uniform tax solution.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Progressive schedule</th>
<th>Regressive schedule</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Inequality aversion=1</td>
<td>Inequality aversn= 0.3</td>
</tr>
<tr>
<td>Average tax rate on necessities</td>
<td>0.22</td>
<td>-0.16</td>
</tr>
<tr>
<td>Average tax rate on luxuries</td>
<td>0.91</td>
<td>-0.13</td>
</tr>
<tr>
<td>% demogrant over average income</td>
<td>42</td>
<td>-2</td>
</tr>
<tr>
<td>% change over uniform in welfare terms</td>
<td>2.8</td>
<td>0.4</td>
</tr>
<tr>
<td>% change over uniform in total output</td>
<td>3.0</td>
<td>0.6</td>
</tr>
</tbody>
</table>
Results that we do not publish here using the above scenarios with the 9 commodity linear Engel curves model (discussed in Revesz (1997)), suggest that with linear Engel curves demand, optimal commodity tax rates will be nearly uniform even when combined with non-linear income tax schedules. This suggests that Deaton’s (1979) theorem is almost valid for non-linear income tax schedules as well. But as explained in section C.1 in appendix 3, in a many-goods model linear Engel curves for all goods represents nearly homothetic preferences, which is not supported by empirical evidence. For econometric evidence against linear Engel curves for all goods refer to Blundell and Ray (1984).

The preliminary results in Table 2 suggest a few points. It appears that generally when Engel curves are not linear then with most exogenously given income tax schedules optimal commodity tax rates will be differentiated and progressive, even in the absence of “real life” complexities. However, there are non-linear income tax schedules which by themselves almost satisfy distributional objectives and where the associated optimal commodity tax rates are negative – in that event the solution tends to move towards uniformity. This can only happen if a non-linear income tax schedule is included in the model.

### 4.3. The two composite goods model

Value added tax (VAT) is nowadays the most widely used indirect tax system in developed countries. Therefore, it is of some interest to examine these taxes in the light of the numerical model under discussion. In many countries VAT is nearly a flat rate tax with little progressivity built into it. In the following discussion we shall focus on VAT systems where significant progressivity has been retained, such as in the UK, France, Italy, Spain, Ireland and Luxembourg [see European Commission (2014)]. VAT in these countries can be characterised in broad terms as a two tiered tax system. Most goods and services are taxed at the standard VAT rate or close to it. A minority of goods (mainly necessities) are taxed at zero or significantly reduced rates. These include: food, medicines, private medical and dental care, passenger transport, books and periodicals and in some countries also domestic fuel, social housing, rented accommodation and children clothing.
To describe these systems in modelling terms, we used a two composite goods approach, where the luxury good is taxed at the standard VAT rate and the necessity is taxed at a much reduced rate. This approach is suitable to examine also indirect tax systems other than VAT (such as sales tax), where separate tax rates are defined for very broad categories of goods and services. Surely the assumption used earlier that 18 separate tax rates can be established for 18 goods is unrealistic. All legal tax codes are based on broad definitions of product groups. Arguably, in the computer age it might be possible to administer indirect tax systems where separate tax rates are determined for individual products based on quality, using measures such as price-weight or price-volume ratios or property valuations, but that is a vision for the future far removed from current practice.

In the two composite goods model the original 18 goods have been retained, but the utility parameters of all nine necessities have been set to the same values and another set of identical values was applied to the nine luxuries (see Table C.2 in appendix 3). Hence, effectively there are only two commodities in the model.

| Table 3 |

<table>
<thead>
<tr>
<th>OPTIMAL COMMODITY TAX RATES WITH TWO COMPOSITE GOODS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Without income tax Public goods = 22% Inequality aversion rates</td>
</tr>
<tr>
<td>0.13</td>
</tr>
<tr>
<td>Average tax rate on the necessity</td>
</tr>
<tr>
<td>Average tax rate on the luxury</td>
</tr>
<tr>
<td>% demogrant to average income</td>
</tr>
<tr>
<td>% change in welfare terms compared to uniform solution</td>
</tr>
<tr>
<td>% change in total output compared to uniform solution</td>
</tr>
</tbody>
</table>

In order to understand better “progressive” VAT systems, we searched for a solution where the tax on luxuries is 23% and the tax on necessities is 2%. It was not easy to find such a solution. Usually the differences between optimal tax rates in the two-goods model are much larger than those between these target figures. But eventually we found a not too realistic combination that yielded these figures as optimal outcomes. This combination corresponds to no income tax, an inequality aversion rate of 0.12 and fixed expenditure on public goods amounting to 22% of total pre-tax output. Under these specifications, given 2% and 23% commodity tax rates, the demogrant turned out to be minus 10% of average income. The first column in table 3 shows the optimal solution under these specifications, the other columns show the solution under higher inequality aversion rates and with progressive income tax.

The figures in table 3 illustrate a few points. First, in the present model even the more “progressive” VAT systems can be justified only under a very low inequality aversion rate,
27 A Numerical Model of Optimal Differentiated Indirect Taxation

in fact lower than those used elsewhere in this paper. But it should be noted that the VAT estimates used here exclude other indirect taxes, such as excises and property taxes. Under higher inequality aversion rates (0.3 and 1) the dispersion of optimal tax rates in the two-goods model is considerably larger than the dispersion of tax rates in the 18 commodities model, reported in Tables 1 and 2. My original expectation was that broad categorisation of taxable items will reduce the dispersion of optimal tax rates. The optimisation results point in the opposite direction. But some caveats should be noted. By construction, segmented LES utility draws a sharp distinction between necessities and luxuries, which enables a clear-cut partitioning in the model between these categories. The broad legal definitions of products for taxation purposes, which ignore qualitative differences between products within the same group, do not allow for similar well defined distinctions between luxuries and necessities. Hence in practice, separate tax rates for broad product groups will not offer as much advantage over uniform taxation as suggested in Table 3.

Given that “progressive” VAT is not much different from a flat rate tax, according to the present model transforming it into uniform taxation will reduce welfare by only 0.2%. But some loss is still there, because of the advantage of differentiated VAT in relation to work incentives, as indicated by the 0.8% gain in total output over the uniform solution.

4.4. Analytical arguments in favour of progressive indirect taxation

At this stage the question arises, why do we get progressive commodity taxes in the optimal solutions instead of uniform taxes? As we shall soon discover, to a large extent the answer is connected with labour supply incentives, even when preferences are weakly separable between commodities and leisure. We start the discussion with the linearised after-income-tax earning parameter defined in (12) and (14). The incentive effects of $w$ and $y$ can be seen from the utility compensated labour supply derivative.

$$\frac{\partial \ell(u_0)}{\partial w} = \frac{\partial \ell}{\partial w} + \frac{\partial \ell}{\partial y} \frac{\partial y(u_0)}{\partial w} = \frac{\partial \ell}{\partial w} - \frac{\partial \ell}{\partial y} \ell$$

(15)

Because lump-sum income discourages work, $\frac{\partial \ell}{\partial y}$ is always negative. The compensated derivative for labour must be definite positive, because labour is a negative good. From (15), this implies that utility compensated variations that increase labour supply require higher $w$ and lower $y$. But higher marginal income tax rates move the earning parameters exactly in the opposite direction. Looking at (12) and (14), higher $T_m$ will reduce $w$ and increase $y$, which means that increasing marginal income tax will discourage work. On the other hand, with commodity taxation, regardless whether progressive or regressive, both $w$ and $y$ will change in the same proportion, determined by the change in the perfect consumer price index. Unlike with progressive income taxation, there is no shift that reduces $w$ and increases $y$. That means that replacing progressive commodity taxation by an equal revenue generating progressive income tax will reduce labour supply, because following such a change $w$ will decrease and $y$ will increase. This had been confirmed by numerical results from an earlier version of the present computational model reported in Revesz (1997).
Revesz (1997) shows that a changeover from progressive indirect taxation to an equal revenue generating income tax schedule, will cause a large decrease in total output and welfare. These results, based on standard consumer theory, confirm the implicit assumption in the “growth friendly taxation” reports (see section 2.3), that replacing income tax by consumption taxes will improve labour incentives.

But there is more to it. The tax burden due to income taxation is more visible than that due to commodity taxation. On behavioural grounds, one would expect that the response to a more visible disincentive will be stronger than the response to a less visible one. This behavioural response also supports the view that progressive direct taxation will have a more detrimental effect on incentives than progressive indirect taxation. The fact that with progressive indirect taxation substantial redistribution can be carried out with little detrimental effect on labour supply, is the prime reason why the optimal indirect tax solutions reported in this paper tend to be progressive rather than uniform or regressive.

Another point in favour of progressive indirect taxation is connected with the modelling of income support. In almost all optimal taxation models it is assumed that distributional objectives are solved by optimising income and commodity taxes combined with a uniform demogrant. But a uniform demogrant (or negative tax) is seldom applied in practice. Support agencies around the world attempt to economise on limited funds by providing targeted support based on household characteristics such as: age, household composition, disabilities, health, employment, participation in workfare or trainfare programs and other observable or semi-observable characteristics [see Akerlof (1978)]. Given that support agencies have access to more information on recipients than tax authorities on taxpayers, it is theoretically possible to get closer to providing support based on abilities than to tax according to abilities. As explained in Mirrlees (1971), taxing or subsidising on the basis of abilities is the first-best optimal solution to the distributional problem, but in practice abilities (ie. fixed endowments or potential) cannot be observed. In any event, provided income support is properly organised and monitored, the provision of targeted welfare payments stands closer to the ideal of ability-based transfers than either income or commodity taxation. The effectiveness of differentiated and targeted support payments to the needy is of central importance in actual tax-transfer systems. The assumption of a uniform demogrant in redistributive models can be justified only on grounds of modelling simplicity.

There have been some attempts in the optimal taxation literature to incorporate differentiated support payments into optimal tax models. Deaton and Stern (1986) examined a model where Engel curves are linear and have their intercepts determined by a linear function of observable household characteristics. Deaton and Stern demonstrate that uniform commodity taxation is optimal only if the lump-sum grants for each demographic group are an optimal function of observable household characteristics. The computational study of Ebrahim and Heady (1988) explored this model with numerical examples, focussing on child benefits. They found that if child benefits are not adequately differentiated according to the number and age of children in the household, then optimal commodity tax rates will not be uniform. Generalising from this result, it can be said that since most income support systems
have to screen partly on the basis of semi-observable characteristics, they will fall short of
the ideal of providing differentiated lump-sum grants based on actual family subsistence
needs and ability to work. In this situation, progressive indirect taxation can be used to com­
pensate for some of the inherent shortcomings in the income support system. These benefits
from progressive indirect taxation are not reflected in the present numerical model, which
assumes a uniform demogrant.

Finally, we come to the subject of sub-optimal demogrants, which may also provide a
rationale for progressive indirect taxation, as mentioned in section 2.2. However, there are a
few problems with this approach. There is an empirical weakness in zero demogrant mod­
els, because even in developing countries where direct support payments are absent, there is
some government support provided to the needy through public education and other in-kind
transfers, such as rudimentary health care and agricultural advisory services. In a simple re­
distributive model, these in-kind transfers can be represented by a demogrant amounting to
the average cost of these services per person. Such an extension of the model does not inval­
icate the idea that progressive indirect taxation could be used to alleviate distributional prob­
lems. Yet, for that purpose there is no need to assume a zero demogrant but only a sub­
optimal demogrant, as a result of insufficient government revenue, due to limited capacity in
tax administration and the proliferation of the “informal economy”. As noted earlier, the
models of Ray (1989) and Asano et al. (2004) assume exogenously given sub-optimal de­
mogrant rather than zero demogrant.

Numerical simulations with the present model, presented in table D.2 in appendix 4,
suggest that when an exogenously given demogrant is set below the endogenously deter­
mined optimal level, then optimal commodity tax rates will be more progressive the further
away the fixed demogrant is located from the optimum. Progressivity is measured by the
ratio of the average tax on luxuries over the average tax of necessities. When the fixed de­
mogrant is set to zero, there is a large increase in indirect tax progressivity compared with
the endogenously determined optimal solution.

Yet despite many modelling possibilities, there is a fundamental problem with this ap­
proach. To what extent a given demogrant is sub-optimal cannot be quantified in an objec­
tive manner. It all depends on the eyes of the beholder, or in mathematical terms, on the util­
ity function and inequality aversion rate employed in the model. Moreover, the situation in
developing countries can be represented also in a model where the demogrant is determined endogenously, by taking out income tax and incorporating into the model high evasion propensities for many products, high administrative costs and low inequality aversion rate. Experiments with the present model along these lines yielded relatively high tax rates on less evasion prone items and marked progressivity in tax rates. Hence, in order to justify progres­
sive indirect taxation in developing countries, there is no need to assume a sub-optimal de­
mogrant. Nonetheless, in many countries where a large percentage of the population are
working in the “informal economy”, it is impossible to generate sufficient revenue for redis­
tribution, in line with the preferences of the median voter or the ruling elite [De Freitas
(2012)]. In order to model such a situation, the sub-optimal demogrant approach has its mer-
its. If one accepts it as a valid argument, then it can be added to a number of other arguments justifying progressive indirect taxation.

4.5. The Laroque-Kaplow proposition

In this section we take a critical look on an extension of the uniform commodity taxation argument that appeared in the last decade. A proposition formulated by Laroque (2005) and Kaplow (2006) asserts that given any income tax function combined with differentiated commodity tax rates, and given identical preferences and weakly separable utility between commodities and leisure, it is possible to carry out a reform involving the replacement of non-uniform commodity taxes by appropriately adjusting the income tax, so that the utility level of all taxpayers will be maintained or improved. The Laroque-Kaplow (LK) proposition extends the Atkinson-Stiglitz theorem to non-optimal income tax functions as well.

At first sight the LK proposition seems to contradict our numerical results, indicating the optimality of differentiated and progressive commodity tax rates in the presence of a linear income tax and a number of non-linear income tax schedules examined in the simulations. But actually this is not the case. What has been examined in this study is the pattern of optimal commodity tax rates in the presence of an exogenously given (fixed) income tax schedule, be it linear or otherwise. A simultaneous change in income and commodity taxation and the issue whether subject to an appropriate change in the income tax schedule optimal commodity tax rates should be uniform or otherwise was not raised in our discussion. Moreover, the LK proposition is based on the assumption of identical preferences. The segmented utility framework employed here assumes that preferences are not the same for all households. Given two different models in terms of perspectives and specifications, it can be said that on purely logical grounds the numerical results presented here neither support nor refute the LK proposition.

At this stage, we could leave the discussion on the LK proposition with this inconclusive statement, however, because this proposition has been invoked in the broader debate about uniform versus differentiated commodity taxation, a few more comments might be appropriate. First, the LK proposition is based on the same strong simplifying assumptions and unrealistic abstract modelling as the Atkinson-Stiglitz theorem, therefore all the objections and qualifications that were raised against that theorem (outlined in section 2.1), apply to the LK proposition as well. But there are a few more.

Boadway (2010) points out that the LK proposition may be valid only if the income tax schedule can be adjusted in an appropriate manner to compensate for changes in commodity taxation. If because of political-administrative constraints an appropriate adjustment to the income tax schedule is not carried out, then such a reform may turn out to be welfare reducing. Given that the LK proposition does not describe the shape of the income tax function that combined with zero or uniform commodity tax rates will yield a better outcome, the doubts raised by Boadway seem pertinent. Boadway (2010) also questions the “two-step de-
composition” approach for separating distributional and efficiency effects, used by Kaplow for policy analysis.

In common with the Atkinson-Stiglitz theorem, the LK proposition suggests that redistributive problems should be solved by appropriate design of the income tax schedule and the demogrant, without having to use indirect taxation for that purpose. Yet the “growth friendly taxation” policy studies mentioned earlier, do not view income taxation in a favourable light, and call for its partial replacement by consumption and property taxes. Apart from labour and saving disincentives involved with income taxation, there is also the problem of high vulnerability to tax evasion and avoidance [see Boadway (2012), DeFreitas (2012)]. Hence, from a practical point of view, income taxation does not appear to be the best choice to solve all distributional problems, as suggested by the LK proposition. The interested reader can find more detailed comments on the LK proposition in Revesz (2014).

4.6. Subsidies for education and health

So far, when we referred to commodities we tacitly assumed that education and health are excluded from this category. But in economic terms education and health are services like all other services, and are unique only in the sense of being almost fully subsidised by the government. Even without government support, there would be considerable demand for these services through the market. The economic rationale for government subsidisation of these services is based on the same principles that justify differentiated commodity taxes/subsidies in general. These are:

- Distributional concerns
- Paternalistic concerns
- Externalities – there are positive externalities associated with education through the unpaid diffusion of knowledge to others. In health positive externalities arise through the prevention of infectious diseases.
- Non-separable utility between commodities and leisure. Education is a strong leisure substitute because it raises the quality of labour supply. Health is a leisure substitute because it increases the size of the working population.

In both education and health, distributional concerns are one of the main motives for public funding [see Hindriks and Myles (2006)]. In view of these facts, those who argue that there is no need for differentiated indirect taxation for distributional purposes, if they wanted to maintain logical consistency, should also advocate the reduction of government expenditure on education and health, in order to use these funds to reduce income taxes and/or increase welfare payments. I have not seen such a suggestion in the optimal taxation literature, which indicates that perhaps the proponents of uniform commodity taxation are not aware to the full implications of their position.
5. Other factors justifying differentiated indirect taxation

In this section we shall examine a number of other factors that can justify differentiated commodity taxation and are not directly related to distributional concerns. These include: compliance costs, government administration costs, tax evasion, externalities, paternalistic concerns and non-separable utility between commodities and leisure. To assess analytically the impact of these factors, a common methodology is used, based on the modified inverse elasticity rule. We assume that each factor can influence optimal tax rates predicted by the modified inverse elasticity rule either through changes in the lump-sum incomes of consumers or changes in tax revenue or both. We examine how the formula will change following the introduction of a factor. Finally, we deduct from the revised formula the original formula to obtain an analytical expression for the net change due to the factor concerned. In following this estimation method of net impact, we ignore possible changes in endogenous variables such as demand elasticities, labour supply elasticities, marginal utilities, other tax rates and the demogrant. Despite the strong assumptions adopted, the numerical results suggest that these analytical approximations of net impact work reasonably well. In each of the following sections we shall compare the analytical approximation with numerical results from iterative calculations.

5.1. Compliance costs

We start the discussion with tax compliance costs. These are defined as administrative and other related costs incurred directly by consumers. They are dead-weight costs \( C_i \) that reduce real output and are a constant portion \( c_i \) of the tax revenue from good \( i \). Symbolically:

\[
C_i = c_i q_i t_i
\]

Taking the indirect utility definition of the social welfare function in (1), then the dead-weight costs defined in (16) can be incorporated into the social welfare function by subtracting them from personal lump-sum incomes ‘\( y \)’. The extended social welfare function will be:

\[
U = \sum_h a_h u_h (p, W_h, [y_{oh} - \sum_i c_i q_{ih} t_i])
\]

Using the modified inverse elasticity rule in appendix 2, we derive in eq. (A.19) the following approximation for the impact of compliance costs on optimal tax rates.

\[
\Delta t_i(c_i) \approx \frac{1 + t_i (1 + \varepsilon_i)}{\varepsilon_i(u_0)} c_i
\]

Assuming that \( \varepsilon_i \) and \( \varepsilon_i(u_0) \) are the same and equal to \(-1\) (see Tables 1 and 4), we can obtain from (18a) the cruder but simpler approximation:

\[
\Delta t_i(c_i) \approx -c_i
\]
Having obtained a theoretical approximation, we can now look in Table 4 at some numerical results. These results were obtained by taking the basic model with or without income tax and adding a compliance cost parameter of 4% to two goods, one a necessity the other a luxury. The resulting $t_i$’s are then compared with the original tax rates obtained without compliance costs, shown in Table 1 and Table D.1 in appendix 4.

<table>
<thead>
<tr>
<th>Good 5 - necessity</th>
<th>Demand elasticities</th>
<th>Good 18 - luxury</th>
<th>Demand elasticities</th>
<th>Good 2 - necessity</th>
<th>Demand elasticities</th>
<th>Good 12 - luxury</th>
<th>Demand elasticities</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>with compliance</td>
<td>without compliance</td>
<td>impact</td>
<td></td>
<td>with compliance</td>
<td>without compliance</td>
<td>impact</td>
</tr>
<tr>
<td>aversion</td>
<td></td>
<td>impact</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>With progressive income tax</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Good 5 - necessity</td>
<td>1</td>
<td>0.03</td>
<td>0.05</td>
<td>−0.02</td>
<td>−1.25</td>
<td>−1.17</td>
<td></td>
</tr>
<tr>
<td>Good 18 - luxury</td>
<td>1</td>
<td>0.80</td>
<td>0.83</td>
<td>−0.03</td>
<td>−1.17</td>
<td>−1.14</td>
<td></td>
</tr>
<tr>
<td>Good 2 - necessity</td>
<td>0.3</td>
<td>−0.13</td>
<td>−0.13</td>
<td>0.00</td>
<td>−0.81</td>
<td>−0.73</td>
<td></td>
</tr>
<tr>
<td>Good 12 - luxury</td>
<td>0.3</td>
<td>−0.15</td>
<td>−0.15</td>
<td>0.00</td>
<td>−1.11</td>
<td>−1.08</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Without income tax</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Good 5 - necessity</td>
<td>1</td>
<td>0.44</td>
<td>0.48</td>
<td>−0.04</td>
<td>−1.26</td>
<td>−1.17</td>
<td></td>
</tr>
<tr>
<td>Good 18 - luxury</td>
<td>1</td>
<td>1.66</td>
<td>1.70</td>
<td>−0.04</td>
<td>−1.17</td>
<td>−1.13</td>
<td></td>
</tr>
<tr>
<td>Good 2 - necessity</td>
<td>0.3</td>
<td>0.23</td>
<td>0.30</td>
<td>−0.07</td>
<td>−0.82</td>
<td>−0.74</td>
<td></td>
</tr>
<tr>
<td>Good 12 - luxury</td>
<td>0.3</td>
<td>0.36</td>
<td>0.40</td>
<td>−0.04</td>
<td>−1.12</td>
<td>−1.10</td>
<td></td>
</tr>
</tbody>
</table>

With negative taxes the impact of compliance costs is zero. The reason is that when taxes are negative, the program sets compliance, administration and evasion costs to zero, because it is hard to put an interpretation to these costs in the presence of subsidies. Otherwise, the results appear to be in line with what is expected according to (18). Overall, the results indicate that compliance costs will reduce optimal tax rates by a similar amount to the share of these costs from commodity taxes.

5.2. Government administration costs

An item closely related to compliance is administration (denoted $s$). In the present model we define compliance to represent costs paid by the taxpayer, while administration is paid by the government. With $s_i$ public revenue will be affected.

Define the total cost of public tax administration as: $S = \sum_h \sum_i q_{ih} t_i s_i \quad (19)$

Assuming fixed expenditure on public goods ($R_0$), the net revenue available for redistribution will be:

$$R = \sum_h \sum_i (q_{ih} t_i - q_{ih} t_i s_i) - R_0 \quad (20)$$

Total real output will be reduced by the dead-weight cost $S = \sum_h \sum_i q_{ih} t_i s_i$
The incidence of administrative costs on consumers or government has an effect on the optimal tax rate. The estimated impact of government administration according to the modified inverse elasticity rule (A.23 in appendix 2)

\[
\Delta t_i(s_i) \approx \frac{s_i \bar{r}_{mi} [(1 + t_i (1 + \varepsilon))]}{\varepsilon(u_0)}
\]

(21a)

Again, assuming that \(\varepsilon_i\) and \(\varepsilon_i(u_0)\) equal –1 (see Tables 1 and 4), we obtain from (21a) the cruder but simpler approximation:

\[
\Delta t_i(s_i) \approx -s_i \bar{u}_{mi}
\]

(21b)

The notable difference between the approximations in (18) and (21) is the multiplication of the administrative cost term in (21) by \(\bar{u}_{mi}\), representing the marginal utility ratio of the product. As indicated in the discussion on Table 1, the marginal utility ratios of products are usually above one and with luxuries even above two. Thus the impact of government administration costs on optimal tax rates is expected to be larger than the impact of identical compliance costs. Table 5 presents results for the same products reported in Table 4. These figures suggest that the impact of administration costs on optimal tax rates is indeed somewhat larger than those of compliance costs, in line with what is predicted from (18) and (21). Like with compliance, administration costs will reduce optimal tax rates by a similar amount to the share of these costs from commodity taxes.

<table>
<thead>
<tr>
<th>Inequality aversion</th>
<th>With admin</th>
<th>Without admin</th>
<th>Impact of admin</th>
<th>Impact of compliance</th>
<th>Marginal utility ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Good 5 - necessity</td>
<td>1</td>
<td>0.02</td>
<td>0.05</td>
<td>–0.03</td>
<td>–0.02</td>
</tr>
<tr>
<td>Good 18 – luxury</td>
<td>1</td>
<td>0.78</td>
<td>0.83</td>
<td>–0.05</td>
<td>–0.03</td>
</tr>
<tr>
<td>Good 2 - necessity</td>
<td>0.3</td>
<td>0.23</td>
<td>0.30</td>
<td>–0.07</td>
<td>–0.07</td>
</tr>
<tr>
<td>Good 12 - luxury</td>
<td>0.3</td>
<td>0.35</td>
<td>0.40</td>
<td>–0.05</td>
<td>–0.04</td>
</tr>
</tbody>
</table>

5.3. Tax evasion

Another factor of interest is tax evasion. It is assumed here that goods have different propensities for evasion depending on observability and marketing arrangements. Generally, goods and services produced by small business are more evasion prone than those produced or marketed through large organisations. We denote this evasion propensity as \(e_i\), representing a fixed portion of taxes evaded on good \(i\). Another important variable is the dead-weight costs of evasion – its constant share from the amount of tax evaded is denoted
Obviously $d_i$ depends on how costly it is to carry out tax evasion on the particular good. Apart from concealment costs, there are sometimes even larger dead-weight losses associated with less efficient production methods employed in smaller less organised firms in the black or grey economy. Taking these definitions, total evasion amounts to

$$E_i = e_i q_i t_i$$  \hspace{1cm} (22a)

and the dead-weight cost of evasion ($D_i$) is given as:

$$D_i = e_i d_i q_i t_i$$  \hspace{1cm} (22b)

This expression is very similar to (16) on compliance and (19) on administration. In this paper we shall model evasion by synthesising the compliance and administration models. In my earlier papers [Revesz (1997, 2014)], I presented an evasion model where the loss in government revenue is partly offset by effective (post-evasion) price reductions and these will benefit all the consumers of the product. However, there are some empirical weaknesses in the assumption that evasion will necessarily be reflected in lower effective prices and these will benefit all the consumers of the product. Arguably, a large part of evasion is not translated into lower effective prices, but instead brings income gains to a small group of tax evaders. For that reason, we shall develop here a simpler model of evasion that does not involve effective price reductions.

It is assumed that evasion on product $i$ causes loss to government revenue. The associated dead-weight costs are defined in (22b). Excluding these dead-weight costs, the loss in government revenue brings benefits to some consumers. In one scenario we assume that the benefits of evasion are spread among the consumers of product $i$ in direct proportion to consumption. Alternatively, we use predetermined distribution weights (denoted $g_{ih}$) to allocate benefits to consumers. By definition:

$$\sum_h g_{ih} = 1 \text{ and } n_{ih} = (1 - d_i) E_i g_{ih}$$  \hspace{1cm} (23)

where $n_{ih}$ represents the gain to taxpayer $h$ from evasion associated with good $i$.

The change in the modified inverse elasticity rule due to evasion will have two components. One is the loss in government revenue, which follows the same formula as government administration costs in (21), with $s_j$ replaced by $e_i$. When the gains are spread among the consumers of the product through higher lump-sum incomes, these gains will be of opposite sign to the lump-sum costs of compliance in (18), with $c_j$ being replaced by $-(1 - d_i) e_i$. Using 18(b) and 21(b), the approximate impact of evasion on the optimal tax rate is given as:

$$\Delta t_i (e_i) \approx (1 - d_i) e_i - e_i \bar{u}_{mi}$$  \hspace{1cm} (24a)

When the benefits are spread according to the distribution weights $g_{ih}$, then the average marginal utility of recipients will be: $\bar{u}_{ig} = \sum_h \partial u_{ih}/\partial y g_{ih}$. In this case the impact on optimal tax rate is approximated as:
\[ \Delta t_i(e_i) \approx (1 - d_i) e_i \frac{\overline{u}_{g_i}}{\overline{u}_i} - e_i \overline{u}_m \]  

(24b)

where \( \overline{u}_i = \frac{\partial \overline{y}}{\partial y} \) is the average marginal utility of the consumers of product \( i \), defined in (A.13) in appendix 2. Here the benefits to consumers are valued according to \( \overline{u}_{ig} \) instead of \( \overline{u}_i \). If the benefits accrue mainly to high income earners, then unless \( q_i \) is a strong luxury, \( \overline{u}_{ig} < \overline{u}_i \) and the decrease in the optimal tax rate due to evasion will be larger than under (24a). If the benefits accrue mainly to low wage earners the opposite applies.

Now let us look at some numerical results. Table 6 is structured the same way as Tables 4 and 5, and the iterative calculations follow the same procedure. It is assumed in these calculations that evasion gains are spread among the consumers of the product, therefore in this case (24a) applies.

### Table 6

**OPTIMAL TAX RATES WITH COMMODITY TAX EVASION**

<table>
<thead>
<tr>
<th>Evasion rate = 0.4 Dead weight cost = 25% of tax evaded</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Product No. and type</strong></td>
</tr>
<tr>
<td>--------------------------</td>
</tr>
<tr>
<td>Without income tax, inequality aversion = 1</td>
</tr>
<tr>
<td>Good 4 - necessity</td>
</tr>
<tr>
<td>Good 14 - luxury</td>
</tr>
<tr>
<td>Without income tax, inequality aversion = 0.3</td>
</tr>
<tr>
<td>Good 4 - necessity</td>
</tr>
<tr>
<td>Good 14 - luxury</td>
</tr>
<tr>
<td>With income tax, inequality aversion = 1</td>
</tr>
<tr>
<td>Good 1 - necessity</td>
</tr>
<tr>
<td>Good 12 - luxury</td>
</tr>
</tbody>
</table>

First, in all cases the decrease in optimal tax exceeds one third of the evasion rate. With two necessities the reduction in tax exceeds the evasion rate itself. Hence, as a conservative rule, it can be said that usually evasion leads to a fall in optimal tax rates exceeding half the rate of evasion, when dead-weight costs amount to a quarter or more of the revenue lost. Evidently, the reduction in tax rates is much larger for necessities than for luxuries. The approximate formula in (24a) indicates the opposite. With the utility ratio (\( \overline{u}_{mi} \)) in the negative term, one would expect the fall in tax rate to be larger for luxuries, given that with these goods \( \overline{u}_{mi} \) will be larger. The outcome from the iterative calculations suggests that with necessities the optimal solution tries to reduce evasion related dead-weight costs on low income households by reducing the tax rate, which from (22b) is directly linked to dead-weight costs. However, the motive to reduce dead-weight costs is much weaker in respect to luxuries, because of the smaller value of these losses in utility terms. Given the larger reduction in taxes on necessities, evasion reinforces the progressivity of optimal indirect taxation. It should be noted that my earlier evasion model, reported in Revesz (1997, 2014), also revealed a highly pro-
gressive impact of evasion on optimal tax rates, provided it involves substantial dead-weight costs.

Needless to say, in order to optimise commodity tax rates one needs to have empirical information on evasion and administration parameters at the level of products or product groups. To the best of my knowledge such information is not yet available. In regard to evasion propensities and dead-weight losses, probably detailed information will not be published in the foreseeable future, because public authorities are not keen to reveal information that might be useful for tax evaders. In any event, the broad picture about evasion propensities is well known. Goods and services passing through large organisations tend to be much less evasion prone than those passing through small business. The experience with petrol and cigarette taxes suggests that tax rates in excess of 100% can be imposed on goods produced and marketed through large organisation without encountering insurmountable evasion difficulties. In section 4.3 we noted that under current legal-institutional arrangements, separate commodity tax rates have to be defined for a small number of broad products groups. In light of the results from the present model, setting significantly higher tax rates for most non-necessities that are produced by large organisations seems to be an advisable approach to increase consumption taxes, in line with the recommendations of the “growth friendly” taxation reports.

5.4. Externalities

This section examines the impact of externalities on optimal tax rates. Externalities can be both negative and positive. Negative externalities include two well-known items – pollution and congestion. Positive externalities include innovation and network externalities as well as some externalities associated with education, culture and public health. For the sake of modelling simplicity, we assume that the costs or benefits of externalities can be quantified in monetary terms and that there is a direct proportional relationship between these costs and benefits and the total consumption of externality generating goods. Denoting the imputed value of the externality as $E_i$ and total consumption of an externality generating good as $Q_i$, then

$$E_i = \mu_i Q_i$$

where $\mu_i$ is the constant Pigovian cost or benefit rate per unit consumption. In this model we also assume that the imputed costs or benefits of externalities enter into the social welfare function through changes in lump-sum incomes. The definition of the social welfare function in this case will be:

$$U = \sum_h a_h u_h (\mathbf{p}, W, \sum_i [v_{oh} + \sum_i z_{ih} \mu_i Q_i])$$

where $z_{ih}$ represents the portion of $E_i$ received by taxpayer $h$. By definition, the distribution weights add up to one, i.e.: $\sum_h z_{ih} = 1$. Total real output decreases or increases by $E_i$. 
In appendix 2 eq. (A.27) we derive an approximate formula for the impact of a single externality on the optimal tax rate of the externality generating good. After taking out the similar $\varepsilon_i$ and $\varepsilon_i(u_0)$ terms from the numerator and denominator (as was done also earlier), we obtain the following approximate formula for the change in tax due to externalities.

$$
\Delta t_i(\mu_i) = -\mu_i \frac{\sum_h z_i h \frac{\partial u_h}{\partial y}}{\bar{u}_i / \partial y}
$$

(27)

The term in the denominator is the average marginal utility of product $i$, that was defined in (10), and more explicitly in appendix 2 (A.13). The term in the numerator is a weighted average of the social marginal utilities of externality recipients using the $z_{ih}$ weights. If the externality has the same effect on every member of the population, then $z_{ih} = 1/H$ for everyone. Given the definition of the marginal utility of the demogrant in (A.2), the numerator in (27) becomes $\mu_i \sum_h 1/H \frac{\partial u_h}{\partial y} = \mu_i \bar{u}/\partial b$. From definition (10) this implies that in this case:

$$
\Delta t_i(\mu_i) \approx -\mu_i \bar{u}_i
$$

(28)

In words, the approximate effect of the externality on the optimal tax rate is the Pigovian rate multiplied by the marginal utility ratio of product $i$. As explained in the discussion about Table 1, unless the product is an inferior good, its marginal utility ratio is above one. This leads us to conclude that if the distribution of the costs or benefits of an externality are shared equally in the population, then the optimal change in the tax rate will be larger than the Pigovian tax or subsidy rate. The average marginal utility of product $i$ in the denominator of (27) suggests that the higher is the average income level of the consumers of the externality generating good the larger will be the change in the optimal tax rate (up or down) due to the externality.

If the $z_{ih}$ are not equal, then provided the higher $z_{ih}$ are concentrated more among low wage earners and provided the marginal utility of income is decreasing with income, then the numerator in (28) will be higher, hence the tax rate change (up or down) will be larger. The opposite argument applies if the $z_{ih}$ tend to be larger among higher wage earners. In summary, the tax or subsidy induced will be larger the more the externality generating good is a luxury and the more the externality affects low income groups. A similar conclusion is presented by Sandmo (1975), who derived a similar inverse elasticity formula to (27) for pollution externalities.

Table 7

<table>
<thead>
<tr>
<th>Product no. and type</th>
<th>Pigovian rate</th>
<th>Marginal utility ratio*</th>
<th>All equal</th>
<th>Type of weights</th>
</tr>
</thead>
<tbody>
<tr>
<td>Good 5 - necessity</td>
<td>-0.2</td>
<td>1.08</td>
<td>0.21</td>
<td>0.17</td>
</tr>
<tr>
<td>Good 17 - luxury</td>
<td>+0.2</td>
<td>2.26</td>
<td>-0.17</td>
<td>-0.20</td>
</tr>
</tbody>
</table>

*Marginal utility ratio of the demogrant

With progressive income tax – inequality aversion = 1
Having established the modified inverse elasticity rule for externalities, we can take a look at some numerical results. Table 7 presents results with constant proportional externality rates of plus or minus 20%. Externality weights refer to the $z_{ij}$ recipient weights discussed earlier. Decreasing weights refer to the situation where the lowest $W$ taxpayer receives five times more externality than the highest $W$ person. Increasing weights refer to the opposite distribution of $z_{ij}$ weights. As expected from the approximate formula in (27) the departure from the Pigovian rates (plus or minus 20%) is larger the more the externality affects low income persons (decreasing $z_{ij}$). Generally luxury goods do not show significantly larger deviations from the Pigovian rate than necessities, contrary to what is predicted from (28). It is interesting to note in Table 7 that without progressive income tax, actual changes in tax rates tend to be well above Pigovian rates, but in line with changes predicted from the modified inverse elasticity formulas in (27) and (28). Recall that zero income tax is a particular form of linear income tax. When progressive income taxation is included in the model, the results are closer to Pigovian rates. These findings suggest that the presence or absence of other distributional instruments (such as progressive income tax) has a strong bearing on the impact of externalities on optimal tax rates.

Turning to the highly publicised subject of environmental taxes, it can be said that the material presented in this paper supports higher environmental taxes. One reason is the finding that tax increases due to negative externalities will exceed the Pigovian tax rates with linear income tax, and to a lesser extent will also tend to exceed them in the presence of progressive income tax. The other reason is connected with the fact that most fossil fuels are marketed through large organisations and are not particularly evasion prone. Also, the administrative and compliance costs of these taxes are relatively low.
5.5. Paternalistic concerns

This section examines the impact on optimal tax rates of paternalistic concerns. These include a number of taxes and subsidies provided in line with political judgments about what is in the interest of consumers in the long run. Examples include taxes on “sin goods”, subsidies for home buying, educational books and software or expenditure on preventative health care. To some extent, the subsidisation of education and health services is also motivated by paternalistic concerns. Needless to say, these concerns provide another reason for differentiated commodity tax rates. Related products are sometimes referred to in the literature as merit (or demerit) goods. Some interesting papers on this subject are presented by Besley (1988) and Pestieau et al. (2012).

Given that these items are related to the life-long utility of the consumer, imputation of such costs and benefits raises inter-temporal issues. In the present model we assume that in the static context these imputed costs and benefits are directly proportional to the quantity consumed from the selected good. The imputed value per unit consumption is denoted $\gamma_i$, therefore total merit goods costs or benefits to taxpayer $h$ equals to: $G_h = \sum_i q_{ih} \gamma_i$. As before, $G_h$ enters into the utility function of consumers through changes in lump-sum income. The modified social welfare function will be:

$$U =\sum_h a_h u_h (p, W_t, \sum_i [y_{oih} + \sum_i q_{ih} \gamma_i]) \tag{29}$$

Comparing this expression with the social welfare function involving externalities defined in (26), we notice that the only difference is that the $\sum_i [y_{oh} + \sum_i z_{ih} \mu_i Q_i]$ term of externalities has been replaced by the $\sum_i [y_{oh} + \sum_i q_{ih} \gamma_i]$ term here. Therefore, there is no need to develop another modified inverse elasticity rule. Suffice is to replace the externality term by the merit good term. As with externalities, we focus on a single good $-q_i$. From (27) it follows that in this case:

$$\Delta t(\gamma_i) = -\frac{\gamma_i \sum_h q_{ih} \partial u_h / \partial y}{\partial u_i / \partial y} = -\gamma_i \partial u_i / \partial y = -\gamma_i \tag{30}$$

The simplification follows from the definition of the average marginal utility of product $i$ in appendix 2 eq. (A. 13). Having obtained a simple approximation for the effect of $\gamma_i$ on the optimal tax rate, we can take a look at some numerical examples in table 8. Generally, the changes in optimal tax rates from the iterative calculations are in line with what is predicted from (30).

<table>
<thead>
<tr>
<th>Product No. and type</th>
<th>Imputed $\gamma_i$ with paternalistic objective</th>
<th>without such objective</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>With income tax – inequality aversion = 1</td>
<td>0.17</td>
<td>0.42</td>
<td>+0.25</td>
</tr>
</tbody>
</table>
Table 8 (Continued)
OPTIMAL TAX RATES WITH MERIT GOODS

<table>
<thead>
<tr>
<th>Product No. and type</th>
<th>Imputed ( \gamma_i ) rate with paternalistic objective</th>
<th>without such objective</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>With income tax – inequality aversion = 1</td>
<td>Good 16 - luxury +0.2 1.48</td>
<td>1.28</td>
<td>–0.20</td>
</tr>
<tr>
<td>Without income tax – inequality aversion = 0.3</td>
<td>Good 2 - necessity +0.2 0.55</td>
<td>0.30</td>
<td>–0.25</td>
</tr>
<tr>
<td></td>
<td>Good 14 – luxury –0.2 1.28</td>
<td>0.38</td>
<td>+0.24</td>
</tr>
</tbody>
</table>

5.6. Leisure complements and substitutes

Complementarity with leisure is one of the central issues in the uniform commodity taxation controversy. All the models favouring uniform taxation assume weakly separable utility between commodities and leisure. It is of some interest to examine the pattern of optimal commodity tax rates when weak separability does not apply, and in particular what is the effect of leisure substitution or complementarity on optimal tax rates. This section will try to provide some answers to these questions.

The starting point for our discussion is the derivative of labour supply with respect to a change in the price of good \( i \). As proved in Revesz (1986, 2005), with weakly separable utility that derivative will be:

\[
\frac{\partial \hat{\ell}}{\partial p_i} (u(\ell, v(q))) = \frac{\partial q}{\partial m} \ell \theta^e - \frac{\partial \ell}{\partial y} q_i
\]

where \( \theta^e \) is the compensated elasticity of labour supply with respect to the wage rate, \( m \) is disposable income, i.e.: \( m = w\ell + y \) and \( \ell \partial \ell / \partial y \) is the derivative of labour supply with respect to lump-sum income. Having established a benchmark equation for weakly separable utility, we can now define deviations from it. We define leisure substitution or complementarity (\( \varepsilon_i \)) as the difference between the actual price derivative of labour and the price derivative corresponding to weakly separable utility:

\[
\varepsilon_i = \frac{\partial \hat{\ell}}{\partial p_i} - \frac{\partial \ell}{\partial p_i} (u(\ell, v(q)))
\]

For leisure substitutes \( \varepsilon_i \) will be positive because they increase labour supply and vice versa for leisure complements. In order to maintain zero homogeneous labour supply as a function of prices and the earning parameters \( w \) and \( y \), the following condition must be met [see Revesz (1986)]:

\[
\sum_i \varepsilon_i p_i = 0
\]

We can use \( \varepsilon_i \) in a modified inverse elasticity calculation to estimate its approximate effect on optimal tax rates. The detailed derivation is presented in appendix 5. The end result is the following approximation from (E.4):
\[
\Delta t_i(\hat{\ell}) \approx -\frac{\partial \hat{\ell}}{\partial q_i} \bar{u}_m W_i T_i' = -\frac{\bar{\varepsilon}_i}{\partial q_i} / \partial t_i \bar{u}_m W_i T_i'
\]

where

\[
\frac{\partial \hat{\ell}}{\partial q_i} = \frac{\partial \hat{\ell}}{\partial p_i} / \partial q_i
\]

\(\bar{\varepsilon}_i\) is the \(q_i\) weighted average value of \(\varepsilon_i\) over \(H\) taxpayers, \(W_i\) is the corresponding average wage rate, \(T_i'\) is the average marginal indirect tax rate of the consumers of product \(i\). It is defined in (A.5b) in appendix 2. \(\bar{u}_{mi}\) is the marginal utility ratio of product \(i\) defined in (A.12). The approximation in (34) suggests that the tax rate on \(i\) will increase (or decrease) depending by how much tax revenue decreased (or increased) as a result of change in labour supply. Another point to notice in (34) is that luxuries will have larger movements in tax rates due to leisure dependency than necessities. This is because with luxuries \(\bar{u}_{mi}, W_i\) and \(T_i'\) are all larger.

Apparently, in this model the main purpose of higher taxes on leisure complements than substitutes is in boosting tax revenue for redistribution, rather than in directly improving the utility position of those paying the taxes. The modified inverse elasticity rule is derived in this case (as was done also with other factors) on the assumption that the introduction of \(\varepsilon_i\) has no first order effect on the marginal utility of income of taxpayers. Only its impact on tax revenue has first order effect. This perspective on the role of \(\varepsilon_i\) in a redistributive model is markedly different from the original model of Corlett and Hague (1953), who examined a single consumer economy and demonstrated utility improvement for the representative consumer due to higher tax on a leisure complement. In an analytical study, Jacobs and Boadway (2014) examine non-separable utility in a many-person model with commodity taxes combined with a Mirrlees (1971) type optimal non-linear income tax function. They reached the same conclusions as Corlett and Hague (1953) in the context of this redistributive model. West and Williams (2007) and Parry and West (2009) present estimates of leisure complementarities and their effect on optimal tax rates in a partial equilibrium setting.

Table 9 compares actual tax rates from a numerical study with the rates predicted using the modified inverse elasticity rule in (34). This numerical study is based on an extended form of LES, which incorporates leisure complements and substitutes. It is explained in detail in appendix 5. The original model we compare with is the one reported in Table 1, with the inequality aversion rate set to one. Notice in Table 9 that larger changes involving luxuries, has led to higher progressivity of optimal tax rates compared with the original model. It is also interesting to note the large improvements in welfare and output compared with the uniform tax solution. However, given that the \(\bar{\varepsilon}_i\) parameters are not based on econometric estimates but were arbitrarily chosen, and the ad hoc nature of extended LES used in these calculations (see appendix 5), perhaps one should observe with caution these striking results.
Other results indicate that the large improvements over the uniform solution arise because of the high inequality aversion rate chosen. Table 10 shows that the utility gains over the uniform solution become much smaller when the inequality aversion rate is reduced from one to 0.3. The explanation is that with lower inequality aversion there is less utility gained by boosting labour supply in order to generate more revenue for redistribution. The scenarios in Table 10 are based on the same parameters reported in Table 9.

### Table 9
**THE EFFECT OF LEISURE COMPLEMENTS AND SUBSTITUTES ON OPTIMAL TAX RATES**

<table>
<thead>
<tr>
<th>good no.</th>
<th>( \frac{\partial \bar{W}}{\partial q_i} )</th>
<th>( \bar{W}_i )</th>
<th>( T_i )</th>
<th>( \bar{u}_{mi} )</th>
<th>original model tax</th>
<th>modified model tax</th>
<th>actual tax difference</th>
<th>predicted tax difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.03</td>
<td>17.6</td>
<td>0.55</td>
<td>1.22</td>
<td>0.98</td>
<td>0.20</td>
<td>-0.78</td>
<td>-0.35</td>
</tr>
<tr>
<td>2</td>
<td>18.3</td>
<td>0.55</td>
<td>1.28</td>
<td>0.85</td>
<td>1.02</td>
<td>1.17</td>
<td><strong>--</strong></td>
<td>--</td>
</tr>
<tr>
<td>3</td>
<td>13.2</td>
<td>0.50</td>
<td>0.95</td>
<td>0.22</td>
<td>0.44</td>
<td>0.23</td>
<td><strong>--</strong></td>
<td>--</td>
</tr>
<tr>
<td>4</td>
<td>-0.02</td>
<td>20.1</td>
<td>0.56</td>
<td>1.41</td>
<td>0.98</td>
<td>1.83</td>
<td>0.85</td>
<td>0.32</td>
</tr>
<tr>
<td>5</td>
<td>17.9</td>
<td>0.53</td>
<td>1.16</td>
<td>0.48</td>
<td>0.71</td>
<td>0.23</td>
<td><strong>--</strong></td>
<td>--</td>
</tr>
<tr>
<td>6</td>
<td>19.7</td>
<td>0.56</td>
<td>1.37</td>
<td>0.98</td>
<td>1.15</td>
<td>0.17</td>
<td><strong>--</strong></td>
<td>--</td>
</tr>
<tr>
<td>7</td>
<td>0.01</td>
<td>15.9</td>
<td>0.53</td>
<td>1.11</td>
<td>0.45</td>
<td>0.51</td>
<td>0.06</td>
<td>-0.09</td>
</tr>
<tr>
<td>8</td>
<td>17.8</td>
<td>0.54</td>
<td>1.23</td>
<td>0.68</td>
<td>0.87</td>
<td>0.20</td>
<td><strong>--</strong></td>
<td>--</td>
</tr>
<tr>
<td>9</td>
<td>-0.04</td>
<td>15.5</td>
<td>0.52</td>
<td>1.09</td>
<td>0.43</td>
<td>0.70</td>
<td>0.27</td>
<td>0.17</td>
</tr>
<tr>
<td>10</td>
<td>0.05</td>
<td>27.8</td>
<td>0.61</td>
<td>2.26</td>
<td>1.49</td>
<td>0.86</td>
<td>-0.62</td>
<td>-1.90</td>
</tr>
<tr>
<td>11</td>
<td>25.9</td>
<td>0.61</td>
<td>2.12</td>
<td>2.68</td>
<td>2.52</td>
<td>-0.16</td>
<td><strong>--</strong></td>
<td>--</td>
</tr>
<tr>
<td>12</td>
<td>27.6</td>
<td>0.61</td>
<td>2.24</td>
<td>1.62</td>
<td>1.77</td>
<td>0.15</td>
<td><strong>--</strong></td>
<td>--</td>
</tr>
<tr>
<td>13</td>
<td>-0.02</td>
<td>28.3</td>
<td>0.61</td>
<td>2.30</td>
<td>1.55</td>
<td>2.83</td>
<td>1.28</td>
<td>0.79</td>
</tr>
<tr>
<td>14</td>
<td>28.81</td>
<td>0.61</td>
<td>2.34</td>
<td>1.32</td>
<td>1.53</td>
<td>0.21</td>
<td><strong>--</strong></td>
<td>--</td>
</tr>
<tr>
<td>15</td>
<td>26.5</td>
<td>0.61</td>
<td>2.17</td>
<td>2.12</td>
<td>2.14</td>
<td>0.02</td>
<td><strong>--</strong></td>
<td>--</td>
</tr>
<tr>
<td>16</td>
<td>26.2</td>
<td>0.61</td>
<td>2.15</td>
<td>2.33</td>
<td>2.29</td>
<td>-0.04</td>
<td><strong>--</strong></td>
<td>--</td>
</tr>
<tr>
<td>17</td>
<td>28.9</td>
<td>0.61</td>
<td>2.35</td>
<td>1.31</td>
<td>1.52</td>
<td>0.21</td>
<td><strong>--</strong></td>
<td>--</td>
</tr>
<tr>
<td>18</td>
<td>-0.14</td>
<td>27.8</td>
<td>0.61</td>
<td>2.26</td>
<td>1.70</td>
<td>3.97</td>
<td>2.27</td>
<td>4.34</td>
</tr>
</tbody>
</table>

Average tax rate on the 9 necessities: 0.70
Average tax rate on the 9 luxuries: 1.83
% change in welfare terms compared to uniform: 3.5
% change in output compared to uniform solution: 1.7

* The complementarity parameter on product 9 applies only to the preferences of the 8 lower W taxpayers.

### Table 10
**OTHER RESULTS WITH LEISURE COMPLEMENTS AND SUBSTITUTES**

<table>
<thead>
<tr>
<th>Inequality aversion rate</th>
<th>% change welfare over uniform</th>
<th>% change output over uniform</th>
</tr>
</thead>
<tbody>
<tr>
<td>Original model</td>
<td>Extended model</td>
<td>Original model</td>
</tr>
</tbody>
</table>

| Without progr. income tax | 0.3 | 0.2 | 1.9 | 1.5 | 5.6 |
| With progr. income tax    | 1   | 2.8 | 3.0 | 9.7 | 12.5|
| With progr. income tax    | 0.3 | 0.4 | 0.6 | 0.8 | 5.9 |
6. Summary and qualifications

6.1. The main results

Taking a broad view on the discussion, the major finding is that given non-linear Engel curves, optimal commodity tax rates tend to be progressive and highly dispersed under logarithmic utility specifications, but quite strong progressivity and dispersion often persists even when the inequality aversion of society is low. Generally speaking, this conclusion applies to most exogenously given income tax schedules. Moreover, the gains in output and welfare of the differentiated over the uniform solution can be quite substantial. These gains can sometimes exceed 3% of total output even in a simple model without any complexities added. However, we discovered in the numerical simulations at least one example of a non-optimal non-linear (progressive) income tax schedule, where the corresponding optimal commodity tax rates turned out to be negative and nearly uniform, when inequality aversion is low. Yet, while exceptions are possible, the large majority of numerical results in purely redistributive models point in favour of differentiated and progressive indirect taxation. This is always the case with linear income tax.

Moreover, the introduction into the model of “real life” complexities, such as tax evasion, administration and compliance costs, externalities and leisure complements and substitutes, tends to increase both tax rate dispersion and the progressivity of optimal tax rates. In addition, they make the gains over the uniform solution much larger. The assumption that these complications are independent factors that are neutral in respect to distributional objectives is shown to be false. The numerical results disprove this assumption, and the modified inverse elasticity formulas of these factors contain marginal utility ratios, which reflect distributional considerations. Tax evasion in particular tends to increase indirect tax progressivity. Among other things, we found that due to distributional considerations, given zero or linear income tax, externality generating goods should be taxed or subsidised well above Pigovian rates.

6.2. Some qualifications

A number of possible objections can be raised against the present model and we shall deal here with three that could be the more important ones.

First, the arguments presented in this paper conflict with the tax uniformity theorems discovered by Atkinson and Stiglitz (1976) and Deaton (1979). As mentioned earlier, in my view these theorems have little practical relevance, since they are based on strong simplifying assumptions, ignore “real life” complexities and the non-optimality of actual income tax schedules and selective support payments to the needy. The extension of the Atkinson-Stiglitz theorem by Laroque (2005) and Kaplow (2006) has been analysed in section 4.5 and its limitations were noted.
Yet another objection can be raised about using arbitrary numbers in the simulations rather than empirical estimates. Given that much of the discussion is based on analytical foundations, this does not appear to be a major problem. Analytical studies from the 1970s have already indicated that with zero or linear income tax, optimal commodity tax rates will be differentiated and progressive (see section 2.1). Other factors studied in this paper were analysed using the modified inverse elasticity rule. Hence the broad conclusions presented here are analytically based and do not depend on the choice of parameter values. The numerical examples illustrate and substantiate analytical arguments. Hopefully, future numerical research on this subject will rely more on empirical estimates.

Another possible objection relates to the non-linear Engel curves in the present model. The segmentation of LES utility has led to a sharp distinction between necessities and luxuries, which could be considered unrealistic and might exaggerate the benefits of progressive indirect taxation. This is probably a significant weakness in the present model. It might be advisable to use in future research utility functions other than segmented LES, or divide LES into more than two segments.

6.3. Policy issues

When examining possible policy implications from the present study I shall focus on European Union (EU) VAT systems, since this is an area that I am a bit familiar with. Currently standard VAT rates in EU countries range between 15% and 27%. A minority of products (mainly necessities) are subject to zero or reduced VAT rates. These concessions are more significant in the UK, France, Italy and Spain. We shall refer to these systems as “progressive” VAT systems.

Considering the legal-institutional requirement to define tax rates for broad product groups, it appears that there is little room to provide further tax relief on necessities beyond what is available in “progressive” VAT systems, apart from setting all concessional rates to zero. In contrast to necessities in “progressive” VAT systems, there appears to be plenty of scope for reform in regard to non-necessities. As noted earlier, tax rates in excess of 100% have been applied for a long time on petrol and cigarettes. Such high tax rates are likely to be administratively manageable also in respect to other non-essential goods and services produced by large organisations. Our numerical results suggest that tax rates on non-necessities in excess of 50% can be justified even under moderate inequality aversion rates. For products that generate negative externalities, taxes should be raised even higher. If the objective is to increase revenues from consumption taxes, as recommended in the “growth friendly” taxation studies, then there seems to be plenty of room to increase VAT rates on non-necessities produced by large organisations. For that purpose, it might be advisable to introduce a few separate VAT rates for non-necessities instead of one standard rate, and assign tax rates to product groups taking into account income elasticities, evasion propensities, administrative and compliance costs, paternalistic concerns, leisure complementarities and legal definitional requirements. Generally speaking, making indirect taxation more progressive
can offset the erosion of equity goals due to income tax cuts, which have been recently im-
plemented or have been proposed in many countries.

In regard to the taxation of housing, it is possible to introduce progressive taxation in
this area through the existing system of property valuations. But given that property taxes
are usually levied by local authorities, who have little interest in distributional issues, pro-
gressive property taxation probably will have to be implemented by central authorities. In
summary, looking at current EU indirect taxation systems, there appears to be considerable
scope for welfare improving and evasion reducing reforms. I prefer not to speculate on the
political feasibility of such reforms.

All in all, the present model yields strong arguments in favour of differentiated and pro-
gressive indirect taxation. But of course, this model offers only a step toward a better under-
standing and further empirical and computational research is needed.

Appendix 1

User’s guide to the computer program titled ‘indirect-tax8’

This program yielded the numerical results examined in this paper. The user can speci-
fy values such as utility function parameters, wage distribution, inequality aversion of soci-
ety, piecewise linear income tax schedule, tax evasion on income and selected commodities
and associated dead-weight losses, administrative and compliance costs related to income
and commodity taxation, externalities, substitution and complementarity with leisure and a
few other factors. The original program was developed more than 18 years ago and was re-
ported in Revesz (1997). The current version is much broader than the original.

Program installation and running

The program is located in the website http://johnlrevesz.com. It is written in a program-
ing language called QBasic, which two decades ago was part of Microsoft Windows. The
program is written as a “source code” in a Word file, without being converted into machine
language, as are most computer programs sold on the market. A special compiler is needed
to convert it into machine code. A QBasic compiler for 64 bit machines is available free of
charge from the following website: http://www.qb64.net/ Another QBasic compiler is avail-
able for US $60 from: http://www.libertybasic.com/index.html

Having a QBasic compiler, the installation is simple. Open the file named qb64 and then
Edit and Paste the entire content of the indirect-tax8 Word file into the QBasic screen. Next,
click Run and then Start. Less than a minute later the first prompt will appear on the screen.
The prompts present a menu of options. After you finished with the prompts, processing will
start automatically and will finish in a couple of minutes. The output file, called tax1.txt, will
be located in the same folder as the QBasic compiler. For the best view, it should be opened with Windows Notepad.

Incidentally, all the library files associated with the compiler should be kept in the same folder as the compiler file. Because the program is a source code in Word format, it could be corrupted accidentally or during data entry. For that reason, it is advisable to keep a couple of copies for backup, preferably in a different folder.

**Data entry**

Virtually all the independent variables and parameters in the model can be changed by the user. All the numbers in the DATA lines close to the beginning of the source code can be changed. Above the DATA lines are text lines (marked at the left with an asterisk to indicate that it is a comment and not a program line) explaining what the DATA lines refer to. These include wage rates, initial lump-sum incomes, utility parameters, portion of income tax or a particular commodity tax evaded, the percentage of dead-weight costs associated with evasion, administrative and compliance costs of income and commodity taxation, five marginal tax rates defining the piecewise linear income tax schedule, targeted lump-sum support grants to the needy, externalities, merit goods and leisure complements and substitutes.

Usually there are a number of data lines for each item. Those marked with an asterisk at the beginning of the line, are previously used lines that are ignored by the program. Only the unmarked DATA line is currently active. Considerable care must be exercised when entering new numbers. If the number of entries in the DATA line does not correspond to the number of READs specified underneath, then the program will be corrupted without warning. If there are too few entries, the READ instruction will assume that the missing numbers are zero. If there are too many entries, the surplus numbers will be picked up by the following READ instruction relating to a different variable or parameter. As a result, the output from the program will be meaningless, Please check the parameter summary tables appearing at the beginning of the printout to ensure that correct numbers are used by the program.

Further details on how to operate the program are presented in the user’s guide located in the [http://johnlrevesz.com](http://johnlrevesz.com) website.

**Appendix 2**

**The modified inverse elasticity rule**

In order to explain better the numerical results, we shall develop an approximate formula for optimal commodity tax rates. While this formula was not used in the numerical calculations, it provides a convenient analytical framework to interpret some of the numerical re-
sults. It starts with the simple distributional model and has been developed further to accommodate additional factors.

In the model without income tax (analysed in section 4.1) suppose that the tax rate $t_i$ is increased by a small amount. The increase has three effects. First, from Roy’s Lemma, consumers buying $q_i$ will have their utility reduced by

$$\sum_h \frac{\partial u_h}{\partial t_i} = - \sum_h q_i \frac{\partial u_h}{\partial y}$$  \hspace{1cm} (A.1)

where we sum up over $H$ taxpayers.

Second, given fixed public expenditure requirements, all the additional tax collected will be redistributed through increase in the uniform demogrant ‘b’. Given equal ‘b’ to all $H$ taxpayers, the social marginal utility of $b$ is

$$\frac{\partial U}{\partial b} = \frac{\sum_{h=1}^{H} \frac{\partial U_h}{\partial y}}{H}$$  \hspace{1cm} (A.2)

The amount transferred depends on the revenue generated by increased $t_i$. In order not to complicate the analysis, we ignore for a moment changes in labour supply. In this situation, deriving government revenue ($R = \sum t_i q_i$) with respect to $t_i$ yields:

$$\frac{\partial R}{\partial t_i} = q_i + \sum_{j=1}^{J} t_j \frac{\partial q_j}{\partial t_i}$$  \hspace{1cm} (A.3)

To simplify matters, we convert the sum on the right side into a single expression. The cross derivatives are usually fairly small terms. The own derivative ($j = i$) can be used to approximate the sum by a single expression. For that purpose, let us look at two polar cases. When all tax rates apart from $t_i$ are zero, then the total revenue derivative reduces to $t_i \frac{\partial q_i}{\partial t_i}$. When all tax rates are the same ($t_i$), and provided labour supply does not change, then

$$\sum_{j=1}^{J} t_j \frac{\partial q_j}{\partial t_i} = t_i \sum_{j=1}^{J} (1 + t_j) \frac{\partial q_j}{\partial p_j} = t_i \sum_{j=1}^{J} p_j \frac{\partial q_j}{\partial p_j} = 0$$

where $q_j$ represents total demand by $H$ taxpayers. These cases suggest that the total revenue derivative may be approximated by the following expression:

$$\frac{\partial R}{\partial t_i} = q_i + \sum_{j=1}^{J} t_j \frac{\partial q_j}{\partial t_i} = q_i + \sum_{j=1}^{J} t_j \frac{\partial q_i}{\partial t_i} (t_i - t_{iA})$$  \hspace{1cm} (A.4)

where $t_{iA}$ is a product specific average indirect tax rate on goods other than $q_i$. It can be smaller or larger than $t_i$. For estimating the net change in total tax revenue, the average $t_{iA}$ on marginal expenditure appears more appropriate than on total expenditure. Finally, we take into account the effect of changing labour supply on tax revenue. Multiplying the cross derivative of labour supply with respect to $p_i$ by $W$ yields the change in gross income due to the tax change. Symbolically: $\frac{\partial m}{\partial p_i} = \frac{\partial l}{\partial p_i} W$. Multiplying the change in income by the average por-
tion of commodity taxes from the last dollar expenditure of the consumers of product $i$ (in other words, the combined marginal indirect tax rate, denoted $\mathcal{T}_i$), we obtain the change in revenue due to labour supply:

$$\frac{\partial R(\ell)}{\partial t_i} = \frac{\partial \ell}{\partial p_i} W \mathcal{T}_i'$$

(A.5a)

where

$$\mathcal{T}_i' = \frac{\sum_j t'_j \frac{\partial q_{ji}}{\partial y}}{(1 + \sum_j t'_j \frac{\partial q_{ji}}{\partial y})}$$

(A.5b)

Here $q_{ji}$ refers to the quantity consumed from $q_j$ by the consumers of $q_i$.

Combining (A.2), (A.3) and (A.5), the net effect on social utility of giving transfers through the demogrant following a change in $t_i$ is defined by the following expression:

$$\frac{\partial U}{\partial R} \frac{\partial R}{\partial U_h} \left( \epsilon_{ih} \mathcal{T}_i \right) \approx -\sum_h \frac{\partial u_h}{\partial y} t_i \frac{\partial q_{ih}(u_0)}{\partial p_i}$$

(A.6)

The third element in this redistributive model is the excess burden loss due to increased price distortion. The excess burden ($D$) is represented by the Harberger triangle under the compensated demand curve:

$$D = -\Delta t_i \Delta q_i / 2 \approx -\Delta t_i \frac{\partial q_i(u_0)}{\partial t_i} \frac{\Delta u_i}{2} = -t_i \frac{\partial q_i(u_0)}{\partial t_i} \frac{\Delta u_i}{2} \approx -t_i \frac{\partial q_i(u_0)}{\partial p_i} / 2$$

(A.7)

We assumed in this approximation that the compensated demand curve is linear, hence its price derivative is constant. It is also assumed that the appropriate Harberger triangle starts from zero tax, thus in (A.7), $\Delta t_i = t_i$. The excess burden loss is borne directly by the taxpayer. Deriving (A.7) with respect to $t_i$ and then multiplying by the consumer’s marginal utility of income term and summing over $h$ taxpayers we obtain:

$$\frac{\partial D}{\partial t_i} \approx -\sum_h \frac{\partial u_h}{\partial y} \frac{\partial q_{ih}(u_0)}{\partial p_i}$$

(A.8)

At the optimum, the three utility effects, represented by (A.1), (A.6) and (A.8) must add up to zero. Summing up while converting derivatives into elasticities ($\varepsilon$) we obtain:

$$0 \approx -\sum_h q_{ih} \frac{\partial u_h}{\partial y} + \frac{\partial U}{\partial b} \left[ \sum_h \left( q_{ih} + q_{ih} \varepsilon_{ih}(t_i-t_{iA}) \right) \frac{\varepsilon_{ih}(\ell) \mathcal{T}_i}{p_i} \right] - \sum_h \frac{\partial u_h}{\partial y} t_i \frac{q_{ih}(u_0)}{p_i}$$

(A.9)

Multiplying all the terms in (A.9) by $p_i / \sum_h q_{ih} = p_i / Q_i$ we arrive at:

$$0 \approx \frac{\partial u_i}{\partial y} p_i + \frac{\partial U}{\partial b} \left[ p_i + \varepsilon_i(t_i-t_{iA}) + \varepsilon_i(\ell) \mathcal{T}_i \frac{m_i}{Q_i} \right] - t_i \frac{\partial u_i}{\partial y} \varepsilon_i(u_0)$$

(A.10)
where \( \frac{\partial \bar{u}}{\partial y} \), \( \varepsilon_i \), \( \bar{\varepsilon}_i(u_0) \), \( \bar{\varepsilon}_i(\ell) \), \( \bar{T}'_i \) are all weighted averages, with the weights given by the \( q_{ih} \). 

purchases of individual consumers. After dividing all terms by \( \frac{\partial \bar{u}}{\partial y} \) and replacing \( p_i \) by 1 + \( t_i \) and \( \varepsilon_i(\ell) \frac{m}{Q_i} \) by \( \varepsilon_i(\ell) \frac{m}{Q_i} \), we arrive at the following elasticities-based formula:

\[
t_i = \frac{1 + t_i - \bar{u}_mi(1 + t_i + \bar{\varepsilon}_i(t_i - t_{iA}) + \bar{\varepsilon}_i(\bar{T}'_i))}{\bar{\varepsilon}_i(u_0)}
\] (A.11)

The term
\[
\bar{u}_mi = \frac{\partial u/\partial b}{\partial \bar{u}_i/\partial y}
\] (A.12)

is called the marginal utility ratio of product \( i \). \( \partial u/\partial b \) has been defined in (A.2). \( \partial \bar{u}_i/\partial y \) is the average marginal utility of the consumers of product \( i \). It is defined as:

\[
\frac{\partial \bar{u}_i}{\partial y} = \frac{\sum h \partial u_h}{\partial \bar{u}_i} q_{ih} / \sum q_{ih}
\] (A.13)

The approximate formula in (A.11) is referred to as the modified inverse elasticity rule. A similar approximation presented in Sandmo (1975) eq. (24) appears in terms of the notations used here as:

\[
t_i \approx - \frac{1 - \bar{u}_mi}{\varepsilon_i} = \frac{1 - \bar{u}_mi}{\bar{u}_mi \bar{\varepsilon}_i}
\] (A.14)

Apparently, the biggest difference between Sandmo’s formula and the one developed here is the absence of the \( \hat{\varepsilon}_i(t_i - t_{iA}) \) and \( \bar{\hat{\varepsilon}}_i(\bar{T}'_i) \) terms in Sandmo’s approximation.

When income tax is included, the modified inverse elasticity rule is changed. It will be:

\[
t_i \approx \frac{1 + t_i - \bar{u}_mi(1 + t_i + \bar{\varepsilon}_i(t_i - t_{iA}) + \bar{\varepsilon}_i(\bar{T}_m + \bar{T}'_i))}{\bar{\varepsilon}_i(u_0)} - \bar{T}'_i (1 - \gamma)
\] (A.15)

where \( \bar{T}_mi \) is the average marginal income tax rate of the consumers of product \( i \), \( \bar{T}_i \) is the corresponding average income tax rate. \( \gamma \) is a general adjustment factor. It ensures that total tax revenue from (A.11), with the \( \bar{T}_mi \) term added, will equal to the combined total direct and indirect tax revenue from the iterations when income tax is included. It follows from this total revenue-equalising definition of \( \gamma \) that the average commodity tax rate from the iterations and the average predicted from (A.15) will be the same. \( \gamma \) is usually below plus or minus 0.15, indicating that the optimal combined tax rate (\( t_i + \bar{T}_i \)) on the consumer group of product \( i \) in the presence of income tax is not much different from the optimal tax rate without it.
Given that (A.11) and (A.15) are used in some analytical explanations, it is of some interest to compare numerical results obtained from these approximations with results from iterations based on (6). Table A.1 displays such comparisons. While some of the predictions from the modified inverse elasticity rule are substantially different from those of the iterations, the results from predictions and iterations are strongly correlated. Despite the differences, these results appear to be sufficiently close to justify the application of the modified inverse elasticity rule for analytical purposes. In the following discussion, when working out approximations for various factors, we use the modified inverse elasticity rule without income tax (A.11), because it is simpler. The same expressions would be obtained by applying the rule with income tax (A.15).

### Table A.1

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<th>With income tax</th>
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<td>1.70</td>
<td>2.12</td>
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<tr>
<td>Average</td>
<td>0.94</td>
<td>0.98</td>
<td>0.31</td>
</tr>
</tbody>
</table>

**Compliance costs**

We start the discussion with compliance costs that are borne by the taxpayer. The definition of these costs in section 5.1 eq. (16) is: $C_i = c_i q_i t_i$. Taking the indirect utility definition of the social welfare function in (1), then the dead-weight costs defined in (16) can be incorporated into the social welfare function by subtracting them from lump-sum income ‘y’. The extended social welfare function will be:
\[ U = \sum_h a_h u_h (\mathbf{p}, W_t, \nu_{oh} - \sum_i c_i q_{ih} t_i) \]  

(A.16)

Applying the modified inverse elasticity rule to (A.16), we may notice that the presence of \( c_i \) has a first order effect only on the personal utility term shown in (A.1). Deriving (A.16) with respect to \( t_i \) we obtain:

\[ \sum_h u_h(t_i, \mathbf{p}) \frac{\partial u_h}{\partial t_i} = -\sum_h a_h \frac{\partial u_h}{\partial y} - \sum_h a_h \left( c_i q_{ih} + t_i \frac{\partial q_{ih}}{\partial t_i} \right) \]  

(A.17)

Combining (A.17) with the unchanged (A.6) and (A.8) terms and performing the operations from (A.4) to (A.10), we arrive at the following extended version of the modified inverse elasticity rule:

\[ t_i \approx \frac{1 + t_i + [(1 + t_i) + \bar{e}_i t_i] c_i - \bar{u}_m (1 + t_i + \bar{e}_i (t_i - \bar{t}_{ih}) + \bar{e}_{ih} \bar{T}_i')}{\bar{e}_i (u_0)} \]  

(A.18)

Subtracting (A.11) from (A.18) yields the estimated impact of \( c_i \) on the optimal tax rate:

\[ \Delta t_i(c_i) \approx \frac{1 + t_i (1 + \bar{e}_i) c_i}{\bar{e}_i (u_0)} \]  

(A.19)

Notice that in this subtraction we implicitly assumed that marginal utilities and demand derivatives with and without \( c_i \) are the same. This is not strictly correct, but judging from the numerical results, this assumption does not distort by much the approximations.

**Administration costs**

Similar exercise can be carried out in regard to the impact of \( s_i \) on the optimal tax rate. With \( s_i \) public revenue will be affected. Assuming fixed expenditure on public goods \( (R_0) \), the net revenue available for redistribution will be:

\[ R = \sum_i (q_{ih} t_i - q_{ih} t_i s_i) - R_0 \]  

(A.20)

Deriving (A.20) with respect to \( t_i \), we obtain a revised transfer term instead of (A.6)

\[ \frac{\partial U}{\partial b} \frac{\partial R}{\partial t_i} = \frac{\partial U}{\partial b} \left[ \sum_h (q_{ih} - s_i q_{ih} + t_i \frac{\partial q_{ih}}{\partial p_i} + \frac{\partial q_{ih}}{\partial p_i} (t_i - \bar{t}_{ih}) + \frac{\bar{e}_{ih}(f) m}{p_i} \bar{T}_i') \right] \]  

(A.21)

Combining (A.21) with (A.1) and (A.8) and performing the operations from (A.6) to (A.10), we arrive at the following extended version of the modified inverse elasticity rule:

\[ t_i \approx \frac{1 + t_i + \bar{u}_m (1 + t_i - s_i (1 + t_i) + \bar{e}_i t_i + \bar{e}_i (t_i - \bar{t}_{ih}) + \bar{e}_{ih} \bar{T}_i')}{\bar{e}_i (u_0)} \]  

(A.22)

Subtracting (A.11) from (A.22) yields the estimated impact of \( s_i \) on the optimal tax rate:

\[ \Delta t_i(s_i) \approx \frac{s_i u_m [(1 + t_i) + \bar{e}_i t_i]}{\bar{e}_i (u_0)} \]  

(A.23)
Externalities

The starting point for deriving the formula for externalities is the extended social welfare function defined by (26) in section 5.4. In order not to complicate the analysis with cross-price effects, we shall concentrate here only on a single externality. In this case the extended social welfare function will be:

\[ U = \sum_h a_h u_h(\mathbf{p}, W_h [r_{oh} + \sum_h z_h \mu_i Q_i]) \]  

(A.24)

where \( Q_i = \sum_h q_{ih} \)

Applying the modified inverse elasticity rule to (A.24), by deriving it with respect to \( t_i \) we obtain:

\[ \sum_h \frac{\partial u_h}{\partial t_i} = -\sum_h q_{ih} \frac{\partial u_h}{\partial y} + \mu_i \sum_h z_h \frac{\partial u_h}{\partial y} \frac{\partial Q_i}{\partial t_i} \]  

(A.25)

Notice that the externality term multiplying \( \mu_i \) is opposite in sign to the utility derivative based on Roy’s Lemma. Combining (A.25) with the unchanged (A.6) and (A.8) terms and performing the operations from (A.6) to (A.10), we arrive at the following extended version of the modified inverse elasticity rule:

\[ \sum_h \frac{\partial u_h}{\partial y} + \frac{\sum_h z_h \frac{\partial u_h}{\partial y}}{\mu_i} \frac{1 + t_i - \bar{e}_i \mu_i}{\bar{u}_m (1 + t_i + \bar{e}_i (t_i - t_{\alpha}) + \bar{e}_i \bar{T}_i')} \]  

(A.26)

Subtracting (A.11) from (A.26) yields the estimated impact of \( \mu_i \) on the optimal tax rate:

\[ \Delta t_i(\mu_i) = -\frac{\bar{e}_i \mu_i \sum_h z_h \frac{\partial u_i}{\partial y}}{\bar{e}_i(u_0) \frac{\partial u_i}{\partial y}} \]  

(A.27)

The numerator contains the Pigovian term multiplied by the weighted marginal utilities of income of externality recipients. The denominator is the average marginal utility of product \( i \) defined in (A.13) multiplied by \( \bar{e}_i(u_0) \).

Appendix 3

Further details about the setup of the numerical model

In this appendix we present more details about the model described in section 3.
C.1. Segmented utility

First, we examine the segmented utility framework. As noted in section 3, in order to establish non-linear Engel curves using LES, nine commodities represent necessities and another nine with different utility parameters represent luxuries. LES being a linear Engel curves demand system, does not ensure automatically that goods defined by (4) will be non-negative. Non-negative consumption at low income levels (say when \( m = b \)) is possible only if the sum total of the positive intercept terms \( \alpha_i \) in (4)) is smaller than \( b \), because the quantities demanded are always greater than \( \alpha_i \). Large negative intercepts are also not compatible with non-negative demand according to (4), when \( w \) and \( y \) are small. Thus, in order to ensure non-negative demand for all goods for consumers at low income levels, the intercept terms of necessities have to be small compared with average income. This has been confirmed by numerical results reported in Revesz (1997). But small intercept terms imply near homothetic preferences (when all \( \alpha_i = 0 \)), hence the utility function of necessities has to be nearly homothetic. With luxuries the intercepts can assume larger positive or negative values without violating the non-negativity requirement, because the earning parameters of consumers (confined only to higher wage earners) are larger. Consequently, the utility function of higher wage earners can represent a wide range of income elasticities. Notice that in this model the consumption of luxury goods starts at intermediate income levels, which is not an entirely unrealistic assumption. This arrangement also ensures that nearly homothetic preferences are confined only to low wage earners.

Segmented LES represents non-identical preferences but not heterogeneous preferences as defined in Saez (2002). Saez examines a model with multiple characteristics where consumers at the same income level choose different bundles of goods. In the present 15 person model there is only one consumer at each income level. Nonetheless, segmented utility does not accord with the assumption of identical preferences adopted in the uniform commodity taxation theorems.

Revesz (1997) reported results from a 9 commodity LES model where no segmentation was used, and all 15 taxpayers consumed only the 9 necessities and shared all utility parameters. The purpose was to test Deaton (1979) theorem that under weakly separable utility, linear income tax and linear Engel curves, optimal commodity tax rates will be uniform. This has been indeed confirmed by the numerical results, but in order to avoid negative demand for any good, the 9 commodity linear Engel curves preferences had to be made nearly homothetic, which is not supported by empirical evidence. In this paper we are concerned only with results from the 18 commodity non-linear Engel curves model.

C.2. The cardinalisation of segmented utility

The application of two different utility functions for low and high wage earners raises some difficulties with cardinalisation. Since utility is an endogenous variable, the cardinal utility of the ninth taxpayer can sometimes turn out to be smaller than that of the eighth tax-
payers who have a lower wage rate. In other cases, there can be a large positive difference between the two. In order to ensure that utility is a smooth function of the wage rate, the program calculates and adds an adjusting constant to the utilities of all members of the higher wage group. Since the same constant is added to the utilities of all members of the second group in the uniform and non-uniform calculations, the utility outcomes in the two situations are comparable. However, the cardinalisation problem makes it difficult to carry out utility comparisons between different scenarios.

C.3. Linear income tax

Given zero homogeneous utility and demand in income and prices, in a model without income tax, a portion of commodity taxes can always be converted into a proportional income tax, without affecting utility or demand. Hence a model without income tax is really a commodity cum linear income-tax model and not purely a commodity tax model. To elaborate on this point, suppose that initially a labour input based redistributive model contains only commodity taxes but no income tax. In this situation, the indirect utility function will be:

\[ u (p, w, y) = u (1 + t_0, W, b) \]  \hspace{1cm} (C.1)

where \( t_0 \) are the initial vector of commodity taxes. Now, let us reduce some commodity taxes and convert them into income tax, \( T \). Suppose that this conversion is done by providing a uniform reduction to all gross prices at the rate \( r \), which is offset by a proportional income tax with a constant marginal income tax \( \ell r \). In order to ensure the same proportional changes of all the terms in (C.1), the demogrant \( b \) also has to be reduced by \( r \). After these adjustments \( u = u[(1 + t_0)(1 – r), W(1 – r), b(1 – r)] \), which yields exactly the same outcomes as \( u \) in (C.1), because of zero homogeneous utility and demand in prices and earning parameters. Following these changes, the commodity tax rates will be: \( t_{11} = (1 + t_{10})(1 – r) – 1 \) and the linear income tax function will be: \( T = rW\ell – rb \). In fact, there are an infinite number of possible linear income tax functions and commodity tax rates defined by different \( r \)'s that will lead to identical outcomes. In the particular case when \( r = 0 \), we have only commodity taxes in the model, which brings us back to our earlier assertion that the baseline model is actually a commodity cum linear income tax model and not purely a commodity tax model.

C.4. Inequality aversion rate

From equations (1) and (3), an easy way to define the social welfare function is to let all \( a_h = 1 \), which implies that \( U \) is represented by the sum of LES utilities. But of course, a broader range of scenarios can be examined by letting the \( a_h \)'s to represent different political value judgments. Here we shall follow an inequality aversion rate approach, based on declining marginal utilities of income. Define the LES utility of taxpayer \( i \) as \( u_i \) with corresponding marginal utility of income \( u_{mi} \), the social welfare function is \( U \) and the inequality
aversion rate is \( z \). Now, a preliminary run with a selected uniform commodity tax rate yielded initial estimates for marginal utilities of income denoted \( u_{m0} \). Taking the inverse of \( u_{m0} \) and multiplying it by a scaling factor ‘c’ to sum up to 15, for the number of taxpayers involved, we obtain fixed social welfare function weights, \( g_i = c/u_{m0i} \). These weights are approximately inversely related to the marginal utility of income of each taxpayer according to LES. Now, let us define the social welfare function as:

\[
U = \sum_i u_i z + u_i (1 - z) c/u_{m0i}
\]  

(C.2)

According to this definition, when \( z = 1 \) then we are back to LES utility. But when \( z = 0 \), then the social welfare function will have nearly constant weighted marginal utilities of income (\( c u_{mi}/u_{m0i} \)), describing a situation where egalitarian objectives are absent. Most of the numerical results discussed in this paper pertain to the original LES utility (\( z = 1 \)), but in some cases a lower inequality aversion rate (0.3) is also examined.

C.5. Compensated elasticity of labour supply

It should be noted that in static redistributive models the optimal tax burden depends not only on the inequality aversion rate but also on the average compensated elasticity of labour supply. The optimal average tax rate is negatively related to the average compensated elasticity of labour supply [see Stern (1976), Tuomala (1984), Revesz (1989), Saez (2001)]. While this item is part of the model (see eq. (5)), apart from setting various values for \( \beta_L \), no attempt has been made here to study systematically the effect of this endogenous variable on optimal tax rates.

C.6. Parameters employed in the model

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<th>High income group</th>
<th>Taxpayer wage rate</th>
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### Table C.1 (Continued)
**LIST OF UTILITY PARAMETERS AND WAGE RATES**

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<th>wage rate</th>
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### Table C.2
**LIST OF UTILITY PARAMETERS IN THE TWO COMPOSITE GOODS MODEL**

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### Appendix 4

**Table D.1**
**OPTIMAL COMMODITY TAX RATES WITH PROGRESSIVE INCOME TAX**

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Table D.1 (Continued)

OPTIMAL COMMODITY TAX RATES WITH PROGRESSIVE INCOME TAX
Extension of Table 2

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<td>–0.94</td>
<td>–0.11</td>
</tr>
<tr>
<td>17</td>
<td>2.26</td>
<td>–1.45</td>
<td>0.57</td>
<td>1.53</td>
<td>–1.21</td>
<td>–0.17</td>
</tr>
<tr>
<td>18</td>
<td>2.16</td>
<td>–1.13</td>
<td>0.83</td>
<td>1.53</td>
<td>–1.05</td>
<td>–0.14</td>
</tr>
</tbody>
</table>

Table D.2 shows average optimal tax rates corresponding to exogenously given demogrants. The fixed demogrants were calculated using the average income level in the zero tax model. When taxation is present, due to the disincentive effect of the demogrant average incomes will decrease, hence the demogrant to average income ratios will be higher, as shown in the bottom row. The scenarios reported in Table D.2 are without income tax, expenditure on public goods amounts to 10% of total income under zero tax and the inequality aversion rate is one.

<table>
<thead>
<tr>
<th>Table D.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>OPTIMAL TAX RATES WITH SUB-OPTIMAL DEMOGRANTS</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Ratio of demogrant to average income in the pre-tax situation</th>
<th>optimal rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>20%</td>
</tr>
<tr>
<td>Average tax on necessities</td>
<td>0.05</td>
</tr>
<tr>
<td>Average tax on luxuries</td>
<td>0.22</td>
</tr>
<tr>
<td>Ratio tax luxuries over necessities</td>
<td>4.4</td>
</tr>
<tr>
<td>% demogrant to average income</td>
<td>0</td>
</tr>
</tbody>
</table>

Appendix 5

The mathematical framework with leisure complements and substitutes

This appendix examines two separate issues. First, we derive the modified inverse elasticity approximation for leisure complements and substitutes presented in eq. (34) in the text. Following this, we outline the extended LES utility function that was used to obtain the numerical results reported in Tables 9 and 10.

In regard to the modified inverse elasticity rule, we may notice that among the three components analysed in appendix 2, only the second component, that is tax revenue, is visibly affected by the presence of $\epsilon$’s. Roy’s Lemma and the excess burden term are not affect-
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ed in any obvious way. For the purpose of obtaining a simple expression for the revenue term, we find it useful to redefine $\epsilon_i$ from (32) as:

$$\epsilon_i = \frac{\partial \ell}{\partial p_i} = \frac{\partial \ell}{\partial q_i} \frac{\partial q_i}{\partial p_i} \quad (E.1)$$

Now, the revised revenue term (based on (A.6)) will be:

$$\frac{\partial U}{\partial b} \frac{\partial R}{\partial t_i} = \frac{\partial U}{\partial b} \sum_h (q_{ih} W_i T_h \frac{\partial \ell}{\partial p_i} + W_i T'_h \frac{\partial \ell}{\partial q_i} \frac{\partial q_i}{\partial p_i} + \frac{\partial q_i}{\partial p_i} (t_i - t_{iA})) \quad (E.2)$$

$W_h$ is the wage rate of taxpayer $h$ and $T'_h$ is the marginal indirect tax rate of consumer $h$ (see A.5b). The first labour term in (E.2) refers to the labour supply derivative under weakly separable utility, as defined in (A.5a). The second term defines the effect of deviation from weak separability on the derivative of labour supply, as defined in (32). It represents the change in labour income due to non-separability multiplied by the marginal indirect tax rate, yielding the change in tax revenue due to the $\epsilon_i$ factor.

Combining (E.2) with the unchanged (A.1) and (A.8) terms and performing the operations from (A.5) to (A.10), we arrive at the following extended version of the modified inverse elasticity rule:

$$t_i(\ell) \approx \frac{1 + t_i - \bar{u}_i}{(1 + \epsilon_i \bar{u}_i) - T_i'} + \frac{\frac{\partial \ell}{\partial q_i} T_i' + \frac{\partial \ell}{\partial p_i} W_i T_i' + \epsilon_i (t_i - t_{iA})}{\epsilon_i (u_0)} \quad (E.3)$$

Subtracting (A.11) from (E.3) and cancelling the similar price elasticity terms in the numerator and denominator, we obtain:

$$\Delta t_i(\ell) \approx \frac{-\frac{\partial \ell}{\partial q_i} p_{iW} T_i W_i}{(E.4)}$$

where all terms on the right hand side represent $q_i$ weighted average values.

Extended LES

To obtain the numerical results in Tables 9 and 10, we used an extended version of LES

With extended LES labour supply in (5) is given as:

$$\ell = Z - q_L = Z(1 - \beta_L) - \alpha_L - \beta_L \frac{p_j q_j - \sum_j k_j q_j}{w - \sum_j p_j \alpha_j} - \sum_j k_j q_j \quad (E.5)$$

where the $k_j$s are constants and $-\sum_j k_j q_j$ is added as an extra term.

This corresponds to the redefinition of utility in (3) as:

$$u = \sum_i \beta_i \log(q_i - \alpha_i) + \beta_L \log(q_{L0} + \sum_i k_i q_i - \alpha_L) \quad (E.6)$$
where $q_{L0}$ is the value of leisure under weakly separable utility, that is in the absence of the $\sum k_i q_i$ term. Notice that from definitions (E.1) and (E.5), \[
\frac{\partial \ell}{\partial q_i} = -k_i
\]

Obviously, the $k_i$s introduce an inter-dependency between commodities and leisure. It is not clear whether (E.6) can be solved to yield explicit global formulae for commodity demand and labour supply. But even without global formulas, it is possible to obtain local formulae in the neighbourhood of a given vector $q_{io}$ obtained from the weakly separable utility model. Suppose the $k_i$s satisfy the following initial condition:

\[
\sum_i k_i q_{i0} = 0 \tag{E.7}
\]

From (E.7) it follows that (E.5) will continue to satisfy zero homogeneous labour supply at $q_{io}$. Moreover, the consumer will be able to make the same choices as under weakly separable utility. But actually the consumer will not choose the previous basket of goods and leisure. Given condition (E.5) his/her new choice of leisure will be:

\[
q_{L} = q_{L0} + \sum_i k_i (q_i - q_{i0}) \tag{E.8}
\]

Given different labour supply ($\ell = Z - q_L$), the consumer’s income will change. To balance the budget constraint, the net change in expenditure will be:

\[
W(\ell - \ell_0) = W\Delta \ell = \sum p_i (q_i - q_{i0}) = \sum p_i \Delta q_i \tag{E.9}
\]

To determine $\Delta q_i$, we employed the marginal consumption propensities $\partial q_i / \partial y$.

From the LES demand equation in (4):

\[
\frac{\partial q_i}{\partial y} = \beta_i / p_i \tag{E.10}
\]

After taking out $\beta_L$ and inflating the $\beta_i$s of commodities by $1 / (1 - \beta_L)$ to equal one, we obtain from (E.10):

\[
\sum_i \frac{\partial q_i}{\partial y} p_i = \sum_i \frac{\hat{\beta_i}}{p_i} p_i = \sum_i \hat{\beta_i} = 1 \tag{E.11}
\]

where $\hat{\beta_i}$ are the recalibrated values of $\beta_i$ for commodities. To maintain proportionality between $\partial q_i / \partial y$ and $\Delta q_i$ and ensure that the budget constraint (E.9) is satisfied:

\[
\Delta q_i = W(\ell - \ell_0) \frac{\hat{\beta_i}}{p_i} = W(\ell - \ell_0) \frac{\hat{\beta_i}}{1 + t_i} \tag{E.12}
\]

Adding the values found for $\Delta \ell$ from (E.8) and for $\Delta q_i$ from (E.12) to the original $q_{L0}$ and $q_{io}$ variables, yields new values for commodities and leisure that can be used to evaluate the utility in (E.6). From then on, the calculations follow the procedure described in section 3.

Although the redefinition of LES presented here is fairly ad hoc, it should be noted that the resulting demand system perfectly satisfies the budget constraint and almost perfectly
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satisfies zero homogeneity of demand and labour supply near the point \( q_{qio} \). This approach was adopted in order to enable numerical testing of the impact of leisure complements and substitutes, without having to write a new program. Arguably, non-separable indirect utility functions, such as those used in the computational studies of Ebrahimi and Heady (1988) and Murty and Ray (1987) are better suited for testing the effect of leisure non-separability on optimal tax rates, but that challenge is left for future research.

Notes

1. Progressive indirect taxation in this paper refers to higher proportional taxes on luxuries and lower taxes or subsidies on necessities. We are not dealing in this paper with non-linear taxes on commodities, which appear in some parts of the theoretical literature.

2. The Laroque-Kaplow proposition (discussed in section 4.5) suggests that the Atkinson-Stiglitz result can be extended to non-optimal income tax schedules.

3. I have also published some papers criticising the proposition about uniform taxation for redistributive purposes [see Revesz (1986, 1997, 2005, 2014)]. The main objections raised in these papers will reappear in a different form here.

4. It should be noted that apart from Deaton and Stern (1986), Ebrahimi and Heady (1988), Naito (1999), Saez (2002) and Jacobs and Boadway (2014), all these mathematical studies rely on the Stiglitz (1982) self-selection model. This is a two-person or a few persons (or groups) approach to tax optimisation that can yield analytical insights. It has been used in a computational model by Bastani et al. (2013).

5. Notice that in this substitution exercise we replace a second-best instrument (VAT concessions) by another instrument (selective welfare payments) that at least in theory could approach the effectiveness of a first-best tax/transfer instrument based on ability. Selective welfare payments are discussed in more detail in section 4.4.

6. More will be said on present VAT systems from a theoretical perspective in section 4.3.

7. LES appears in many publications. For one, see Thomas (1987). The present version uses instead of income \((m)\) the earning parameters \(w\) and \(y\), as well as total time \((Z)\) and leisure \((L)\).

8. With LES utility constraint \((2b)\) is redundant. The LES demand equations are defined so that the budget equation \(\sum p_i q_i - (W \ell + b) = 0\) is always satisfied. Summing over \(h\), it is not difficult to see that in this situation provided constraint \((2a)\) is satisfied so will \((2b)\).

9. Whether these should be referred to as heterogeneous preferences or otherwise is discussed in section C.1 in appendix 3.

10. The indirect utility functions employed in earlier computational optimal commodity tax studies involving labour supply [Ebrahimi and Heady (1988), Murty and Ray (1987)], can represent non-linear Engel curves, but at a more limited range than segmented utility.

11. The convergence proved to be fairly rapid in models without income tax. When income tax is present, the convergence is generally slower and more erratic.

12. The approximate formula for optimal tax rates developed by Sandmo (1975) also suggests that in a model without income tax, optimal tax rates tend to be progressive. This formula is presented in appendix 2 eq. (A.14).

13. Their model is similar to the present one, in the sense that they employ linear Engel curves demand that is not identical across all households.

14. The main source for disincentives in the commodity tax model is the demogrant, which being lump-sum income will reduce labour supply. The demogrant also reduces labour supply in income tax models, in addition to other disincentives.
15. These calculations are also included in the current version of the program but are not reported here in order to save on space.

16. Another weakness in these models is the absence of variable labour supply. As far as I can see, apart from Murty and Ray (1987), labour supply does not play a role in these models.

17. Note the small differences between $\tilde{\varepsilon}_i$ and $\tilde{\varepsilon}_i(u_0)$ in Table 4. With LES utility the difference between the two is $\tilde{\beta}_i$. Given the condition that $\sum_i \tilde{\beta}_i = 1$, then with 18 goods plus leisure that means that on the average $\tilde{\beta}_i$ is below 0.06, and so is the difference between $\tilde{\varepsilon}_i$ and $\tilde{\varepsilon}_i(u_0)$.

18. We are not dealing here with a traditional tax evasion model concerned with evasion, frequency of auditing, penalties and probabilities of detection. For a review of this literature refer to Myles (1995) and Hindriks and Myles (2006). Cremer and Gahvari (1993), cited in section 2.1, also follow this approach.

19. There is no need to assume that all consumers will benefit from evasion. As long as the gains are spread randomly within the group, the average marginal utility of beneficiaries will be the same as that of the consumer group as a whole.


21. Sandmo (1975) uses similar specifications to the present model, including no income tax and a single externality tax. Sandmo did not claim that the optimal pollution tax rate will usually exceed the Pigovian rate in the absence of income tax, because he did not test numerically his analytical model.

22. With weakly separable utility demand is independent of the composition of $m$ in terms of $w\ell$ and $y$ (see Revesz 1986). Therefore, $\frac{\partial q_i}{\partial m} = \frac{\partial q_i}{\partial y}$

23. It is likely that factors with large variations between products, such as externalities, leisure complementarities or evasion propensities, will have a larger effect on tax rate differentiation than factors with smaller differences between products, such as administrative and compliance costs.

24. In fact, the utility parameters for the low wage group (see Table C.1) are based on econometric estimates by Deaton, and Muellbauer (1980). For further details see Revesz (1997). The parameters chosen for the higher wage group are arbitrary.

25. As explained in section 2.3, for the purpose of providing support to those at the bottom end of the scale, the targeting and monitoring of welfare payments and in-kind transfers is probably a more critical economic issue than the taxation of necessities.

26. Note, here utility is derived with respect to lump-sum income ($y$) rather than income ($m = w\ell + y$), but with weakly separable utility the two derivatives are the same [see Revesz (1986)].

References


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Resumen

Este estudio examina la estructura óptima de tipos impositivos aplicados a los productos básicos con un modelo computacional estático "muchos agentes, muchos bienes" usando modelos segmentados de utilidad LES (Sistema de Gasto Lineal). Uno de las principales hallazgos es que con curvas de Engel no lineales y el impuesto sobre la renta lineal, los tipos impositivos óptimos sobre productos básicos serán progresivas y altamente dispersas bajo especificaciones de utilidad logarítmica. La dispersión de los tipos impositivos se reduce si la tasa de aversión a la desigualdad de la sociedad es baja. Con escalas del impuesto sobre la renta no optimas y no lineales exógenamente dadas, usualmente hay todavía necesidad de imposición indirecta diferenciada y progresiva. Estos resultados están en evidente contraste con la continua preocupación, de gran parte de la literatura, por la tributación indirecta uniforme con propósitos redistributivos. Los resultados también indican que si la evasión fiscal incurre en costes muertos sustanciales, generalmente se reducen los tipos impositivos óptimos en más de la mitad de la proporción evasión/ingreso del producto, con una reducción que es mayor para necesidades y menor para los lujos. Los costes de cumplimiento privados y los costes de gestión del gobierno reducen los tipos impositivos óptimos en una cantidad similar a la cuota de estos costes en los impuestos. En un modelo de impuesto sobre la renta lineal, el efecto de las externalidades sobre las tasas impositivas óptimas supera sustancialmente las correspondientes tipos impositivos pigouvianos o subsidios. El principal beneficio de impuestos más altos sobre los complementos de ocio que sobre los substitutos de ocio, aparenta impulsar los ingresos fiscales de redistribución, en lugar de mejorar la posición de utilidad de los contribuyentes. El efecto de complejidades tales como la evasión de impuestos, los costes administrativos, las externalidades y los complementos/sustitutos de ocio sobre la redistribución no es neutral. Generalmente, estos factores tienden a incrementar la progresividad de los tipos impositivos óptimos sobre productos básicos.

Palabras clave: Tributación óptima, tributación de los productos básicos, impuestos indirectos, evasión fiscal, externalidades, bienes preferentes.