The Economics of Advice with Endogenous Information

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Abstract

This paper extends the standard cheap talk, optimal delegation and centralization models to the case where the agent’s private information is endogenously learned instead of given by nature. We first assume overt learning. We find that compared to centralization, cheap talk generates under-investment in learning, while optimal delegation generates over-investment in learning. The principal either delegate both the learning task and decision authority to the agent, or retain both. It is never optimal for her to delegate one and retain the other. We then assume covert learning. We find that in the cheap talk model with covert learning, the agent invests less in learning and both parties have lower welfare than when learning is overt. In the optimal delegation model with covert learning, the principal has to eliminate a set of “intermediate” projects in order to induce an ex ante uninformed agent to learn.

Keywords: cheap talk, strategic communication, optimal delegation, endogenous information.

JEL Numbers: C70, D82, D83, D86.

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1 Introduction

Institutions and individuals (principals) rely heavily on the advice of experts (agents) in making decisions, because of their particular knowledge, information and skills. One characteristic of the interaction between an uninformed principal and an informed agent is that the agent is often biased – the objectives of the agent often diverge from those of the principal. For instance, investors in financial markets often rely on the advice of financial advisers and mutual fund managers to decide which financial products to purchase. While investors always want to choose the products that best fit their needs, the interests of financial advisers and fund managers may be distorted by kickbacks, fees or commissions. As a result of the biased interests, financial advisers and fund managers may strategically distort their information by recommending products that maximize their own payoff.

Much research has been devoted to studying the consequence of information asymmetry and divergent interests on decision making and a number of solutions have been suggested to improve the efficiency of decision making. Among the main predictions of the literature is the “delegation principle,” which says that “the power to make decisions should reside in the hands of those with the relevant information” (Krishna and Morgan, 2008. See also Ottaviani, 2000 and Dessein, 2002). In theoretical words, optimal delegation dominates cheap talk.

This “delegation principle” is built upon a standard assumption that the agent’s private information is endowed and perfect, and the only information channel available to the principal is to consult the agent. Real life situations, however, often seem to be different. The agent has no superior information ex ante, and both parties are able to gain relevant information through a costly learning process. In particular, a principal can either learn information directly and make informed decisions without consulting the agent, or assign the learning task to the agent. In the later case, the principal can further decide whether
to retain the decision authority or delegate it to the agent. For instance, an investor in the financial market can make investment decisions by him/herself alone, or seek help from an expert. In the later case, the investor can further choose to invest in a mutual fund and delegate the decision authority to the fund manager, or to consult a financial advisor and retain the decision authority.

Questions arise naturally in these situations: Does the standard “delegation principle” still hold when the agent’s information is endogenously learned (instead of given by nature) and imperfect? If the principal can invest in learning directly, should she allocate the learning task to the agent or retain it?

To answer these and related questions, this paper extends the cheap talk model of Crawford and Sobel (1982) and the optimal delegation model of Holmstrom (1984) to the case where the agent’s private information is endogenously learned. A third model studied in this paper is the centralization model where the principal invests in learning directly and makes informed decision without consulting the agent. In the language of organizational theory, the three schemes or models correspond to different organizational structures: In the cheap talk scheme the principal allocates the learning task to the agent but retains the decision authority. In the optimal delegation scheme the principal delegates both the learning task and decision authority to the agent and imposes some constraints on the choice of projects. In the centralization scheme the principal retains both the learning task and the decision authority. In all three schemes, learning is modeled as a choice of costly effort that increases the informativeness (or quality) of a signal the learner (principal or agent) privately receives.

We first study the three models under the overt learning assumption, by which the agent’s learning behavior is observed by the principal. In particular, we rank the three models in terms of the learning effort and the principal’s welfare at the equilibrium. We then study the cheap talk and optimal delegation models under the covert learning assumption, by which the agent’s invest in learning is unobservable to the principal. In particular, we contrast the
cheap talk equilibrium under covert learning to that under overt learning. We also contrast the optimal delegation set under covert learning to that under overt learning.

There are many real life examples of overt and covert learning. For instance, the education levels and working experience of financial advisers and mutual fund managers are often observable to their clients. However, their time and effort spent on a particular financial product are often unobservable by their clients.

In the overt learning case, we find that if the agent is highly biased, little informed ex ante and inefficient in learning, then he has no incentive to learn and can not credibly reveal information. This result holds for both the cheap talk and optimal delegation schemes, suggesting that no matter whether an investor chooses to consult a financial adviser or to delegate the investment decision to a mutual fund manager, the investor should always choose a “real” expert who is little biased, well educated, highly experienced and efficient in obtaining new information.

Compared to centralization, the cheap talk scheme generates under-investment in learning, while optimal delegation results in over-investment in learning. Unlike centralization, in the cheap talk scheme much information is lost in communication, which weakens the agent’s incentives to learn. On the other hand, little information is lost in the optimal delegation scheme. In addition, in the optimal delegation scheme, the more the agent invests in learning, the less restricted he is in selecting the project. This mechanism gives the agent additional incentives to learn relative to the centralization scheme.

Optimal delegation always dominates cheap talk. That is, the standard “delegation principle” still holds when information is endogenously learned and learning is overt. The reason is the same as in the exogenous information case: The principal prefers biased but informed decisions (in optimal delegation) to unbiased and uninformed decisions (in cheap talk). On the other hand, optimal delegation may or may not be preferred to centralization from the principal’s perspective: optimal delegation is better if preference bias is small and
learning is inefficient. In all cases, the principal either delegates the learning task and the
decision authority or retains both. It is never optimal for the principal to delegate one and
retain the other.

Observability of the learning process, or the lack thereof, impacts the outcome of the
cheap talk and optimal delegation schemes. In the cheap talk scheme with covert learning,
the agent invests less in learning and both parties have lower welfare than when learning
is overt. In short, cheap talk with overt learning Pareto dominates cheap talk with covert
learning.

The optimal delegation scheme in the covert learning case shares some similarities with
that in the overt learning case. For instance, in both cases the optimal delegation set –
within which the agent can freely choose his preferred projects – has no lower bound but has
a binding upper threshold. The upper threshold, however, is higher in the covert learning
case, in order to optimally induce the agent to learn. In addition, when learning is covert, to
motivate an ex ante uninformed agent to learn requires elimination of a set of “intermediate”
projects, leaving the agent with only extreme choices.

1.1 Literature Review

Different models have been developed in the literature with different combinations of assump-
tions on the principal’s commitment power, as shown in Table 1. The cheap talk model was
introduced by Crawford and Sobel(1982) and further studied by Ottaviani (2000), Krishna
and Morgan (2001 a, b; 2004), Ottaviani and Sorensen (2006 a, b) and Inderst and Otta-
viani (2009, 2011). The optimal delegation model was first studied by Holmstrom (1984),
followed by Melamud and Shibano (1991), Ottaviani (2000), Dessein (2002) and Alonso and
and full commitment models. In this paper, we focus on the cheap talk and delegation cases,
by abstracting away from the possibility of monetary transfers.

Table 1: Assumptions of the principal’s commitment power in the literature

<table>
<thead>
<tr>
<th>Commit to monetary compensations</th>
<th>Yes</th>
<th>No</th>
</tr>
</thead>
<tbody>
<tr>
<td>Commit to project selection</td>
<td>Yes</td>
<td>Full commitment</td>
</tr>
<tr>
<td></td>
<td>No</td>
<td>Compensation contract</td>
</tr>
</tbody>
</table>

A more recent strand of literature is concerned with the economic implications of endogenous information in different schemes. Closely related to the current study are Ottaviani (2000), Szalay (2005) and Argenziano, Severinov, and Squintani (2011). By introducing the possibility of a naive principal, Ottaviani (2000) proves the existence of full revealing equilibria in the cheap talk model and that the learning effort is not affected by the observability/unobservability of learning activity. By assuming a sophisticated principal, this paper instead shows that loss of information is inevitable in the cheap talk model and the agent invests less in learning if learning is covert instead of overt. This paper differs from Szalay (2005) in two main aspects: First, Szalay (2005) focuses on delegation with covert learning, while we cover both communication and delegation models with overt and covert learning. Second, by assuming no conflict of interest between the principal and agent, Szalay (2005) abstracts from the miscommunication problem and focuses on the information acquisition problem. The present paper assumes a constant conflict of interest, and therefore studies both problems. We find that some properties of the optimal delegation set in Szalay (2005) extend to the current context (i.e., removal of intermediate choices) and some don’t (i.e., unlimited extreme choices). This paper also differs from Argenziano, Severinov, and Squintani (2011) in three main aspects: First, they focus on the cheap talk scheme. Second, they model learning as discrete Bernoulli experiments, while we assume continuous learning effort. Third, they assume that the babbling equilibrium is played off equilibrium path. We instead assume that the most informative equilibrium is always played and find that their
main results – i.e., overinvestment in learning and dominance of cheap talk over delegation – do not survive this extension.

The remainder of the paper is organized as follows. Section 2 sets up the model. Section 3 studies the over learning case, and Section 4 studies the covert learning case. Section 5 concludes. Proofs are collected in the Appendices.

2 The Model

There are a principal and an agent who have to make a decision on projects together. The principal has the legal right to choose the project $y \in R$, while the agent has the ability (or channel) to acquire information about a state of nature $\theta \in [\theta, \bar{\theta}]$, which is necessary to choose the “right” project.

The payoff function of the principal is $u(y, \theta) \equiv -(y-\theta)^2$. With a slight abuse of notation, the agent’s payoff is $u(y, \theta, b) \equiv -(y - \theta - b)^2$, where $b$ is a bias parameter measuring the congruence of the parties’ preferences, and is commonly known. Neither the principal nor the agent know the precise value of $\theta$, but they share a common prior about it.

The agent may learn additional information about $\theta$. Learning is modeled as a choice of effort $a \in [0, \overline{a}]$, that increases the informativeness (or quality) of a one-dimensional signal $S$ he privately receives. We consider both the overt learning case where $a$ is observed by the principal, and the covert learning case where $a$ is unobservable to the principal. The agent’s private cost of effort is common knowledge: $c(a) = \kappa a^2$. Without loss of generality, we assume $\kappa = 1$.

The marginal distributions of $\theta$ and $S$ are both independent of $a$, so effort affects only the joint distribution of the two variables. This assumption has two implications: first, the principal cannot make any inference about $a$ from observing $S$, and second even if $a$ is commonly observable, it is not informative about $\theta$. Without loss of generality, we assume
that the signal $S$ is uniformly distributed on $[0, 1]$.\footnote{For any one-dimensional signal $S$, we can always standardize it by taking an increasing transformation, $F(S)$, where $F$ is the accumulative distribution function of $S$. The standardized signal $Z \equiv F(S)$ always follows a uniform distribution on $[0, 1]$.}

After exerting $a$ and observing $s$, the agent offers some “advice” to the principal. Upon hearing the advice from the agent, the principal chooses a project $y$. The principal’s expected utility for a given set of $a, s$ and $y$ is $-\int_\theta^\theta (y - \theta)^2 f(\theta|s, a) d\theta = -y^2 + 2yE[\theta|s, a] - E[\theta^2|s, a]$. Therefore, the principal’s optimal project given $a$ and $s$ is $E[\theta|s, a]$, which is the posterior estimate of $\theta$. Accordingly, the agent’s optimal project given a signal $s$ is $E[\theta|s, a] + b$. To facilitate the analysis, we assume that $E[\theta|s, a]$ is an affine function,

$$E[\theta|s, a] = (\alpha a + \beta)(2s - 1) + E[\theta],$$

where $\alpha > 0$ and $\beta \geq 0$.

Figure 1 shows the principal’s and the agent’s ideal projects as a function of $s$ for a give $a$. The two parameters in (1), $\alpha$ and $\beta$, measure the agent’s learning efficiency and ex ante information level (before learning) respectively. Loosely speaking, a more informative $S$ means $E[\theta|S, a]$ is closer to $\theta$, and therefore more variant is $E[\theta|S, a]$. In the current study, $V(\theta|S, a) = \alpha^2 < 3$ to avoid extreme solutions. It follows from (1) that $\partial E[\theta|s, a]/\partial s = 2(\alpha a + \beta) \geq 0$. That is, $\theta$ and $S$ are positively affiliated so that a larger value of $s$ indicates a larger posterior estimate of $\theta$. In addition, $E[\theta|a] = E[\theta]$. That is, the expectation of $\theta$ is independent of $a$ – a result consistent with the assumption that effort cannot affect the marginal distribution of $\theta$. 
Example: Truth-or-Noise

The state of nature $\theta$ is drawn from a uniform distribution on $[0, 1]$. With probability $2(\alpha a + \beta)$, $S$ perfectly matches $\theta$, and with probability $1 - 2(\alpha a + \beta)$, $S$ is a noise independently drawn from the standard uniform distribution. It is evident that $\alpha$ measures the agent’s learning efficiency, and $\beta$ measures the agent’s ex ante information level (before learning). The posterior estimate of $\theta$ given $s$ and $a$ is $E[\theta | s, a] = (\alpha a + \beta)(2s - 1) + \frac{1}{2}$.

3 Overt Learning

In this section, we assume overt learning – the principal observes the agent’s learning effort $a$, but not his private signal $s$. We study three different schemes/models, depending on whether the principal allocates the task of learning and the authority of decision-making to the agent, or retain them for herself. In the *cheap talk* scheme, the principal assigns the costly learning
task to the agent, but retains the decision-making authority. In the *optimal delegation* scheme, the principal delegates both the learning task and decision-authority to the agent, and put some constraints on the choices. In the *centralization* scheme, the principal retains both the learning task and decision authority; she invests in learning directly, bares the cost of learning, and makes an informed decision without consulting the agent.

We first identify sufficient conditions for zero learning in each scheme. We then compare schemes according to the agent’s learning effort and the principal’s welfare at the equilibria. Proofs of all the lemmas and propositions in this section are gathered in Appendix B.

### 3.1 Cheap talk with Overt Learning

The cheap talk (CT) or communication scheme is a natural extension of the standard Crawford and Sobel (1982) uniform-quadratic model (see also Ottaviani, 2000). There are two sequential stages. The first stage is the agent’s voluntary choice of $a$ which is commonly observed. The second stage is a cheap talk subgame, in which the agent decides on his reporting strategy as a function of the privately observed signal, and the principal decides on her decision strategy as a function of the message received from the agent. Each party can not observe the other’s strategy, but forms consistent belief of it in a perfect Bayesian equilibrium. We use backward deduction to characterize the pure strategy perfect Bayesian equilibria. We denote the learning effort and the agent’s and principal’s payoffs at the equilibria by $a_{CT}$, $U_{A}^{CT}$ and $U_{P}^{CT}$, respectively.

The analysis of the second stage is based on a given learning effort $a$ and borrows heavily from Crawford and Sobel (1982) and Ottaviani (2000). There exists a partition, $S \equiv (s_0, s_1, ..., s_n)$, determined by the agent’s learning behavior. The agent reports which element of the partition his observation lies in, say $[s_{i-1}, s_i]$. Accordingly, the principal chooses

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2Section 7 in Ottaviani (2000) proves how the quadratic preferences can be derived from first principles in a model of investment and independent financial advice.
the project \( y \) that maximizes her welfare:

\[
y([s_{i-1}, s_i], a) = \arg\max_y \left[ - \int_{s_{i-1}}^{s_i} \int_\theta (y - \theta)^2 f(\theta|s, a)d\theta \frac{1}{s_i - s_{i-1}} ds \right] = E[\theta] \frac{s_{i-1} + s_i}{2}, a].
\]

(2)

That is, the principal will choose her ideal project as if the realized signal is the middle point of the reported partition.

The condition of indifference for agent observing \( s_i \) between messages \([s_{i-1}, s_i]\) and \([s_i, s_{i+1}]\) is

\[
- \int_\theta (y([s_i, s_{i+1}], a) - \theta - b)^2 f(\theta|s_i, a)d\theta = - \int_\theta ((y([s_{i-1}, s_i], a) - \theta - b)^2 f(\theta|s_i, a)d\theta,
\]

solved by

\[
s_{i+1} - s_i = s_i - s_{i-1} + \frac{2b}{a + \beta}.
\]

(3)

Equation (3) implies that messages are of increasing length, with \( \frac{2b}{a + \beta} \) as the increase in step size. This result implies an upper limit on \( n \). In particular, set \( s_0 = 0 \), we get

\[
s_n = s_1 n + n(n - 1) \frac{b}{a + \beta}.
\]

Because \( s_n \leq 1 \), we must have \( n(n - 1) \frac{b}{a + \beta} < 1 \), or equivalently

\[
n(a) \leq \left[ -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{a + \beta}{b}} \right],
\]

(4)

where \( [\cdot] \) stands for “the smallest positive integer greater or equal to.”

Equation (3) is a standard second-order difference equation. By setting \( s_0 = 0 \) and \( s_n = 1 \), one can easily solve (3) and get

\[
s_i = \frac{i}{n} - \frac{b}{a + \beta} i(n - i).
\]

(5)

Then as calculated in Appendix A, the principal’s and agent’s expected utilities (given
\[ U_{CT}^P(a) = -V(\theta) + \frac{b^2(1 - n^2)}{3} + \frac{(\alpha a + \beta)^2}{3} \left( 1 - \frac{1}{n^2} \right). \quad (6) \]
\[ U_{CT}^A(a) = -V(\theta) + \frac{b^2(1 - n(a)^2)}{3} + \frac{(\alpha a + \beta)^2}{3} \left( 1 - \frac{1}{n(a)^2} \right) - b^2 - a^2. \quad (7) \]

Note that \( U_{CT}^P(a) \) is increasing in \( n \) within the range of \( n \) defined by (4). That is, equilibria are Pareto-ranked according to \( n \). We will focus on the most informative equilibrium so that (4) takes strict equality. This completes the analysis of the second stage information communication subgame.

In the first stage, the agent chooses the learning effort, \( a_{CT} \), that maximizes (7). This optimization problem is complicated by the discontinuity of \( n(a) \) in \( a \) at certain points of \( a \): according to (4), \( n(a) \) jumps from \( i \) to \( i + 1 \) at \( a = a_i \), where \( a_i \) is such that \(-\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\alpha a_i + \beta}{b}} = i \). As a result, \( U_{CT}^P(a) \) may be discontinuous in \( a \) as well. However, we have the following lemma.

**Lemma 1.** \( U_{CT}^A \) is continuous in \( a \).

We can further rule out the possibility that (7) is maximized at a certain \( a_i \). If this were the case, we must have \( \frac{\partial_- U_{CT}^A(a_i)}{\partial a_i} \geq 0 \geq \frac{\partial_+ U_{CT}^A(a_i)}{\partial a_i} \). However, \( \frac{\partial_- U_{CT}^A(a_i)}{\partial a_i} = \frac{2\alpha(\alpha a_i + \beta)}{3} \left( 1 - \frac{1}{i^2} \right) - 2ka_i < \frac{2\alpha(\alpha a_i + \beta)}{3} \left( 1 - \frac{1}{(i+1)^2} \right) - 2ka_i = \frac{\partial_+ U_{CT}^A(a_i)}{\partial a_i} \). Therefore, the agent’s optimal effort level must be in the interior of a certain \([a_i, a_{i+1}]\). In addition, \( U_{CT}^A(a) \) is concave on \([a_i, a_{i+1}]\) due to the assumption that \( \alpha^2 < 3 \). Therefore the first order necessary condition of a positive \( a_{CT} \) is as follows,

\[ \frac{\partial U_{CT}^A(a_{CT})}{\partial a} = \frac{2\alpha(\alpha a_{CT} + \beta)}{3} \left( 1 - \frac{1}{n(a_{CT})^2} \right) - 2a_{CT} = 0. \quad (8) \]

\(^3\partial_- \) is the left derivative, and \( \partial_+ \) is the right derivative.
Proposition 3.1. In the cheap talk scheme with overt learning,

(i) if \( b \leq \frac{\beta}{2} \), then \( a_{CT} > 0 \) and there are multiple equilibria.

(ii) if \( b \geq \frac{\sqrt{3}\beta}{2\sqrt{3-\alpha}} \) and \( \alpha^2 \leq \frac{3}{4} \), or if \( b \geq \frac{\beta}{\sqrt{3-\alpha^2}} \) and \( \alpha^2 > \frac{3}{4} \), then \( a_{CT} = 0 \) and the only equilibrium is the “babbling” equilibrium where the principal completely ignores the agent’s advice.

Proposition 3.1 shows that in the cheap talk scheme with overt learning, the principal benefits from consulting with a little-biased (small \( b \)) and ex ante well informed (big \( \beta \)) agent, who always has an incentive to learn at equilibria.

More interestingly, the principal cannot benefit from communicating with a “non-expert,” who is highly biased (big \( b \)), little informed ex ante (small \( \beta \)) and inefficient in learning (small \( \alpha \)). Such a “non-expert” has no incentive to learn, and cannot credibly reveal information at the equilibrium.

Note that Conditions (i) and (ii) are exclusive but not exhaustive. Out of these two cases, the agent’s effort at the equilibrium is undetermined.

3.2 Optimal Delegation with Overt Learning

Similar to the cheap talk scheme, the optimal delegation (OD) scheme has two sequential stages: the first stage is the agent’s voluntary choice of \( a \) which is commonly observed. The second stage is the principal’s design of an optimal decision strategy (or contract) as a function of the message received from the agent. Different from cheap talk, the principal reveals her decision strategy to the agent and commits to it. Along the lines of Holmstrom (1984), Melamud and Shibano (1991), Ottavania (2000), and Alonso and Matouschek (2008), we can think of the second stage as a delegation scheme: the principal optimally designs a delegation set, \( P \), which is a limited subset of projects in which the agent can choose his preferred one. Therefore, in the optimal delegation scheme, the principal delegates to the
agent both the learning task and the decision authority, but places constraints on his choice of projects. We denote the learning effort and the principal’s and agent’s expected utilities in this case $a_{OD}$, $U_{P}^{OD}$ and $U_{A}^{OD}$ respectively.

Again, we use backward deduction to characterize the optimal delegation set and learning effort. Since the second stage is the standard delegation model as in Holmstrom (1984), Melamud and Shibano (1991), Ottaviania (2000), and Alonso and Matouschek (2008), we follow their line of reasoning and distinguish between the following two cases.

Case 1: If the preference bias is big enough that $b \geq \alpha a + \beta$, then $P = \{E[\theta]\}$. That is, the only allowed project is the principal’s ex ante optimal project. Accordingly, the agent’s and principal’s expected utilities are

$$U_{A}^{OD}(a) = -V(\theta) - b^2 - a^2, \quad (9)$$
$$U_{P}^{OD}(a) = -V(\theta).$$

Case 2: If the two parties are well aligned so that $b < \alpha a + \beta$, then $P = (-\infty, \alpha a + \beta + E[\theta] - b]$. That is, choices are up to a maximum of $\alpha a + \beta + E[\theta] - b$. Accordingly we have

$$U_{A}^{OD}(a) = -V(\theta) + \frac{(\alpha a + \beta)^2}{3} - \frac{4b^3}{3(\alpha a + \beta)} - a^2, \quad (10)$$
$$U_{P}^{OD}(a) = -V(\theta) + \frac{(\alpha a + \beta)^2}{3} + \frac{2b^3}{3(\alpha a + \beta)} - b^2. \quad (11)$$

Lemma 2. If $a_{OD} > 0$, then $\alpha a_{OD} + \beta > b$, and $U_{A}^{OD}$ and $U_{P}^{OD}$ satisfy (10) and (11) respectively.

Proof. In Case 1, $\frac{\partial U_{A}^{OD}(a)}{\partial a} = -2a < 0, \forall a > 0$. Therefore, $a_{OD} > 0$ only in Case 2. This completes the proof. \qed

Proposition 3.2. In the optimal delegation scheme with overt learning, if $b \leq \beta$ then
\( a_{OD} > 0 \). If \( b \geq \frac{\beta}{1-\alpha^2} \geq 0 \), then \( a_{OD} = 0 \).

Optimal delegation is equivalent to optimal contracting. (Holmstrom, 1984.) That is, from the principal’s perspective, giving the agent a delegation set is equivalent to designing a contract of project selection: both generate the same project selection at the optimum and the same welfare for the principal. Proposition 3.2 therefore says that in the optimal delegation scheme, the principal benefits from contracting ex post (i.e., after learning) with a little-biased and well pre-informed agent, who always have an incentive to learn. On the other hand, the principal gains nothing from contracting with a “non-expert,” who is much biased, little informed ex ante, efficient in learning, and has a high learning cost. Such a “non-expert” always fails to learn and reveals no information.

### 3.3 Centralization

In the centralization scheme, the principal does everything herself – she invests in learning directly and makes a decision without consulting the agent. To facilitate a comparison between schemes later on, we assume that the principal has the same learning technology as the agent. That is, (1) holds for the principal as well, with \( s \) being the signal observed by the principal and \( a \) being the principal’s learning effort. The principal makes two sequential decisions: the optimal learning effort \( a_{Central} \) and then the optimal project \( y_{Central} \). Apparently, the principal always chooses her ideal project, \( y_{Central}(s, a) = E[\theta|s, a] \) and her corresponding expected utility is

\[
U_P^{Central}(a) = \frac{(\alpha a + \beta)^2}{3} - V(\theta) - a^2.
\]  

(12)
Since $\alpha^2 < 3$, $U_{Central}^P(a)$ is concave in $a$. We have

$$a_{Central} = \arg\max_a U_{Central}^P(a) = \frac{\alpha\beta}{3 - \alpha^2},$$  \hspace{1cm} (13)$$

$$U_{Central}^P = \frac{\beta^2}{3 - \alpha^2} - V(\theta).$$ \hspace{1cm} (14)

**Proposition 3.3.** In the centralization scheme, $a_{Central} > 0$ if and only if $\beta > 0$.

*Proof.* It follows directly from (13)

Proposition 3.3 says that in the centralization scheme, as long as the principal is not completely uninformed ex ante, she will invest in learning. This result is in accordance with the finding of Radner and Stiglitz (1984) and Ottaviani (2000) that the marginal return of information is increasing and is zero when the principal (agent) is uninformed.

### 3.4 Scheme Comparison

In this section, we compare the three schemes, in terms of the learning effort and the principal’s welfare at equilibria. To facilitate the comparison, we assume that the principal has the same learning technology as the agent.

#### 3.4.1 A Comparison of Learning Effort

**Proposition 3.4.** Assume that both the principal and the agent have the same learning technology. If $a_{CT} > 0$, then $a_{OD} > a_{Central} > a_{CT}$.

The optimal delegation scheme generates over-investment in learning, more than in the centralization scheme where the principal retains both the learning task and decision authority. Note that the agent’s private information is imperfect but can be improved by learning. In the optimal delegation scheme, the agent reaps much of the benefit of improved
information, as his ex post ideal project is always chosen up to a limit. In this aspect, the agent’s incentive to learn in optimal delegation is close to the principal’s incentive of learning in centralization. In addition, when learning is overt, the optimal delegation set hinges on the agent’s learning effort: the more the agent invests in learning, the less restricted he is in selecting projects. This mechanism gives the agent extra incentives to learn. Overall, in optimal delegation the agent has more incentives to learn than the principal does in centralization.

Cheap talk instead results in under-investment in learning, compared to centralization. This result is a consequence of two opposing effects. On one hand, the agent’s incentive to learn is negatively affected by the fact that a great part of information is lost in communication. Therefore the agent benefits little from improved information. On the other hand, similar to the optimal delegation scheme with overt learning, the agent gets extra incentives to learn due to the fact that the second-stage communication game hinges on the agent’s first-stage learning behavior. Proposition 3.4 suggests that the extra incentives can not fully cover the disincentives caused by loss of information.

3.4.2 A Comparison of the Principal’s Welfare

The principal’s welfare is affected by three key factors: first, who bears the learning cost. Second, how informed the decision is. And third, how biased the decision is (from the principal’s ideal project). We will see how the interaction among these three factors, especially between the last two factors, affect the comparison of schemes in terms of the principal’s welfare.

Proposition 3.5. \( U_{OD}^P \geq U_{CT}^P \).

Proposition 3.5 says that optimal delegation dominates cheap talk in term of the principal’s welfare. In models with exogenous information, Ottaviani (2000) and Dessein (2002)
prove a “delegation principle” on decision authority, which says that “the power to make
decisions should reside in the hands of those with the relevant information,” (Krishna and
Morgan, 2008.) or equivalently optimal delegation dominates cheap talk. Proposition 3.5
shows that the “delegation principle” also holds in the current context with endogenous
information.

In both the optimal delegation and cheap talk schemes, the learning cost is borne by
the agent, so the trade-off is between the informed decision and the biased decisions. In
optimal delegation, by credibly promising to choose the agent’s ex post ideal projects up to
a limit, the principal greatly motivates the agent to learn in the first stage, and induces him
to completely reveal what he learns in the second stage. Therefore, the principal makes well
informed decisions in the optimal delegation scheme. The trade-off is that the principal has
to choose projects biased from her ex post favorites, because of the conflict of interests. In
the cheap talk scheme, such a mechanism inducing full revelation is unavailable. Therefore,
much information is lost in the communication, and the principal makes unbiased decisions
based on limited information from the agent. In short, the principal prefers biased but
informed decisions (optimal delegation) to unbiased decisions based on limited information
(cheap talk).

**Proposition 3.6.** \( U_{P}^{Central} \geq U_{P}^{CT} \) if \( b \geq \frac{\alpha \beta}{3-\alpha^2} \). On the other hand, \( U_{P}^{Central} < U_{P}^{CT} \), if \( b \) is

Proposition 3.6 says that if the agent is very biased (i.e., \( b \) is large), little informed ex ante
(i.e., \( \beta \) is small) and inefficient in learning (i.e., \( \alpha \) is small), centralization outperforms cheap
talk; the principal would be better off retaining the learning task rather than delegating it,
even if the principal has no better learning technology. On the other hand, if the preference
bias is tiny, both parties are little informed ex ante and inefficient in learning, cheap talk
outperforms centralization; the principal would better to delegate the learning task to the
agent instead of retaining it, even if the agent is not more efficient in learning.

In both the cheap talk and centralization schemes, the principal makes unbiased decisions based on the received information. The trade-off is instead between the learning cost and the informed decision. Unlike centralization where the principal makes informed decisions, cheap talk leads to uninformed decisions: the agent not only acquires less information but also communicates less information to the principal. On the other hand, in cheap talk the learning cost is borne by the agent and not internalized by the principal. When the preference bias is small and learning cost is high, the beneficial effect of cheap talk dominates, and cheap talk can outperform centralization from the principal’s perspective.

**Proposition 3.7.** $U^\text{Central}_P > U^\text{OD}_P$ if $b \geq \frac{\beta}{1-\alpha^2} > 0$. On the other hand, $U^\text{Central}_P < U^\text{OD}_P$ if $b \leq \min\left(1, \frac{\alpha}{3-\alpha^2}\right) \beta$.

Proposition 3.7 implies that if the preference bias is large, the principal is better off retaining both the learning task and the decision authority instead of delegating both to the agent. On the other hand, if the preference bias is small, the principal is better off by delegating both the learning task and the decision authority, instead of retaining both.

In both optimal delegation and centralization schemes the principal’s decisions are informed to a similar extent: compared to centralization, the agent over-invests in learning in optimal delegation and therefore is better informed, but he also pools his suggestions on the top and therefore loses some information in communication. The comparison therefore mainly involves a trade-off between the learning cost and the biased decision. On one hand, the learning cost in optimal delegation is borne by the agent and not internalized by the principal. On the other hand, the principal almost always makes biased decisions in optimal delegation. Proposition 3.7 intuitively suggests that the trade-off depends on the preference bias.

Propositions 3.6, 3.5 and 3.7 in combination lead to the following corollary.
Corollary 1. When learning is endogenous and overt, the “delegation principle” still holds. The principal either delegates both the learning task and the decision authority, or retains both. It is never optimal for the principal to delegate one and retain the other.

4 Covert Learning

In many cases the agent’s learning behavior is unobservable. For instance, when an investor relies on an agent – either a financial adviser or a mutual fund manager – for such decisions as which stocks to invest in and how much to invest in the stocks, the agent’s knowledge about each individual stock is important and is improved by the agent’s time and effort spent on that stock, which is often unobservable to the investor.

In this section, we study the cheap talk and optimal delegation schemes under the covert learning assumption, by which the principal observes neither the agent’s learning effort nor his private signal. We make comparisons to the previous results for the over learning case. The proofs of all the lemmas and propositions in this section are in Appendix E.

4.1 Cheap Talk with Covert Learning

In a cheap talk game with covert learning, the principal cannot observe $a$, neither can she make any inference about $a$ from the agent’s suggestion. Instead she has to make a conjecture about the agent’s effort, evaluate the credibility of the agent’s advice based on this conjecture, and choose a project accordingly. In this case, a pure strategy perfect Bayesian equilibrium consists of an effort level $\hat{a}_{CT}$ and a reporting strategy $\hat{r}_{CT}(\cdot)$ for the agent, and

---

4This is because the marginal distribution of $S$ is independent with $a$. 
a decision rule \( \hat{y}_{CT}(\cdot) \) for the principal such that

\[
(\hat{a}_{CT}, \hat{r}_{CT}) = \arg\max_{a,r} \left[ -\int_0^1 \int_{\theta} \left( \hat{y}_{CT}(r(s)) - \theta - b \right)^2 f(\theta|s,a) d\theta ds - c(a) \right],
\]

\[
\hat{y}_{CT} = \arg\max_y \left[ -\int_0^1 \int_{\theta} \left( y(\hat{r}_{CT}(s)) - \theta \right)^2 f(\theta|s,\hat{a}_{CT}) d\theta ds \right].
\]

Different from the overt learning case, \( \hat{y}_{CT} \) is not a function of \( a \). That is, the principal can not adjust her decision rule ex post upon observing \( a \). Instead, the principal has to make a conjecture of the agent’s learning effort. At the equilibrium, the principal has correct belief of \( \hat{a}_{CT} \) and plays a Crawford and Sobel cheap talk game with the agent. In particular, there is always a “babbling” equilibrium in which the agent does not invest in learning and the principal completely ignores the agent’s advice. In what follows, we instead focus on the case where \( \hat{a}_{CT} > 0 \). More specifically, assume that the agent’s reporting strategy is characterized by a partition \( \hat{S} = \{\hat{s}_0, \hat{s}_1, ..., \hat{s}_n\} \). Then the principal’s optimal responding strategy is \( \hat{y}_{CT}([\hat{s}_{i-1}, \hat{s}_i]) = E[\theta|\hat{s}_{i-1}+\hat{s}_i, \hat{a}_{CT}] \), where \( \hat{s}_i = \frac{i}{n} - \frac{b}{a\hat{a}_{CT}+\beta} i(n - i) \) and \( n \leq \left\lceil -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{aa_{CT}+\beta}{b}} \right\rceil \). The principal’s decision rule in effect generates a “delegation” set with discrete elements, \( \hat{y}_i = \hat{y}_{CT}([\hat{s}_{i-1}, \hat{s}_i]) \).

Facing the “delegation” set \( \{\hat{y}_i\}_{i=1}^{n} \), the agent optimally chooses a learning effort and a reporting strategy. Denote the agent’s and principal’s payoffs at the equilibria \( \hat{U}_{A}^{CT} \) and \( \hat{U}_{P}^{CT} \) respectively. The difficulty in solving this problem is the need to optimize over sets, which precludes us from using standard optimization techniques. Nevertheless, as computed in Appendix D, the agent’s expected utility is

\[
\hat{U}_{A}^{CT}({\{\hat{y}_i\}_{i=1}^{n}, a}) = \frac{(aa + \beta)^2}{3} - V(\theta) - a^2 - \frac{\sum_{i=2}^{n}(\hat{y}_i - \hat{y}_{i-1})^3}{24(aa + \beta)} \]

\[
- \frac{(\hat{y}_1 + aa + \beta - E[\theta] - b)^3}{6(aa + \beta)} - \frac{(aa + \beta + E[\theta] + b - \hat{y}_n)^3}{6(aa + \beta)},
\]

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where all the $\hat{y}_i$'s are constant in $a$, and this is a key difference from the over learning case. Accordingly,

$$\frac{\partial \hat{U}_{CT}}{\partial \hat{a}} = \alpha \left[ \frac{2(\alpha a + \beta)}{3} + \frac{\sum_{i=2}^{n}(\hat{y}_i - \hat{y}_{i-1})^3}{24(\alpha a + \beta)^2} \right. \nonumber \\
- \frac{3(\alpha a + \beta)(\hat{y}_1 + \alpha a + \beta - E[\theta] - b)^2 - (\hat{y}_1 + \alpha a + \beta - E[\theta] - b)^3}{6(\alpha a + \beta)^2} \nonumber \\
- \frac{3(\alpha a + \beta)(\alpha a + \beta + E[\theta] + b - \hat{y}_n)^2 - (\alpha a + \beta + E[\theta] + b - \hat{y}_n)^3}{6(\alpha a + \beta)^2} \right] - 2a. \quad (15)$$

The following computation is useful,

$$\sum_{i=2}^{n}(\hat{y}_i - \hat{y}_{i-1})^3 = \frac{8(n-1)}{n^3}(\alpha \hat{a}_{CT} + \beta)^3 + 8b^2(n-1)(n-2)(\alpha \hat{a}_{CT} + \beta),$$

$$\hat{y}_1 + \alpha \hat{a}_{CT} + \beta - E[\theta] - b = \frac{\alpha \hat{a}_{CT} + \beta}{n} - nb,$$

$$\alpha \hat{a}_{CT} + \beta + E[\theta] + b - \hat{y}_n = \frac{\alpha \hat{a}_{CT} + \beta}{n} + nb.$$

By substituting the above computation into (15), we get the first order necessary condition for $\hat{a}_{CT} > 0$ as follows

$$\frac{\partial \hat{U}_{CT}}{\partial \hat{a}_{CT}} = \left[ \frac{2\alpha(\alpha \hat{a}_{CT} + \beta)}{3} \left( 1 - \frac{1}{n^2} \right) - 2k \hat{a}_{CT} \right] - \frac{2\alpha b^2(n^2 - 1)}{3(\alpha \hat{a}_{CT} + \beta)} = 0. \quad \text{The first order condition in cheap talk with overt learning}$$

The terms in the brackets are (8) – the first order condition in the overt learning case. If $\hat{a}_{CT} > 0$ as we assumed, then $n \geq 2$, because otherwise the agent could not affect the principal’s decision and would have no incentive to learn. It then follows that the last term in the above equation is negative. Therefore $\hat{a}_{CT} < a_{CT}$. That is, the agent invests less in learning if the learning process is unobservable.

In the covert learning case, the principal has correct belief of $\hat{a}_{CT}$ and the two parties still play a Crawford and Sobel cheap talk game. Therefore the principal’s and the agent’s
payoffs functions in terms of \( \hat{a}_{CT} \) remain the same as in the overt learning case: (6) and (7) respectively. Because \( \hat{a}_{CT} < a_{CT} \) and \( a_{CT} \) maximizes (7), it follows that \( \hat{U}_A^{CT} < U_A^{CT} \). Moreover, because the principal’s payoff function (6) is increasing in both \( a \) and \( n \) (within the boundary of \( n \)) and because \( n \) is also increasing in \( a \) (as evident in (4)), \( \hat{U}_P^{CT} < U_P^{CT} \).

We summarize these results in the following proposition.

**Proposition 4.1.** If \( \hat{a}_{CT} > 0 \), then \( \hat{a}_{CT} < a_{CT} \), \( \hat{U}_P^{CT} < U_P^{CT} \) and \( \hat{U}_A^{CT} < U_A^{CT} \). That is, in the cheap talk scheme, the agent invests less in learning and both parties get lower welfare if learning is covert. In short, cheap talk with overt learning Pareto dominates cheap talk with covert learning.

By introducing the possibility of a naive principal who blindly follows the agent’s advice, Ottaviani (2000) proves the existence of a full revealing equilibrium and that the learning effort at equilibria is not affected by the observability/unobservability of the learning process. Proposition 4.1 shows that Ottaviani’s novel results rely on the critical assumption of a possibly native principal: when the principal is always sophisticated, loss of information is inevitable in communication and the agent invests less in learning if learning is covert instead of overt.

### 4.2 Optimal Delegation with Covert Learning

The principal offers the agent a delegation set before the agent acquires information. The agent then voluntarily acquires information, and chooses his ex post preferred project within the delegation set. As displayed in Figure 2, any delegation set can be characterized by

\[
P = \{ m, \bar{m}, (m_i, \bar{m}_i)_{i=1..n} \},
\]

5A sophisticate principal is “never fooled in equilibrium into taking an action which is not to her best advantage ex post” (Ottavini, 2000).
where $m$ and $\overline{m}$ are the lower and upper bounds of the delegation set respectively, and $(m_i, \overline{m_i})$ is the $i$th *eliminated* interval of projects.

![Diagram](image)

**Figure 2**: The agent’s reporting strategy $y(s)$ given a delegation set $P \equiv \{m, \overline{m}, (m_i, \overline{m_i})_{i=1..n}\}$.

Denote the learning effort at equilibria by $\hat{a}_{OD}$ and the payoffs of the agent and principal by $\hat{U}_{OD}^A$ and $\hat{U}_{OD}^P$. Define $\hat{y}_{OD}(s)$ as the choice of projects (as a function of $s$) at the equilibria. Also define

$$M_i \equiv \frac{\overline{m_i} - m_i}{2}, \quad \text{i.e., half length of the } i\text{th eliminated interval of projects,}$$

$$\overline{M}(a) \equiv \max(m + aa + \beta - E[\theta] - b, \ 0), \quad \text{and}$$

$$\underline{M}(a) \equiv \max(aa + \beta + E[\theta] + b - \overline{m}, \ 0).$$
Then as computed in Appendix D, the agent’s and principal’s payoffs given effort $a$ are

$$
\hat{U}^{OD}_A(P, a) = \frac{(\alpha a + \beta)^2}{3} - V(\theta) - a^2 - \sum_{i=1}^{n} \frac{M_i^3}{3(\alpha a + \beta)} - \frac{M^3}{6(\alpha a + \beta)} - \frac{3bM^2}{6(\alpha a + \beta)}. 
$$

(16)

$$
\hat{U}^{OD}_P(P, a) = \hat{U}^{OD}_A(P, a) + c(a) - 2bE[\hat{y}_{OD}] + 2bE[\theta] + b^2
$$

$$
= \frac{(\alpha a + \beta)^2}{3} - V(\theta) - b^2 - \sum_{i=1}^{n} \frac{M_i^3}{3(\alpha a + \beta)} - \frac{M^3 + 3bM^2}{6(\alpha a + \beta)} - \frac{3bM^2}{6(\alpha a + \beta)}. 
$$

(17)

We focus on the case where the optimal learning is positive, $\hat{a}_{OD} > 0$. The optimal delegation set in the case where $\hat{a}_{OD} = 0$ is straight-forward and does not add much to the discussion.\footnote{It can be shown that the optimal delegation set for $\hat{a}_{OD} = 0$ is that $P = (-\infty, \beta + E[\theta] - b]$ if $\beta \geq b$ and $P = \{E[\theta]\}$ if $\beta < b$.} Recall that when learning is overt, the optimal delegation set has no lower limit but has a upper threshold of $\alpha a_{OD} + \beta + E[\theta] - b$, or equivalently $M = 0$ and $\overline{M} = 2b$. (See Section 4.2.) In the following proposition, we show that when learning is covert the optimal delegation set has similar properties, but with a higher upper threshold.

**Proposition 4.2.** When learning is covert, the optimal delegation set has no lower bound and the upper bound is higher than when learning is overt. That is, $\underline{m} = -\infty$ and $\overline{m} > \alpha \hat{a}_{OD} + \beta + E[\theta] - b$, or equivalently $M = 0$ and $\overline{M} < 2b$. In addition, if the optimal learning effort is low so that $\hat{a}_{OD} < \frac{b(3-a^2)}{3a}$, then the upper bound is binding. That is, $\underline{m} < \alpha \hat{a}_{OD} + \beta + E[\theta] + b$ or equivalently $\overline{M} > 0$.

Similar to the overt learning case, Proposition 4.2 says that the optimal delegation set has no (binding) lower bound when learning is covert, for two reasons. First, a binding lower bound results in sub-optimal choices of projects that are even worse than the agent’s ex post ideal choices. Second, a binding lower bound weakens the agent’s incentives to learn: the more the agent invests in learning, the more likely the agent’s suggestions get rejected.

Different from the overt learning case, the optimal delegation set has a higher upper
threshold when learning is covert. In the covert learning case, an increase of the upper bound \( \overline{m} \) at \( \overline{m} = \alpha a_{OD} + \beta + E[\theta] - b \) has two opposite effects on the principal’s welfare: first, it increases incentives for learning ex ante.\(^7\) Second, it leads to more biased choices of projects ex post. Proposition 4.2 suggests that the beneficial incentive effect more than offsets the harmful biased selection effect.

**Lemma 3.** Any optimal delegation set with multiple removed project intervals is equivalent to a delegation set with a single removed project interval.

Proposition 4.2 and Lemma 3 suggest that without loss of generality, we can focus on delegation sets with a single removed project interval and a upper bound, i.e., \( P = \{-\infty, \overline{m}, (\overline{m}_1, \overline{m}_1)\} \).

An interesting question is whether the principal should remove any “intermediate” projects. That is, if we define \( M_1 = \frac{m_1 - m_1}{2} \), then should \( M_1 \) be zero or positive? It is hard to answer this question, mainly due to the two opposing effects of removing “intermediate” projects on the principal’s welfare. On one hand, removing “intermediate” projects enhances the agent’s motivation to learn, which is beneficial to the principal. Indeed, it is evident from (16) that \( \frac{\partial^2 \hat{a}_{OD}}{\partial a_{OD} M_1} \frac{\alpha M_1^2}{(\alpha a_{OD} + \beta)^2} \geq 0 \), implying that \( \hat{a}_{OD} \) increases with \( M_1 \). On the other hand, restricted choices lead to ex post biased choices of projects (at intermediate values of the signal) and reduce the principal’s welfare. The trade-off between the two opposite effects is undeterminate. We can, however, prove the following proposition, which shows that to motive an ex ante uninformed agent to learn, the principal has to prohibit the agent from choosing any “intermediate” projects, including his ex ante optimal choice.

**Proposition 4.3.** To motivate an ex ante uninformed agent to learn, the principal has to eliminate a set of “intermediate” projects, including the agent’s ex ante optimal project. That is, if \( \beta = 0 \) and \( \hat{a}_{OD} > 0 \), then \( M_1 > 0 \) and \( E[\theta] + b \in (m_1, \overline{m}_1) \).

\(^7\)As shown by (34) in Appendix E, the higher the upper threshold is, the more investment is in learning.
Szalay (2005) studies a similar optimal delegation model with covert learning. By assuming perfect alignment of interests between the two parties, Szalay (2005) abstracts from the miscommunication problem and focuses on the learning-motivation problem. In this paper, we assume a constant conflict of interest, and therefore studies both problems. Propositions 4.2 and 4.3 show that some properties of the optimal delegation set in Szalay (2005) extend to the current context (i.e., removal of intermediate choices) and some properties may not (i.e., unlimited extreme choices).

5 Conclusion

For the principal-agent interaction in various activities such as buying a house and investing in financial markets, the agent’s private information is often endogenously learned instead of given by nature. In this paper, we study the economic implications of costly and endogenous learning in the cheap talk model of Crawford and Sobel (1982) and the optimal delegation model of Holmstrom (1984). In the cheap talk scheme the principal assigns the learning task to the agent but retain the decision authority; In the optimal delegation scheme the principal delegates both the learning task and decision authority to the agent, and imposes some constraints on the choice of projects. We study both models under the over learning assumption and the covert learning assumption respectively.

Under the overt learning assumption, we find that if the agent is highly biased, little informed ex ante and inefficient in learning, then he has no incentive to learn and can not credibly reveal information. This result holds for both the cheap talk and the optimal delegation schemes. We then compare the cheap talk and the optimal delegation schemes to the centralization scheme where the principal retain both the learning task and decision authority. Compared to centralization, the cheap talk scheme generates under-investment in learning, while optimal delegation results in over-investment in learning. The reason is that
much information is lost in cheap talk but little is lost in optimal delegation. And loss of information weakens the agent’s incentives to learn. Optimal delegation always dominates cheap talk. That is, the standard “delegation principle” still holds when information is endogenously learned and learning is overt. The principal either delegate both the learning task and decision authority to the agent, or retain both. It is never optimal for her to delegate one and retain the other.

Observability of the learning process, or the lack thereof, impacts the outcome of the cheap talk and the optimal delegation schemes. In the cheap talk scheme, the agent invests less in learning and both parties have lower welfare if learning is covert instead of overt. The optimal delegation set in the covert learning case shares some similarities with that when learning is overt. For instance, in both the overt and covert learning cases, the optimal delegation set has no lower bound but has a binding upper threshold. The upper threshold, however, is higher in the covert learning case to induce the agent to learn. In addition, when learning is covert, the principal has to eliminate a set of “intermediate” projects, including the agent’s ex ante optimal projects, to induce an ex ante uninformed agent to learn.
6 References


Argenziano, R., S. Severinov, and F. Squintani, Strategic Information Acquisition and Transmission, mimeo (2011)


A Computation of Formulas (6) and (7)

In Appendix A, we derive formulas (6) and (7) in the main text, which are the principal’s and agent’s expected utilities in the cheap talk scheme with overt learning.

From (2), we have

\[ U_{P}^{CT}(a) = \int_{0}^{1} \int_{\theta}^{\theta} u(y(s), \theta) f(\theta|s,a) d\theta ds \]

\[ = -\sum_{i=1}^{n} \int_{s_{i-1}}^{s_{i}} \int_{0}^{\theta} \left( y([s_{i-1},s_{i}], a) - \theta \right)^{2} f(\theta|s,a) d\theta ds \]

\[ = -\int_{0}^{1} E[\theta^{2}|s,a] ds + \sum_{i=1}^{n} y([s_{i-1},s_{i}], a)^{2} (s_{i} - s_{i-1}) \]

\[ = E^{2}[\theta] - E[\theta^{2}] + (\alpha a + \beta)^{2} \sum_{i=1}^{n} (s_{i} - s_{i-1})(s_{i} + s_{i-1} - 1)^{2} \]

\[ + 2E[\theta](\alpha a + \beta) \sum_{i=1}^{n} (s_{i} - s_{i-1})(s_{i} + s_{i-1} - 1). \]

According to (5), we have

\[ s_{i} - s_{i-1} = \frac{1}{n} + \frac{b}{\alpha a + \beta} (2i - 1 - n). \]  

\[ s_{i} + s_{i-1} = \frac{2i - 1}{n} + \frac{b}{\alpha a + \beta} [(2i - 1)(i - n) - (i - 1)]. \]  

It is useful to compute

\[ \sum_{i=1}^{n} (s_{i} - s_{i-1}) = 1. \]

\[ \sum_{i=1}^{n} (s_{i}^{2} - s_{i-1}^{2}) = 1. \]

\[ \sum_{i=1}^{n} (s_{i} - s_{i-1})(s_{i} + s_{i-1})^{2} = \frac{1}{3} \left[ \frac{b^{2}(1 - n^{2})}{(\alpha a + \beta)^{2}} + 4 - \frac{1}{n^{2}} \right]. \]

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Thus
\[
\sum_{i=1}^{n} (s_i - s_{i-1})(s_i + s_{i-1} - 1)^2 = \sum_{i=1}^{n} (s_i - s_{i-1})(s_i + s_{i-1})^2 - 2 \sum_{i=1}^{n} (s_i^2 - s_{i-1}^2) + \sum_{i=1}^{n} (s_i - s_{i-1})
\]
\[
= \frac{1}{3} \left[ b^2(1 - n^2) + \frac{(\alpha a + \beta)^2}{n^2} \right].
\]
\[
\sum_{i=1}^{n} (s_i - s_{i-1})(s_i + s_{i-1} - 1) = \sum_{i=1}^{n} (s_i^2 - s_{i-1}^2) - \sum_{i=1}^{n} (s_i - s_{i-1}) = 0.
\]
Thus the principal’s welfare is
\[
U_{CT}^P(a) = -V(\theta) + \frac{b^2(1 - n^2)}{3} + \frac{(aa + \beta)^2}{3} \left(1 - \frac{1}{n^2}\right).
\]
And the agent’s welfare is
\[
U_{CT}^A(a) = -\int \int (y(s) - \theta - b)^2 f(\theta|s,a) d\theta ds - c(a)
\]
\[
= U_{CT}^P(a) + 2bE[y(s)] - 2bE[\theta] - b^2 - c(a)
\]
\[
= U_{CT}^P(a) - b^2 - c(a)
\]
\[
= -V(\theta) + \frac{b^2(1 - n(a)^2)}{3} + \frac{(aa + \beta)^2}{3} \left(1 - \frac{1}{n(a)^2}\right) - b^2 - a^2.
\]

B Proofs for Section 3

Appendix B includes proofs of all the lemmas and propositions in Section 3.

Proof of Lemma 1

Proof. From (7), it is obvious that $U_{CT}^A(a)$ is continuous in $a$ on the intervals $(a_i, a_{i+1})$ for all $i$, because $n(a)$ is constant on $(a_i, a_{i+1})$. Then we are left to prove that $U_{CT}^A(a)$ is continuous at $a = a_i$ for all $i$. That is, we need to prove that $\lim_{a \to a_i} U_{CT}^A(a) = U_{CT}^A(a_i) =$
lim_{a \to a_i^-} U^A_{CT}(a)^8 The first equality is obvious as lim_{a \to a_i^-} n(a) = n(a_i). The second equality is less apparent because lim_{a \to a_i^+} n(a) = n(a_i) + 1. However, we have the following

$$
\lim_{a \to a_i^+} U^A_{CT}(a) - U^A_{CT}(a_i) = \frac{1 + 2i}{3} \left[ \frac{(aa_i + \beta)^2}{i^2(1 + i)^2} - b^2 \right] = 0,
$$

where the last equality follows from $-\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{aa_i + \beta}{b}} = i$ by definition. This completes the proof.

\[ \square \]

**Proof of Proposition 3.1**

**Proof.** If $\beta \geq 2b$, then according to (4) (taking strict equality), $n(a) \geq 2$ for all $a > 0$. Therefore, $\frac{\partial_{1}U^A_{CT}(a)}{\partial a}_{a=0} = \frac{2a \beta}{3} \left( 1 - \frac{1}{n(a)^2} \right) > 0$. That is, a marginal increase of $a$ at zero increases the agent’s expected utility, and therefore $a_{CT}$ must be positive. Since $n(a) \geq 2$, there are multiple equilibria.

On the other hand, if $\beta < 2b$, then $n(a) = 1$ and $\frac{\partial U^A_{CT}(a)}{\partial a} = -2ka < 0$ for all $a \leq \frac{2b - \beta}{\alpha}$. Then to prove that $a_{CT} = 0$, it suffices to prove that $U^A_{CT}(0) \geq U^A_{CT}(a), \forall a > \frac{2b - \beta}{\alpha}$. That is,

$$
U^A_{CT}(0) = -V(\theta) - b^2 \geq -V(\theta) + \frac{b^2(1 - n(a)^2)}{3} + \frac{(aa + \beta)^2}{3} \left( 1 - \frac{1}{n(a)^2} \right) - b^2 - a^2 \equiv U^A_{CT}(a),
$$

or equivalently,

$$
\frac{b^2(1 - n(a)^2)}{3} + \frac{(aa + \beta)^2}{3} \left( 1 - \frac{1}{n(a)^2} \right) - a^2 \leq 0, \quad \forall a \geq \frac{2b - \beta}{\alpha}.
$$

Since $n(a) \geq 2$ for all $a > \frac{2b - \beta}{\alpha}$, it further suffices to prove that

$$
-b^2 + \frac{(aa + \beta)^2}{3} - a^2 \leq 0, \quad \forall a > \frac{2b - \beta}{\alpha}.
$$

---

8 $a \rightarrow a_i^-$ means $a$ approaches $a_i$ from below, and $a \rightarrow a_i^+$ means $a$ approaches $a_i$ from above.
Rearranging the left hand side of the above inequality gives

$$(\alpha^2 - 3)a^2 + 2\alpha \beta a + \beta^2 - 3b^2 \leq 0, \quad \forall a > \frac{2b - \beta}{\alpha}.$$  

For the above inequality to hold, either of the following two sets of conditions must hold:

(a) $$(\alpha^2 - 3)a^2 + 2\alpha \beta a + \beta^2 - 3b^2 \leq 0, \quad \forall a. \text{ (Left hand side of Figure 3.)}$$ Or equivalently

$$b \geq \frac{\beta}{\sqrt{3 - \alpha^2}}.$$  

(b) $$(\alpha^2 - 3)a^2 + 2\alpha \beta a + \beta^2 - 3b^2 \leq 0 \text{ at } a = \frac{2b - \beta}{\alpha}, \text{ and } \frac{\alpha \beta}{3 - \alpha^2} \leq \frac{2b - \beta}{\alpha}. \text{ (Right hand side of Figure 3.)}$$ Or equivalently,

$$b \geq \frac{\sqrt{3} \beta}{2 \sqrt{3 - \alpha}} \quad \text{and} \quad b \geq \frac{3 \beta}{2(3 - \alpha^2)}.$$  

Condition (a) or (b) above holds if and only if

$$b \geq \min \left( \frac{1}{\sqrt{3 - \alpha^2}}, \max \left( \frac{3}{2(3 - \alpha^2)}, \frac{\sqrt{3}}{2 \sqrt{3 - \alpha}} \right) \right) \beta.$$  

(19)

\[ 
\begin{align*}
\text{Figure 3: } (\alpha^2 - 3)a^2 + 2\alpha \beta a + \beta^2 - 3b^2. 
\end{align*} \]
Then applying the following inequalities,

\[
\frac{1}{\sqrt{3} - \alpha^2} > \frac{\sqrt{3}}{2\sqrt{3} - \alpha}, \quad \forall \alpha^2 < 3.
\]

\[
\frac{3}{2(3 - \alpha^2)} < \frac{\sqrt{3}}{2\sqrt{3} - \alpha}, \quad \text{if and only if} \quad \alpha^2 < \frac{3}{4},
\]

\[
\frac{1}{\sqrt{3} - \alpha^2} > \frac{3}{2(3 - \alpha^2)}, \quad \text{if and only if} \quad \alpha^2 < \frac{3}{4}.
\]

Condition (19) is equivalent to Condition (ii) in the lemma. Moreover, since \(\frac{\sqrt{3}}{2\sqrt{3} - \alpha} > \frac{1}{2}\) and \(\frac{1}{\sqrt{3} - \alpha^2} > \frac{1}{2}\), the condition that \(b > \frac{\beta}{2}\) is redundant.

Since \(a_{CT} = 0\) and \(b > \frac{\beta}{2}\), \(n(a_{CT}) = 1\) according to (4). Therefore, the only equilibrium is the “babbling” equilibrium where the principal completely ignores the agent’s advice. \(\Box\)

**Proof of Proposition 3.2**

*Proof.* If \(b \leq \beta\), then Case 2 is relevant for all values of \(a\). At \(a = 0\) we have

\[
\frac{\partial U^A_{OD}(a)}{\partial a} = \frac{2\alpha(\alpha a + \beta)}{3} - 2a + \frac{4\alpha^3 b}{3(\alpha a + \beta)^2}
\]

\[
= \frac{2\alpha\beta}{3} + \frac{4\alpha b^2}{3\beta^2} > 0.
\]

That is, the agent benefits from a marginal increase of \(a\) from zero, and therefore \(a_{OD} > 0\).

On the other hand, if \(b \geq \frac{\beta}{1 - \alpha^2} > \beta\), then as shown in Figure 4, the model switches from Case 1 to Case 2 as \(a\) increases and exceeds \(\frac{b - \beta}{\alpha}\). In Case 1, the agent’s expected utility (9) strictly decreases with \(a\), therefore to prove that the agent’s expected utility is maximized at \(a = 0\) it suffices to prove that \(U^A_{OD}\) in Case 2 is strictly concave and that \(\frac{\partial U^A_{OD}(a)}{\partial a} \leq 0\) at
Figure 4: $U_A^{OD}(a)$ when $b \geq \frac{\beta}{1-\alpha^2} > \beta$.

$a = \frac{b-\beta}{\alpha}$. Indeed, we have

$$\frac{\partial^2 U_A^{OD}(a)}{\partial a^2} = 2\left(\frac{\alpha^2}{3} - 1\right) - \frac{8\alpha^2 \beta^3}{3(\alpha a + \beta)^3} < 0,$$

$$\frac{\partial U_A^{OD}(a)}{\partial a} \bigg|_{a = \frac{b-\beta}{\alpha}} = 2\left(\frac{\alpha^2}{3} - 1\right)\left(\frac{b-\beta}{\alpha}\right) + \frac{2\alpha \beta}{3} + \frac{4b \alpha}{3} < 0,$$

where the inequality in the first formula follows from the assumption that $\alpha^2 < 3$ and the inequality in the second formula follows from the condition that $b \geq \frac{\beta}{1-\alpha^2}$. This completes the proof.

**Proof of Proposition 3.4**

Proposition 3.4 can be proved by a sequence of lemmas as follows.

**Lemma 4.** If $a_{Central} = 0$, $a_{CT} = 0$. If $a_{CT} > 0$, $a_{Central} > a_{CT}$.

**Lemma 5.** If $a_{OD} = 0$, $a_{CT} = 0$. If $a_{CT} > 0$, $a_{OD} > a_{CT}$.

**Lemma 6.** If $a_{Central} > 0$ and $a_{OD} > 0$, $a_{OD} > a_{Central}$.

**Proof of Lemma 4**
Proof. If \( a_{Central} = 0 \), then by Proposition 3.3, \( \beta = 0 \), implying that Condition (ii) in Proposition 3.1 is satisfied, and therefore \( a_{CT} = 0 \). If \( a_{CT} > 0 \), then

\[
\frac{\partial U_P^{Central}(a)}{\partial a}_{|a=a_{CT}} = \frac{2\alpha(a a_{CT} + \beta)}{3} - 2a_{CT} > \frac{2\alpha(a a_{CT} + \beta)}{3} \left( 1 - \frac{1}{n(a_{CT})^2} \right) - 2a_{CT} = \frac{\partial U_A^{CT}(a)}{\partial a}_{|a=a_{CT}} = 0,
\]

where the first equality follows from (12) and the second equality follows from (8). Therefore, \( a_{Central} > a_{CT} \).

\[ \square \]

Proof of Lemma 5

Proof. Part 1. If \( a_{OD} = 0 \), then by Proposition 3.2, \( \beta < b \). Then according to (9), \( U_A^{OD}(0) = -V(\theta) - b^2 \). Assume for contradiction that \( a_{CT} > 0 \), then \( n(a_{CT}) \geq 2 \), or equivalently \( \alpha a_{CT} + \beta \geq 2b \) according to (4) (taking strict equality).

\[
U_A^{CT}(a_{CT}) = -V(\theta) + \frac{(\alpha a_{CT} + \beta)^2}{3} \left( 1 - \frac{1}{n(a_{CT})^2} \right) + \frac{b^2(1 - n(a_{CT})^2)}{3} - b^2 - a_{CT}^2
\]

\[
< -V(\theta) + \frac{(\alpha a_{CT} + \beta)^2}{3} - \frac{a_{CT}^2}{3} - 2b^2
\]

\[
< -V(\theta) + \frac{(\alpha a_{CT} + \beta)^2}{3} - \frac{a_{CT}^2}{3} - \frac{4b^3}{3(\alpha a_{CT} + \beta)}
\]

\[
= U_A^{OD}(a_{CT})
\]

\[
\leq U_A^{OD}(0)
\]

\[
= U_A^{CT}(0),
\]

where the first equality is (7), the first inequality follows from the fact \( n(a_{CT}) \geq 2 \) if \( a_{CT} > 0 \), the second inequality follows from the fact that \( \alpha a_{CT} + \beta \geq 2b \) if \( a_{CT} > 0 \), the second equality is (10) since \( \alpha a_{CT} + \beta \geq 2b > b \), the third inequality follows from the condition that \( a_{OD} = 0 \),
and the last equality follows from the fact that $U_A^{OD}(0) = U_A^{CT}(0) = -V(\theta) - b^2$.

We thus arrive at a contradiction with the assumption that $a_{CT} > 0$. And it must be true that $a_{CT} = 0$.

**Part 2.** If $a_{CT} > 0$, then by the first part of lemma 5 we know $a_{OD} > 0$. Therefore, $U_A^{CT}$ and $U_A^{OD}$ are characterized by (7) and (10) respectively. The corresponding first order conditions for $a_{CT}$ and $a_{OD}$ are

$$\frac{\partial U_A^{CT}(a)}{\partial a} \Big|_{a=a_{CT}} = \frac{2\alpha(\alpha a_{CT} + \beta)}{3} \left(1 - \frac{1}{n^2}\right) - 2a_{CT} = 0,$$

$$\frac{\partial U_A^{OD}(a)}{\partial a} \Big|_{a=a_{OD}} = \frac{2\alpha(\alpha a_{OD} + \beta)}{3} - 2a_{OD} + \frac{4\alpha b^3}{3(\alpha a_{OD} + \beta)^2} = 0.$$

Since $\frac{\partial U_A^{CT}(a)}{\partial a} < \frac{\partial U_A^{OD}(a)}{\partial a}$, $\forall a$ and $n$, it must be true that $a_{CT} < a_{OD}$.

**Proof of Lemma 6**

**Proof.** From Lemma 2 and (10), we have

$$\frac{\partial U_A^{OD}(a)}{\partial a} = \frac{2\alpha(\alpha a + \beta)}{3} - 2a + \frac{4\alpha b^3}{3(\alpha a + \beta)^2} > \frac{2\alpha(\alpha a + \beta)}{3} - 2a = \frac{\partial U_P^{Central}(a)}{\partial a}, \forall a > 0.$$

Therefore, $a_{OD} > a_{Central}$.

**Proof of Proposition 3.5**

**Proof.** If $a_{CT} = a_{OD} = 0$, then according to Proposition 3.2 $\beta < b$. Then (4) implies that $n(a_{CT}) = 1$ and (6) implies that $U_P^{CT} = U_P^{OD} = -V(\theta)$. 

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If $a_{CT} = 0$ and $a_{OD} > 0$, then according to Proposition (3.1) $b > \frac{\beta}{2}$. Then (4) implies that $n(a_{CT}) = 1$ and (6) implies that $U^C_P = -V(\theta)$. On the other hand, since $a_{OD} > 0$, according to Lemma 2 we have $\alpha a_{OD} + \beta > b$ and

$$U^O_P(a) = -V(\theta) + \frac{(\alpha a + \beta)^2}{3} - b^2 = -V(\theta) = U^C_P,$$  if $\alpha a + \beta = b.  $

$$\frac{\partial U^O_P(a)}{\partial (\alpha a + \beta)} = \frac{2(\alpha a + \beta)}{3} - \frac{2b^3}{3(\alpha a + \beta)^2} > 0, \ \forall \alpha a + \beta > b.$$  

Therefore $U^O_P(a_{OD}) \geq U^C_P(0).$

If $a_{CT} > 0$ and $a_{OD} > 0$, then $n(a_{CT}) \geq 2$ because otherwise the “babbling” equilibrium is the only equilibrium in the cheap talk scheme, and $a_{CT} > 0$ can not be a solution. In addition , according to Lemma 5 $a_{OD} > a_{CT}$. Then we have

$$U^C_P(a_{CT}) = -V(\theta) + \frac{(\alpha a_{CT} + \beta)^2}{3} \left(1 - \frac{1}{n(a_{CT})^2}\right) + \frac{b^2(1 - n(a_{CT})^2)}{3}$$

$$< -V(\theta) + \frac{(\alpha a_{CT} + \beta)^2}{3} - b^2 + \frac{2b^3}{3(\alpha a_{OD} + \beta)}$$

$$< -V(\theta) + \frac{(\alpha a_{OD} + \beta)^2}{3} - b^2 + \frac{2b^3}{3(\alpha a_{OD} + \beta)}$$

$$= U^O_P(a_{OD}),$$  (21)

where the first equality is (6), the first inequality follows from the fact that $2 \leq n_{CT} < \infty$, the second inequality from the result that $a_{OD} > a_{CT}$, and the last equality is (11).  

**Proof of Proposition 3.6**

**Proof.** Part 1. Prove that if $b \geq \frac{\alpha \beta}{3-\alpha^2}$, then $U^C_P(\text{Central}) \geq U^C_P$. We only need to prove this result for the case with $a_{CT} > 0$. This is because if $a_{CT} = 0$, then according to Proposition (3.1), $b > \frac{\beta}{2}$. Then (4) implies that $n(a_{CT}) = 1$. Therefore $U^C_P = -V(\theta) \leq -V(\theta) + \frac{\beta^2}{3-\alpha^2} = 39$
Assume that $a_{CT} > 0$. Because $a_{CT} > 0$ and because

$$
\frac{\partial U_A^{CT}(a)}{\partial a} = 2\alpha(\alpha a + \beta) \left(1 - \frac{1}{n^2}\right) - 2a < \frac{2\alpha(\alpha a + \beta)}{3} - 2a, \quad \forall n,
$$

we must have $a_{CT} < \left\{ a \left| \frac{2\alpha(\alpha a + \beta)}{3} - 2a = 0 \right. \right\} = \frac{\alpha\beta}{3 - \alpha^2}$, or equivalently,

$$
\alpha a_{CT} + \beta < \frac{3\beta}{3 - \alpha^2}.
$$

(22)

Then we have

$$
U_P^{CT} = -V(\theta) + \frac{(\alpha a_{CT} + \beta)^2}{3} \left(1 - \frac{1}{n^2}\right) + \frac{\beta^2(1 - n^2)}{3}
$$

$$
< -V(\theta) + \frac{3\beta^2}{(3 - \alpha^2)^2} \left(1 - \frac{1}{n^2}\right) + \frac{\beta^2(1 - n^2)}{3}
$$

$$
< -V(\theta) + \frac{3\beta^2}{(3 - \alpha^2)^2} - \beta^2
$$

$$
\leq -V(\theta) + \frac{\beta^2}{3 - \alpha^2} = U_P^{Central},
$$

where the first equality is (6), the first inequality follows from (22), the second inequality follows from the fact that $2 \leq n < \infty$ if $a_{CT} > 0$, the third inequality follows from the condition that $b \geq \frac{\alpha\beta}{3 - \alpha^2}$, and the last equality follows from (14).

**Part 2.** Prove that if $b$ is close to zero, $\beta > 2b$ and $\alpha^2 < \frac{8}{3}$, then $U_P^{Central} < U_P^{CT}$.

Because $\beta > 2b$, according to Proposition 3.1, $a_{CT} > 0$. Then $n(a_{CT}) \geq 2$, because otherwise the only equilibrium is the “babbling” equilibrium and $a_{CT} > 0$ can not be optimal.
Then (7) implies that
\[
\frac{\partial U_C^A(a)}{\partial a} = \frac{2\alpha (\alpha a + \beta)}{3} \left( 1 - \frac{1}{n^2} \right) - 2a \\
\geq \frac{2\alpha (\alpha a + \beta)}{3} \left( 1 - \frac{1}{2^2} \right) - 2a \\
= \frac{\alpha (\alpha a + \beta)}{2} - 2a, \quad \forall a > 0 \text{ and } \forall n \geq 2.
\]

Therefore, \( a_{CT} \geq \left\{ a \left| \frac{2(\alpha a + \beta)}{2} - 2a = 0 \right\} = \frac{\alpha \beta}{4 - \alpha^2} \right. \) or equivalently
\[
\alpha a_{CT} + \beta \geq \frac{4\beta}{4 - \alpha^2}. \tag{23}
\]

In addition, \( n(a_{CT}) = \left\lceil -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\alpha a_{CT} + \beta}{b}} \right\rceil < \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\alpha a_{CT} + \beta}{b}} \), therefore
\[
\lim_{b \to 0} \frac{b^2(1 - n(a_{CT})^2)}{3} \geq \lim_{b \to 0} \frac{b^2 \left[ 1 - \left( \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\alpha a_{CT} + \beta}{b}} \right)^2 \right]}{3} = 0. \tag{24}
\]
\[
\lim_{b \to 0} n(a_{CT}) = \lim_{b \to 0} \left\lceil -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\alpha a_{CT} + \beta}{b}} \right\rceil = \infty. \tag{25}
\]

Therefore,
\[
\lim_{b \to 0} U_P^C = \lim_{b \to 0} \left\{ -V(\theta) + \frac{b^2(1 - n(a_{CT})^2)}{3} + \frac{(\alpha a_{CT} + \beta)^2}{3} \left( 1 - \frac{1}{n(a_{CT})^2} \right) \right\} \\
\geq -V(\theta) + \frac{(4\beta)^2}{3(4 - \alpha^2)^2} \\
> -V(\theta) + \frac{\beta^2}{3 - \alpha^2} \\
= U_P^{Central}, \tag{26}
\]

where the first equality is (6), the first inequality follows from (23), (24) and (25), the
second inequality follows from the fact that \( \frac{(4\beta)^2}{3(4-\alpha^2)} > \frac{\beta^2}{3-\alpha^2} \) if and only if \( \alpha^2 < \frac{8}{3} \), and the last equality follows from (14).

\[ \square \]

**Proof of Proposition 3.7**

**Proof. Part 1.** Prove that if \( b \geq \frac{\beta}{1-\alpha^2} > 0 \), \( U_P^{Central} > U_P^{OD} \).

According to Proposition 3.2, if \( b \geq \frac{\beta}{1-\alpha^2} > 0 \), then \( a_{OD} = 0 \) and therefore \( U_P^{OD} = -V(\theta) \). On the other hand, according to (14), \( U_P^{Central} = -V(\theta) + \frac{\beta^2}{3-\alpha^2} \). Therefore, \( U_P^{Central} > U_P^{OD} \).

**Part 2.** Prove that if \( b \leq \min\left(1, \frac{\alpha}{3-\alpha^2}\right) \beta \), \( U_P^{OD} > U_P^{Central} \).

If \( \beta \geq b \), then according to Propositions 3.2 and 3.3 and Lemma 6, \( a_{OD} > a_{Central} > 0 \). Therefore

\[
U_P^{OD}(a_{OD}) = -V(\theta) + \frac{(\alpha a_{OD} + \beta)^2}{3} + \frac{2b^3}{3(\alpha a_{OD} + \beta)} - b^2 \\
> -V(\theta) + \frac{(\alpha a_{OD} + \beta)^2}{3} - b^2 \\
> -V(\theta) + \frac{(\alpha a_{Central} + \beta)^2}{3} - b^2 \\
\geq -V(\theta) + \frac{(\alpha a_{Central} + \beta)^2}{3} - \left( \frac{\alpha \beta}{3-\alpha^2} \right)^2 \\
= U_P^{Central}(a_{Central}),
\]

where the first equality follows from (11), the second inequality follows from the result that \( a_{OD} > a_{Central} > 0 \), the third inequality follows from the condition that \( b \leq \frac{\alpha \beta}{3-\alpha^2} \), and the last equality follows from (12) and (13). \[ \square \]
C The Full Delegation Scheme

In Appendix C, we study the full delegation case where the principal simply delegates both the learning task and the decision authority to the agent without any restrictions. The agent makes two sequential decisions: first on the learning effort, and then on the project. In the second stage, the agent always chooses his ideal project, \( y(s, a) = E[\theta|s, a] + b \), and receives an expected utility of \( U_A(a) = \frac{(\alpha a + \beta)^2}{3} - V(\theta) - a^2 \). In the first stage, the agent chooses \( a \) to maximize his expected utility above. Because \( \alpha^2 < 3 \), we get

\[
a_{FD} = \frac{\alpha \beta}{3 - \alpha^2},
\]

\[
U_P^{FD} = 3\left(\frac{\beta}{3 - \alpha^2}\right)^2 - V(\theta) - b^2,
\]

\[
U_P^{FD} = \frac{\beta^2}{3 - \alpha^2} - V(\theta).
\]

D The Agent’s and the Principal’s Expected Utilities

When the Agent is Given a Delegation Set

In Appendix D, we compute the agent’s and the principal’s expected payoff functions when the agent is given a delegation set, which is characterized by \( P \equiv \{m, \overline{m}, (m_i, \overline{m}_i)_{i=1..n}\} \), where \( m \) and \( \overline{m} \) are the lower and upper bounds of the delegation set respectively, and \( (m_i, \overline{m}_i) \) is the \( i \)th interval of eliminated projects.

We start with a relatively simple case. Assume that there is a single eliminated set of intermediate projects \( (m, \overline{m}) \) with width \( 2M = \overline{m} - m \). Assume that the agent’s learning effort \( a \) is large enough so that \( (m, \overline{m}) \) is within the range of the agent’s ex post ideal projects, as displayed in Figure 5. Then there exists a \( \tau \in [0, 1] \), such that \( E[\theta|\tau, a] + b = \frac{m + \overline{m}}{2} \). Accordingly, \( m = E[\theta|\tau - \frac{M}{2(\alpha a + \beta)}, a] + b \) and \( \overline{m} = E[\theta|\tau + \frac{M}{2(\alpha a + \beta)}, a] + b \).
Then the agent’s expected utility can be written as

\[
U_A = -\int_0^{\tau - \frac{M}{2(\alpha a + \beta)}} (E[\theta|s,a] - \theta)^2 f(\theta|s,a) d\theta \, ds
- \int_{\tau - \frac{M}{2(\alpha a + \beta)}}^{\tau} (E[\theta|\tau - \frac{M}{2(\alpha a + \beta)},a] - \theta)^2 f(\theta|s,a) d\theta \, ds
- \int_{\tau}^{\tau + \frac{M}{2(\alpha a + \beta)}} (E[\theta|\tau + \frac{M}{2(\alpha a + \beta)},a] - \theta)^2 f(\theta|s,a) d\theta \, ds
- \int_{\tau + \frac{M}{2(\alpha a + \beta)}}^{1} (E[\theta|s,a] - \theta)^2 f(\theta|s,a) d\theta \, ds
- c(a)
= -\int (E[\theta|s,a] - \theta)^2 f(\theta|s,a) d\theta \, ds
+ \int_{\tau - \frac{M}{2(\alpha a + \beta)}}^{\tau} (E[\theta|s,a] - \theta)^2 f(\theta|s,a) d\theta \, ds
- \int_{\tau}^{\tau - \frac{M}{2(\alpha a + \beta)}} (E[\theta|\tau - \frac{M}{2(\alpha a + \beta)},a] - \theta)^2 f(\theta|s,a) d\theta \, ds
- \int_{\tau}^{\tau + \frac{M}{2(\alpha a + \beta)}} (E[\theta|\tau + \frac{M}{2(\alpha a + \beta)},a] - \theta)^2 f(\theta|s,a) d\theta \, ds
- c(a)
\]
The first term to the right of the second equality can be written as

\[ V_S(E[\theta|s, a]) - V(\theta) = \frac{(\alpha a + \beta)^2}{3} - V(\theta). \]

The second term can be written as

\[ \int_{\tau - \frac{M}{2(\alpha a + \beta)}}^{\tau + \frac{M}{2(\alpha a + \beta)}} (E[\theta^2|s, a] - E^2[\theta|s, a]) \; ds. \]

The third term can be written as

\[ -E^2[\theta|\tau - \frac{M}{2(\alpha a + \beta)}, a] \frac{M}{2(\alpha a + \beta)} + 2E[\theta|\tau - \frac{M}{2(\alpha a + \beta)}, a] \int_{\tau - \frac{M}{2(\alpha a + \beta)}}^{\tau} E[\theta|s, a] \; ds \]

\[ - \int_{\tau - \frac{M}{2(\alpha a + \beta)}}^{\tau} E[\theta^2|s, a] \; ds. \]

The forth term can be written as

\[ -E^2[\theta|\tau + \frac{M}{2(\alpha a + \beta)}, a] \frac{M}{2(\alpha a + \beta)} + 2E[\theta|\tau + \frac{M}{2(\alpha a + \beta)}, a] \int_{\tau}^{\tau + \frac{M}{2(\alpha a + \beta)}} E[\theta|s, a] \; ds \]

\[ - \int_{\tau}^{\tau + \frac{M}{2(\alpha a + \beta)}} E[\theta^2|s, a] \; ds. \]

Altogether, we have

\[ U_A = \frac{(\alpha a + \beta)^2}{3} - V(\theta) - \int_{\tau - \frac{M}{2(\alpha a + \beta)}}^{\tau + \frac{M}{2(\alpha a + \beta)}} E^2[\theta|s, a] \; ds \]

\[ - \frac{M}{2(\alpha a + \beta)} \left( E^2[\theta|\tau - \frac{M}{2(\alpha a + \beta)}, a] + E^2[\theta|\tau + \frac{M}{2(\alpha a + \beta)}, a] \right) \]

\[ + 2E[\theta|\tau - \frac{M}{2(\alpha a + \beta)}, a] \int_{\tau - \frac{M}{2(\alpha a + \beta)}}^{\tau} E[\theta|s, a] \; ds \]

\[ + 2E[\theta|\tau + \frac{M}{2(\alpha a + \beta)}, a] \int_{\tau}^{\tau + \frac{M}{2(\alpha a + \beta)}} E[\theta|s, a] \; ds - c(a). \]
The following calculations are useful.

\[-\int_{\tau - \frac{M}{2(\alpha a + \beta)}}^{\tau + \frac{M}{2(\alpha a + \beta)}} E^2[\theta|s, a]ds\]

\[= - \frac{M}{3(\alpha a + \beta)} \left( + E \left[ \theta|\tau - \frac{M}{2(\alpha a + \beta)}, a \right] E^2 \left[ \theta|\tau + \frac{M}{2(\alpha a + \beta)}, a \right] + E^2 \left[ \theta|\tau - \frac{M}{2(\alpha a + \beta)}, a \right] \right) \]

\[= - \frac{M}{3(\alpha a + \beta)} \left( 3E^2[\theta|\tau, a] + M^2 \right).\]

\[2E \left[ \theta|\tau - \frac{M}{2(\alpha a + \beta)}, a \right] \int_{\tau - \frac{M}{2(\alpha a + \beta)}}^{\tau} E[\theta|s, a]ds\]

\[= \frac{M}{2(\alpha a + \beta)} E \left[ \theta|\tau - \frac{M}{2(\alpha a + \beta)}, a \right] E[\theta|\tau, a] + \frac{M}{2(\alpha a + \beta)} E^2 \left[ \theta|\tau - \frac{M}{2(\alpha a + \beta)}, a \right].\]

\[2E \left[ \theta|\tau + \frac{M}{2(\alpha a + \beta)}, a \right] \int_{\tau}^{\tau + \frac{M}{2(\alpha a + \beta)}} E[\theta|s, a]ds\]

\[= \frac{M}{2(\alpha a + \beta)} E \left[ \theta|\tau + \frac{M}{2(\alpha a + \beta)}, a \right] E[\theta|\tau, a] + \frac{M}{2(\alpha a + \beta)} E^2 \left[ \theta|\tau + \frac{M}{2(\alpha a + \beta)}, a \right].\]

By substituting the above formula into (27), one gets

\[U_A = \left[ \frac{(\alpha a + \beta)^2}{3} - V(\theta) - a^2 \right] - \frac{M^3}{3(\alpha a + \beta)}.\]

An interesting finding from the above formula is that the agent’s welfare is affected only by the width of the removed set of projects (but not the position of the set) as long as the set is within the range of the agent’s ex post ideal projects. In addition, the terms in the brackets are the agent’s expected payoff in the full delegation case (see Appendix C), where he is given unlimited choice of projects. Therefore, for a given effort \(a\) the impact of removing an interval of projects with length \(2M\) on the agent’s payoff is \(-\frac{M^3}{3(\alpha a + \beta)}\). Following this line of reasoning,
if there is a lower bound of the delegation set \( m \) with \( M(a) = \max(m + \alpha a + \beta - E[\theta] - b, 0) \), then given \( a \) the impact of this lower bound on the agent’s welfare is \(-\frac{M^3}{6(aa+\beta)}\).

In general, a delegation set can be characterized by a set \( P \equiv \{m, \overline{m}, (m_i, \overline{m}_i)_{i=1..n}\} \), where \( m \) and \( \overline{m} \) are the lower and upper bounds of the delegation set respectively, and \((m_i, \overline{m}_i)\) is the \( i \)th interval of eliminated projects (see Figure 2).

Define \( y(s) \) as the choices of projects (as a function of \( s \)) at the equilibria. Also define

\[
M_i \equiv \frac{\overline{m}_i - m_i}{2} \quad \text{i.e., half the length of the} \; i \; \text{th interval of eliminated projects},
\]

\[
M(a) \equiv \max(m + \alpha a + \beta - E[\theta] - b, 0),
\]

\[
\overline{M}(a) \equiv \max(aa + \beta + E[\theta] + b - \overline{m}, 0).
\]

Then following the same line of reasoning, the expected payoffs of the agent and of the principal given effort \( a \) are

\[
U_A(P, a) = \frac{(aa + \beta)^2}{3} - V(\theta) - a^2 - \sum_{i=1}^{n} \frac{M_i^3}{3(aa + \beta)} - \frac{M^3}{6(aa + \beta)} - \frac{\overline{M}^3}{6(aa + \beta)}.
\]

\[
U_P(P, a) = U_A(P, a) + c(a) - 2bE[y(s)] + 2bE[\theta] + b^2
= \frac{(aa + \beta)^2}{3} - V(\theta) - b^2 - \sum_{i=1}^{n} \frac{M_i^3}{3(aa + \beta)} - \frac{M^3 + 3bM^2}{6(aa + \beta)} - \frac{\overline{M}^3 - 3b\overline{M}^2}{6(aa + \beta)},
\]

where the second equality in the above equation follows from substituting the following formula

\[
E[y(s)] = E[\theta] + b + \frac{M^2 - \overline{M}^2}{4(aa + \beta)}.
\] (28)

**E Proofs for Section 4**

To prove Proposition 4.2, we need to use the following two lemmas.
Lemma 7. \( M \leq \overline{M} \) at all equilibria of the optimal delegation scheme with covert learning. That is, the upper threshold of the optimal delegation set is more binding than the lower threshold.

Proof. Assume for contradiction that the optimal delegation is \( P = \{m, \overline{m}, (m_i, \overline{m}_i)_{i=1...n}\} \) with \( M > \overline{M} \) at the equilibria. The lemma is proven by showing that there is another delegation set \( \tilde{P} \) that generates higher principal’s welfare. Let \( \tilde{P} = \{\tilde{m}, \tilde{\overline{m}}, (m_i, \overline{m}_i)_{i=1...n}\} \) such that \( E[\theta] + b - m = \overline{m} - E[\theta] - b \) and \( \tilde{m} - E[\theta] - b = E[\theta] + b - m \). That is, \( M = \overline{M} \), \( \tilde{M} = M \) and \( M < \tilde{M} \) for all \( a \). Thus \( \tilde{P} \) induces the same effort level as \( P \) does and generates the same welfare for the agent. In addition, according to (28), \( E[y(s)] \) with \( \tilde{P} \) is lower than \( E[y(s)] \) with \( P \). In sum, we have shown that \( \tilde{P} \) generates the same \( \hat{U}_A^{OD} \) as \( P \) does, but a lower \( E[y(s)] \). Then according to (17), the principal’s welfare is larger with \( \tilde{P} \) than with \( P \). This is in contradiction with the assumption that \( P \) is the optimal delegation set. Therefore \( M \leq \overline{M} \) at all equilibria. \( \square \)

Lemma 8. \( \hat{U}_A^{OD} \) is concave in \( a \).

Proof. According to (16), we have

\[
\frac{\partial \hat{U}_A^{OD}}{\partial a} = \frac{2\alpha(\alpha a + \beta)}{3} - 2a + \sum_{i=1}^{n} \frac{\alpha M_i^3}{3(\alpha a + \beta)^2} - \frac{\alpha M^2 M_{3/2}}{6(\alpha a + \beta)^2} - \frac{\alpha \overline{M}^2 \overline{M}_{-1/2}}{6(\alpha a + \beta)^2},
\]

(29)

where \( M_{3/2} = E[\theta|\frac{3}{2}, a] + b - m > 0 \) and \( \overline{M}_{-1/2} = \overline{m} - E[\theta| - \frac{1}{2}, a] - b > 0 \).

The following calculations are useful,

\[
\frac{\partial \alpha M^2 M_{3/2}}{6(\alpha a + \beta)^2} = \frac{\alpha^2 M}{9(\alpha a + \beta)^2} \left( M_{3/2}^2 + M^2 - M_{3/2}^2 \right) > 0,
\]

\[
\frac{\partial \alpha \overline{M}^2 \overline{M}_{-1/2}}{6(\alpha a + \beta)^2} = \frac{\alpha^2 \overline{M}}{9(\alpha a + \beta)^2} \left( \overline{M}_{-1/2}^2 + \overline{M}^2 - \overline{M}_{-1/2}^2 \right) > 0.
\]
Therefore, we have

\[
\frac{\partial^2 U_{A}^{OD}}{\partial a^2} = 2 \left( \frac{\alpha^2}{3} - 1 \right) - \sum_{i=1}^{n} \frac{2\alpha^2 M_i^3}{3(\alpha a + \beta)^3} - \frac{\partial \alpha M^2 M_{1/2}}{6(\alpha a + \beta)^2} - \frac{\partial \alpha^2 M}{6(\alpha a + \beta)^2} < 0.
\]

\[
\square
\]

**Proof of Proposition 4.2**

*Proof. Part 1: Proof of \( m = -\infty \) or equivalently \( M = 0 \).*

From (17), we have

\[
\frac{d\hat{U}_{P}^{OD}}{dm} = \frac{d\hat{U}_{A}^{OD}}{dm} + \frac{dc(a)}{dm} - 2b \frac{dE[y(s)]}{dm} = \frac{\partial \hat{U}_{A}^{OD}}{\partial m} + \frac{\partial c(a)}{\partial a} \frac{\partial a}{\partial m} - 2b \frac{dE[y(s)]}{dm},
\]

where the second equality follows from the envelop theorem.

The following calculations based on (16) are useful,

\[
\frac{\partial \hat{U}_{A}^{OD}}{\partial m} = -\frac{M^2}{2(\alpha a + \beta)}, \quad \text{(31)}
\]

\[
\frac{\partial^2 \hat{U}_{A}^{OD}}{\partial a \partial m} = -\frac{\alpha MM_1}{2(\alpha a + \beta)^2} < 0, \quad \forall M > 0, \quad \text{(32)}
\]

where \( M_1 = E[\theta | 1, a] + b - m > 0. \)

Since \( \hat{a}_{OD} > 0 \), the first order necessary condition \( \partial \hat{U}_{A}^{OD}/\partial a = 0 \) must hold for all values of \( m \). That is, we have

\[
\frac{d\hat{U}_{A}^{OD}}{dm} = \frac{\partial^2 \hat{U}_{A}^{OD}}{\partial a^2} \frac{\partial a}{\partial m} + \frac{\partial^2 \hat{U}_{A}^{OD}}{\partial a \partial m} = 0,
\]

or equivalently

\[
\frac{\partial a}{\partial m} = -\frac{\partial^2 \hat{U}_{A}^{OD}}{\partial a \partial m} / \partial a^2 \frac{\partial^2 \hat{U}_{A}^{OD}}{\partial a^2} < 0, \quad \forall M > 0, \quad \text{(33)}
\]

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where the inequality follows from (32) and Lemma 8.

According to (28) we have

\[
\frac{dE[y(s)]}{dm} = \frac{\partial E[y(s)]}{\partial m} + \frac{\partial E[y(s)]}{\partial a} \frac{\partial a}{\partial m} = \frac{M}{2(\alpha a + \beta)} - \frac{\alpha (\overline{M} - M)(\overline{m} - m)}{4(\alpha a + \beta)^2} \frac{\partial a}{\partial m}.
\]

By substituting the above formula and (31) into (30) and making use of (33), we eventually get

\[
\frac{d\hat{U}^{OD}_P}{dm} = -\frac{M(M + 2b)}{2(\alpha a + \beta)} + \left[2a + \frac{b\alpha (\overline{M} - M)(\overline{m} - m)}{2(\alpha a + \beta)^2}\right] \frac{\partial a}{\partial m} < 0, \quad \forall M > 0,
\]

where the inequality follows from Lemma 7. Therefore, at all equilibria, we must have \( M = 0 \), or equivalently \( m = -\infty \).

**Part 2: Proof of** \( \overline{m} > \alpha \hat{a}_{OD} + \beta + E[\theta] - b \) or equivalently \( \overline{M} < 2b \).

Similarly, we have

\[
\frac{d\hat{U}^{OD}_P}{dm} = \frac{d\hat{U}^{OD}_A}{dm} + \frac{dc(a)}{dm} - 2b \frac{dE[y(s)]}{dm} = \frac{\partial \hat{U}^{OD}_A}{\partial m} + \frac{\partial c(a)}{\partial a} \frac{\partial a}{\partial m} - 2b \frac{dE[y(s)]}{dm},
\]

where the second equality follows from the envelop theorem.

The following calculations based on (16) are useful,

\[
\frac{\partial \hat{U}^{OD}_A}{\partial m} = \frac{\overline{M}^2}{2(\alpha a + \beta)}, \quad \frac{\partial a}{\partial m} = -\frac{\alpha \overline{M} \overline{m}}{2(\alpha a + \beta)^2} \frac{\partial^2 \hat{U}^{OD}_A}{\partial a^2} > 0, \quad \text{if} \quad \overline{M} > 0, \quad (34)
\]
where $\overline{M}_0 = \overline{m} - E[\theta|0, a] - b > 0$.

Because $\overline{M} = 0$ as we proved above, (28) leads to

$$\frac{dE[y(s)]}{dm} = \frac{\overline{M}}{2(\alpha a + \beta)} - \frac{\alpha \overline{M}_0 \overline{M}}{4(\alpha a + \beta)^2} \frac{\partial a}{\partial m}.$$ 

Finally, we have

$$\frac{d\hat{U}^{OD}}{dm} = \frac{\overline{M}(\overline{M} - 2b)}{2(\alpha a + \beta)} \left(2a + \frac{\alpha b \overline{M}_0 \overline{M}}{2(\alpha a + \beta)^2}\right) \frac{\partial a}{\partial m}. \tag{35}$$

From (35) it is clear that $\frac{d\hat{U}^{OD}}{dm} > 0$ if $\overline{M} \geq 2b$ or equivalently if $\overline{m} \leq \alpha \hat{a}_{OD} + \beta + E[\theta] - b$. Therefore, we must have $\overline{M} < 2b$ or equivalently $\overline{m} > \alpha \hat{a}_{OD} + \beta + E[\theta] - b$.

**Part 3: Proof of $\overline{m} < \alpha \hat{a}_{OD} + \beta + E[\theta] + b$ or equivalently $\overline{M} > 0$.**

By substituting (34) into (35), we get

$$\frac{d\hat{U}^{OD}}{dm} = \frac{\overline{M}}{2(\alpha a + \beta)} \left\{ \overline{M} - 2b - \left[2a + \frac{\alpha b \overline{M}_0 \overline{M}}{2(\alpha a + \beta)^2}\right] \frac{\alpha \overline{M}_0}{(\alpha a + \beta) \frac{\partial^2 \hat{U}^{OD}}{\partial a^2}} \right\}.$$ 

If $a < \frac{b(3-\alpha^2)}{3\alpha}$, then

$$\lim_{\overline{M} \to 0^+} \left\{ \overline{M} - 2b - \left[2a + \frac{\alpha b \overline{M}_0 \overline{M}}{2(\alpha a + \beta)^2}\right] \frac{\alpha \overline{M}_0}{(\alpha a + \beta) \frac{\partial^2 \hat{U}^{OD}}{\partial a^2}} \right\} \leq -2b - 2a \frac{\alpha \cdot 2(\alpha a + \beta)}{(\alpha a + \beta) \cdot 2(\frac{\alpha^2}{3} - 1)} < 0,$$

where the first inequality follows from the fact that $\lim_{\overline{M} \to 0^+} \overline{M}_0 = E[\theta|1, a] - E[\theta|0, a] = 2(\alpha a + \beta)$ and the fact that $\frac{\partial^2 \hat{U}^{OD}}{\partial a^2} \leq 2(\frac{\alpha^2}{3} - 1)$ (see the proof of Lemma 8). The second inequality follows from the condition that $a < \frac{b(3-\alpha^2)}{3\alpha}$. Then there exists an $\overline{M}^* > 0$ such...
that
\[ \frac{d\hat{U}^{OD}_P}{d\overline{m}} = \frac{\overline{M}}{2(\alpha a + \beta)} \left\{ \overline{M} - 2b - \left[ 2a + \frac{\alpha b M_0 \overline{M}}{2(\alpha a + \beta)} \right] \frac{\alpha M_0}{(\alpha a + \beta)} \right\} < 0, \ \forall 0 < \overline{M} < M^*. \]

Therefore, if \( \hat{a}_{OD} < \frac{b(3-a^2)}{3a} \), then we must have \( \overline{M} > 0 \) or equivalently \( m < \alpha \hat{a}_{OD} + \beta + E[\theta] + b \).

\[ \square \]

**Proof of Lemma 3**

Proof. Assume that \( P = \{ -\infty, \overline{m}, (m_i, \overline{m}_i)_{i=1,..n} \} \) with \( n \geq 2 \) is an optimal delegation set, and the corresponding optimal learning effort is \( a^P \). We prove the lemma by showing that another delegation set, \( Q = \{ -\infty, \overline{m}, (l_i, \overline{l}_i) \} \) with \( L^3 \equiv \left( \frac{l_i - l_i}{2} \right)^3 = \sum_{i=1}^n \left( \frac{m_i - m_i}{2} \right)^3 \equiv \sum_{i=1}^n M_i^3 \), induces the same learning effort, i.e., \( a^P = a^Q \), and generates the same expected utility for the principal.

Firstly, it is evident that \( a^P = a^Q \) because

\[ \frac{\partial \hat{U}^{OD}_A(Q_a)}{\partial a} = 2a(\alpha a + \beta) - 2a + \frac{\alpha L^3}{3(\alpha a + \beta)} \frac{\partial - \overline{M}^3}{\partial a} \]

\[ = 2a(\alpha a + \beta) - 2a + \sum_{i=1}^n \frac{\alpha M_i^3}{3(\alpha a + \beta)} \frac{\partial \overline{M}^3}{\partial a} \]

\[ = \frac{\partial \hat{U}^{OD}_A(P_a)}{\partial a}, \ \forall a. \]

Then because \( L^3 = \sum_{i=1}^n M_i^3 \) and \( a^P = a^Q \), from (17) we have \( U_P(P, a^P) = U_P(Q, a^Q) \).

\[ \square \]

**Proof of Proposition 4.3**

Proof. Assume that \( \beta = 0 \) and \( \hat{a}_{OD} > 0 \). Assume for contradiction that \( M_1 = 0 \). From (29)
we have

$$\frac{\partial \hat{U}_A^{OD}}{\partial a} = 2 \left( \frac{\alpha^2}{3} - 1 \right) a - \frac{\alpha M^2 M_{-1/2}}{6 \alpha^2 a^2} < 0, \; \forall a > 0,$$

because $M_{-1/2} = \bar{m} - E[\theta] - \frac{1}{2}, a] - b > 0$. We thus arrive at a contradiction with $\hat{a}_{OD} > 0$ by assuming $M_1 = 0$.

To prove that $E[\theta] + b \in (\bar{m}_1, \bar{m}_1)$, it suffices to prove that $E[\theta] + b$ must be eliminated and is lower than the upper threshold: $E[\theta] + b < \bar{m}$.

We know from Appendix C that if $\beta = 0$ then $a_{FD} = 0$ and $y(s) = E[\theta] + b$ in the full delegation scheme. Therefore if the project $E[\theta] + b$ were available for choose in the optimal delegation scheme, then we would have $\hat{a}_{OD} = 0$ and $y(s) = E[\theta] + b$ in the optimal delegation scheme. This contradicts with the assumption that $\hat{a}_{OD} > 0$, and therefore $E[\theta] + b$ must be eliminated.

Then we need to prove that $E[\theta] + b < \bar{m}$. Assume for contradiction that $E[\theta] + b \geq \bar{m}$. Because $\beta = 0$, (16) becomes

$$\begin{align*}
\hat{U}_A^{OD}(P, a) &= -V(\theta) - (E[\theta] + b - \bar{m})^2, & \text{if } a = 0, \\
\hat{U}_A^{OD}(P, a) &= -V(\theta) - (E[\theta] + b - \bar{m})^2 - \left( 1 - \frac{\alpha^2}{3} \right) a^2, & \text{if } a > 0.
\end{align*}$$

Because $\alpha^2 < 3$, this is in contradiction with the assumption that $\hat{a}_{OD} > 0$.

We have therefore proven that $E[\theta] + b$ must be eliminated and is lower than the upper threshold: $E[\theta] + b < \bar{m}$. Then it follows naturally that $E[\theta] + b \in (\bar{m}_1, \bar{m}_1)$. \qed