Fractional calculus for nanoscale flow and heat transfer

Hong-Yan Liu
Ji-Huan He
Zheng-Biao Li
Fractional calculus for nanoscale flow and heat transfer

Hong-Yan Liu
School of Fashion Technology, Zhongyuan University of Technology, Zhengzhou, China

Ji-Huan He
National Engineering Laboratory for Modern Silk, Soochow University, Suzhou, China, and
Zheng-Biao Li
College of Mathematics and Information Science, Qujing Normal University, Qujing, China

Abstract

Purpose – Academic and industrial researches on nanoscale flows and heat transfers are an area of increasing global interest, where fascinating phenomena are always observed, e.g. admirable water or air permeation and remarkable thermal conductivity. The purpose of this paper is to reveal the phenomena by the fractional calculus.

Design/methodology/approach – This paper begins with the continuum assumption in conventional theories, and then the fractional Gauss’ divergence theorems are used to derive fractional differential equations in fractal media. Fractional derivatives are introduced heuristically by the variational iteration method, and fractal derivatives are explained geometrically. Some effective analytical approaches to fractional differential equations, e.g. the variational iteration method, the homotopy perturbation method and the fractional complex transform, are outlined and the main solution processes are given.

Findings – Heat conduction in silk cocoon and ground water flow are modeled by the local fractional calculus, the solutions can explain well experimental observations.

Originality/value – Particular attention is paid throughout the paper to giving an intuitive grasp for fractional calculus. Most cited references are within last five years, catching the most frontier of the research. Some ideas on this review paper are first appeared.

Keywords Fractional derivative, Variational iteration method, Air permeability, Fractal medium, Fractional heat conduction, Fractional Navier-Stokes equations

Paper type General review

1. Introduction

Fractional differential equations have been caught much attention recently due to exact description of nonlinear phenomena (Abdou and Yildirim, 2012; Erturk et al., 2012; Gupta et al., 2012; Hristov, 2013; Khan et al., 2012; Merdan et al., 2013; Wei et al., 2013; Yildirim and Kocak, 2012; Zielinski and Voller, 2013), on the other hand, understanding the behavior of a flow and heat transfer at the nanoscale has been a great interest in recent years (Aminossadati and Ghasemi, 2012; Bourantas et al., 2013; Chamkha and Rashad, 2012; Cho et al., 2012a, b; Hummer, 2007; Majumder et al., 2005; Mohammed

This work is supported by Priority Academic Program Development of Jiangsu Higher Education Institutions (PAPD), National Natural Science Foundation of China under grant No. 61303236 and No.11372205 and Project for Six Kinds of Top Talents in Jiangsu Province under grant No. ZBZZ-035, Science & Technology Pillar Program of Jiangsu Province under grant No. BE2013072, Yunnan Province NSF grant No. 2011FB090.
et al., 2012; Shanthi et al., 2012; Tham et al., 2012; Whitby and Quirke, 2007; Yahyazadeh et al., 2012), however, our understanding is rare and very preliminary, and its progress is greatly blocked by conventional assumptions, especially the continuum assumption.

Water, for example, on any observable scales is of continuity, and Newton’s laws can be applied, however, on nanoscales or even on Angstrom scales, water becomes intrinsically discontinuous. On such small scales, prediction by any conventional theories results in remarkable error up to six orders, this fascinating phenomenon is generally called as the nano-effect (He et al., 2007), which means nanoscale flow or heat transfer has admirable water or air permeation (Fan and He, 2012a-c; Fan and Shang, 2013a, b) and remarkable thermal conductivity and diffusion property Cho et al. (2012a, b).

Majumder et al. (2005) found that liquid flow through a membrane composed of an array of aligned carbon nanotubes is four to five orders of magnitude faster than would be predicted from conventional fluid-flow theory, similar phenomena were observed by other researchers (Hummer, 2007; Whitby and Quirke, 2007). This does not mean that the conservation laws in nanoscale flows are broken, but the governing equations should be derived by means of the fractional calculus, otherwise enormous error will be caused. Cho et al. (2012a, b) found that heat transfer performance can be remarkably enhanced by either copper-water nanofluid or Al₂O₃-water nanofluid; Fan and Liu (2010) revealed that wool keratin fiber reveals excellent heat transfer capability due to the nanoscale hierarchical structure. He et al. (2012) elucidated nanoscale fluid can be used to fabricate various nanomaterials.

When flow tends to nanoscales (e.g. flow in carbon nanotube, nanoscale turbulence, fractional Brownian motion), fractional calculus has to be adopted. Although the fractional calculus was invented by Leibnitz over three centuries ago, it only became a hot topic recently owing to the development of the computer and nanotechnology and its exact description of many real-life problems.

2. Smooth boundary vs fractal boundary
A smooth boundary is necessary to establish a differential equation to describe a phenomenon. Consider a steady flow in continuum mechanics, the mass conservation requires that:

\[ \oint_S \rho \mathbf{u} \cdot d\mathbf{s} = 0 \]  \hspace{1cm} (1)

where \( \mathbf{u} = (u,v,w) \), \( \rho \) is the density.

According to Gauss theory, Equation (1) becomes:

\[ \iiint_V \nabla \cdot (\rho \mathbf{u}) dV = 0 \]  \hspace{1cm} (2)

where \( S \) is the boundary of \( V \).

From Equation (2) the governing equation for the mass conversation is obtained, which is:

\[ \nabla \cdot (\rho \mathbf{u}) = 0 \]  \hspace{1cm} (3)
or:
\[
\frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) + \frac{\partial}{\partial z} (\rho w) = 0
\] (4)

In the above derivation, we assume that the boundary (S) of the volume (V) is smooth, this is required in continuum mechanics.

Governing equations for continuum media are well established by using the Gauss’ divergence theorems:

\[
\iiint_V \nabla \varphi dV = \oiint_{\partial V} n \varphi dS
\] (5)

\[
\iiint_V \nabla \cdot A dV = \oiint_{\partial V} n \cdot A dS
\] (6)

\[
\iiint_V \nabla \times A dV = \oiint_{\partial V} n \times A dS
\] (7)

Mathematically the above Gauss’ divergence theorems are invalid for fractal media, such as porous media and weaves (He, 2008). So the governing equations for noncontinuum media should be derived by fractal derivative (Fan and He, 2012a-c) or fractional calculus (Yang, 2011, 2012).

Divergence theorem of fractal media can be expressed as (Yang, 2011, 2012):

\[
\iiint_{V^{(\alpha)}} \nabla^\alpha \cdot udV^{(\gamma)} = \oiint_{S^{(\beta)}} u \cdot dS^{(\beta)}
\] (8)

where \(V^{(\alpha)}\) is the fractal medium, \(S^{(\beta)}\) is its boundary, \(\alpha\) is the fractal dimensions of axes, \(\beta = 2x, \gamma = 3x\).

The mass of fluid in a fractal medium \(V^{(\gamma)}\) is:

\[
M = \iiint_{V^{(\gamma)}} \rho dV^{(\gamma)}
\] (9)

and the mass conservation of fluid in a fractal medium is:

\[
\frac{\partial^\alpha M}{\partial t^\alpha} = - \oiint_{S^{(\beta)}} \rho u \cdot dS^{(\beta)}.
\] (10)

Using divergence theorem of local fractional field, we obtain (Yang, 2011, 2012):

\[
\iiint_{V^{(\gamma)}} \frac{\partial^\alpha \rho}{\partial t^\alpha} dV^{(\gamma)} + \oiint_{S^{(\beta)}} \rho u \cdot dS^{(\beta)} = \iiint_{V^{(\gamma)}} \left[ \frac{\partial^\alpha \rho}{\partial t^\alpha} + \nabla^\alpha \cdot (\rho u) \right] dV^{(\gamma)} = 0
\] (11)
which implies the following fractional differential equation:

$$\frac{\partial^2 \rho}{\partial t^2} + \nabla^2 \cdot (\rho \mathbf{v}) = 0.$$  \hfill (12)

The momentum equation in a fractal medium is (Yang, 2011, 2012):

$$D_a^x \int \int \int_{V^{(x)}} \rho \mathbf{v} \mathbf{d}V^{(x)} = \int \int \int_{V^{(x)}} \rho \mathbf{b} \mathbf{d}V^{(x)} + \int \int \int_{S^{(p)}} \mathbf{J} \cdot \mathbf{d}S^{(p)},$$  \hfill (13)

where \( \mathbf{J} \) denotes the fractal Cauchy stress tensor and \( \mathbf{b} \) denotes the specific fractal body force.

Equation (13) can be re-written in the form:

$$D_a^x \int \int \int_{V^{(x)}} \rho \mathbf{v} \mathbf{d}V^{(x)} = \int \int \int_{V^{(x)}} \nabla^2 (\rho \mathbf{v}) \mathbf{d}V^{(x)} + \int \int \int_{S^{(p)}} \rho \mathbf{v} \mathbf{b} \cdot \mathbf{d}S^{(p)} = \int \int \int_{V^{(x)}} \rho \mathbf{b} \mathbf{d}V^{(x)}$$

$$+ \int \int \int_{S^{(p)}} \mathbf{J} \cdot \mathbf{d}S^{(p)}$$

from which a fractional partner for Navier-Stokes can be obtained (Yang, 2011, 2012):

$$\frac{\partial^2 (\rho \mathbf{v})}{\partial t^2} + \nabla^2 \cdot (\rho \mathbf{v}) = \rho \mathbf{b} + \nabla^2 \cdot \mathbf{J}.$$  \hfill (15)

Using mass equation, Equation (12), fractional Navier-Stokes can be simplified as (Yang, 2011, 2012):

$$\rho \frac{D^x \mathbf{v}}{Dt^2} = \rho \mathbf{b} + \nabla^2 \cdot \mathbf{J}.$$  \hfill (16)

In the 3D Cantorian coordinates, systems of Navier-Stokes equations for incompressible flow on fractal medium can be written as (Yang, 2011, 2012; Yang et al., 2013):

$$\begin{cases}
\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} = 0 \\
\rho \frac{D^x v_x}{Dt^2} = -\frac{\partial \rho}{\partial x} + \mu \left( \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho b_x \\
\rho \frac{D^y v_y}{Dt^2} = -\frac{\partial \rho}{\partial y} + \mu \left( \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho b_y \\
\rho \frac{D^z v_z}{Dt^2} = -\frac{\partial \rho}{\partial z} + \mu \left( \frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right) + \rho b_z 
\end{cases}$$

\hfill (17)

The above derivation is valid for an isotropic fractal spacetime, that means the time is discontinuous with fractional dimensions of \( t \), and the isotropic fractal medium has fractional dimensions of \( x, y, z \) on each axis. On quantum scale, time does be discontinuous, however, in many practical applications, time can be considered to be continuous, and a porous medium is generally of anisotropic fractal. Considering these facts, the mass equation in a porous medium can be expressed as:

$$\frac{\partial \rho}{\partial t} + \frac{\partial^2 \rho x_v}{\partial x^2} + \frac{\partial^2 \rho y_v}{\partial y^2} + \frac{\partial^2 \rho z_v}{\partial z^2} = 0$$

\hfill (18)
where \( a_1, a_2 \) and \( a_3 \) are, respectively, the fractional dimensions in x-, y- and z-directions.

The heat conduction equation in fractal media can be expressed as (Yang and Baleanu, 2013):

\[
K^{2x} \nabla^{2x} T - \rho \alpha^{2x} \frac{\partial^2 T}{\partial t^2} = 0
\]  

or:

\[
K^{2x} \left( \frac{\partial^{2x} T}{\partial x^{2x}} + \frac{\partial^{2x} T}{\partial y^{2x}} + \frac{\partial^{2x} T}{\partial z^{2x}} \right) - \rho \alpha^{2x} \frac{\partial^2 T}{\partial t^2} = 0
\]

where \( \nabla^{2x} = \partial^{2x}/\partial x^{2x} + \partial^{2x}/\partial y^{2x} + \partial^{2x}/\partial z^{2x} \) is local fractional Laplace operator (Yang, 2011, 2012). In practical applications, e.g. the governing equation for heat conduction in a porous medium is:

\[
K_x \frac{\partial^{2x} T}{\partial x^{2x}} + K_y \frac{\partial^{2x} T}{\partial y^{2x}} + K_z \frac{\partial^{2x} T}{\partial z^{2x}} - \rho \alpha^{2x} \frac{\partial T}{\partial t} = 0
\]

where \( a_1, a_2 \) and \( a_3 \) are, respectively, the fractional dimensions in x-, y- and z-directions, \( K_x, K_y \) and \( K_z \) are, respectively, thermal conductivity coefficients in x-, y- and z-directions.

3. Definition of the fractional derivatives

There are many definitions on fractional derivatives, including Riemann-Liouville fractional derivative, Caputo fractional derivative, Jumarie fractional derivative. A complete review on various definitions is given by Yang (2011) in his monograph. Hereby we want to introduce the fractional derivatives by the variational iteration method (He, 1998, 1999a, b, 2007; He and Wu, 2007).

We consider the following linear equation of n-th order:

\[
u^{(n)} = f(t)
\]

By the variational iteration method (He, 1998, 1999a, b, 2007; He and Wu, 2007), we can construct the following iteration formulation:

\[
u_{m+1}(t) = \nu_m(t) + (-1)^n \int_{t_0}^t \frac{1}{(n-1)!} (s-t)^{n-1} \left[ \nu_m^{(n)}(s) - f_m(s) \right] ds.
\]

For a linear equation, from Equation (22), we have the following exact solution:

\[
u(t) = \nu_0(t) + (-1)^n \int_{t_0}^t \frac{1}{(n-1)!} (s-t)^{n-1} \left[ \nu_0^{(n)}(s) - f(s) \right] ds.
\]

where \( \nu_0(t) \) satisfies the boundary/initial conditions.

Introduce an integration operator \( I^n \) defined by:

\[
I^n f = \frac{1}{(n-1)!} \int_{t_0}^t (s-t)^{n-1} \left[ \nu_0^{(n)}(s) - f(s) \right] ds = \frac{1}{\Gamma(n)} \int_{t_0}^t (s-t)^{n-1} \left[ f_0(s) - f(s) \right] ds
\]

where \( f_0(t) = \nu_0^{(n)}(t) \).
We can define a fractional derivative in the form:
\[
D^a_t f = D^a_t \frac{d^n}{dt^n} (I^{n-a} f) = \frac{1}{\Gamma(n-a)} \frac{d^n}{dt^n} \int_{t_0}^{t} (s-t)^{n-a-1} [f_0(s) - f(s)] ds
\]

(26)

If \(f_0(t)\) is continuous but not differentiable anywhere, we have (Jumarie, 2006):
\[
f_0(t) = f(t_0) + \frac{(t-t_0)}{\Gamma(1+a)} f^{(a)}(t_0) + \frac{(t-t_0)^2}{\Gamma(1+2a)} f^{(2a)}(t_0) + \cdots + \frac{(t-t_0)^{n-1}}{\Gamma(1+(n-1)a)} f^{(n-1)a)}(t_0)
\]

(27)

where \(f^{(n)x)}(t) = \underbrace{D^x D^x \cdots D^x f(t)}_{n \text{ times}}\).

Equation (26) becomes:
\[
D^a_t f = \frac{1}{\Gamma(n-a)} \frac{d^n}{dt^n} \int_{t_0}^{t} (s-t)^{n-a-1} \left[ \sum_{i=0}^{n-1} \frac{(s-t_0)^i}{\Gamma(1+ia)} f^{(ia)}(t_0) - f(s) \right] ds
\]

(28)

Keeping only the first term of \(f_0(s)\), we give another definition of fractional derivative in the form:
\[
D^a_t f = \frac{1}{\Gamma(n-a)} \frac{d^n}{dt^n} \int_{t_0}^{t} (s-t)^{n-a-1} [f(t_0) - f(s)] ds
\]

(29)

Hereby \(f\) can be continuous and possibly not differentiable anywhere.

Equation (24) is the variational iteration algorithm-I (He et al., 2010; He, 2012b). The variational iteration algorithm-II is (He et al., 2010; He, 2012a, b):
\[
u_{m+1}(t) = u_0(t) - (-1)^n \int_{t_0}^{t} \frac{1}{(n-1)!} (s-t)^{n-1} f_m(s) ds = u_0(t) - (-1)^n \int_{t_0}^{t} (s-t)^{n-1} f_m(s) ds
\]

(30)

Note: \(u_0\) must satisfy the initial/boundary conditions.

For a linear equation, we have:
\[
u(t) = u_0(t) - (-1)^n \int_{t_0}^{t} (s-t)^{n-1} f(s) ds
\]

(31)
We can define another fractional derivative in the form:

\[
D^a f(t) = \frac{1}{\Gamma(n - a)} \frac{d^n}{dt^n} \int_{t_0}^{t} (s - t)^{n-a-1} f(s) ds
\]  

(32)

Recently the local fractional derivative has been caught much attention due to its simple chain rule, which is defined as (Yang, 2011, 2012):

\[
f^{(x)}(x_0) = \left. \frac{d^k f(x)}{dx^k} \right|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^x (f(x) - f(x_0))}{(x - x_0)^x},
\]

(33)

where \(\Delta^x (f(x) - f(x_0)) \cong \Gamma(1 + x)\Delta(f(x) - f(x_0)).\)

The local fractional derivative has the following simple chain rules:

\[
\frac{\partial^k f(x)}{\partial x^{\alpha x}} = \frac{\partial^2}{\partial x^2} \cdots \frac{\partial^k}{\partial x^k} f(x)
\]

(34)

\[
\frac{d^k f(g(x))}{dx^k} = f^{(1)}(g(x))g^{(x)}(x)
\]

(35)

4. Definition of the fractal derivatives

The fractal derivative can be categorized as a special local fractional derivative. There are also several definitions. Chen’s definition is as follows (Chen et al., 2010):

\[
\frac{du(x)}{dx^a} = \lim_{s \to x} \frac{u(x) - u(s)}{x^a - s^a},
\]

(36)

where \(a\) is the order of the fractal derivative.

The fractal derivative (He, 2011a) can be understood as a derivative in a fractal medium. We consider a fractal medium illustrated in Figure 1, and assume the smallest measure is \(L_0\), any discontinuity less than \(L_0\) is ignored, then the distance between two

![Figure 1](image-url)  

The distance between two points in a discontinuous spacetime.
The fractal derivative can be defined as:

\[
\frac{Du}{Dx^a} = \lim_{\Delta x \to L_0} \frac{u(A) - u(B)}{k(\Delta x)^a} = \lim_{\Delta x \to L_0} \frac{u(A) - u(B)}{k(x_A - x_B)^a}
\]  

(37)

where \(k\) is a constant, \(a\) is the fractal dimension, the distance between two points in a discontinuous space can be expressed as \(ds = kL_{0}^{\alpha}\), where \(k\) is a function of fractal dimension, \(k = k(\alpha)\), and it follows:

\[
k(1) = 1, \quad k(x > L_0) = 1 \quad \text{and} \quad \alpha(x > L_0) = 1
\]

(38)

The fractal dimension is defined as:

\[
\alpha = \frac{\ln M}{\ln N}
\]

(39)

where \(M\) is the number of new units within the original unit with a new dimension, \(N\) is the ratio of the original dimension to the new dimension. In most practical applications, \(\alpha \approx 1\), and it can be approximated as:

\[
\alpha = \frac{\ln M}{\ln N} \approx \frac{M - 1}{N - 1} \approx \frac{M}{N}
\]

(40)

To show this, we consider a Koch curve, where \(M = 4\) and \(N = 3\), \(\alpha = \ln 4/\ln 3 = 1.26 \approx 4/3\). As illustrated in Figure 1, \(kL_0^\alpha\) is the length of zigzag line on dimension of \(L_0\) (in practical applications, \(L_0\) can be considered as a unit length, \(L_0 = 1\)), according to fractal geometry, we have:

\[
kL_0^\alpha = \frac{M}{N} \approx \alpha
\]

(41)

The fractal derivative is then updated as:

\[
\frac{Du}{Dx^a} = \lim_{\Delta x = x_A - x_B \to L_0} \frac{u(A) - u(B)}{\frac{M}{N}(x_A - x_B)^a}
\]

(42)

or

\[
\alpha \frac{Du}{Dx^a} = \lim_{\Delta x = x_A - x_B \to L_0} \frac{u(A) - u(B)}{(x_A - x_B)^a}
\]

(43)

Considering the fact that \(\alpha\) approaches to unit, sometimes we also use \(Du/Dx^a = \lim_{x_A - x_B \to L_0} (u(A) - u(B))/(x_A - x_B)^a\) for simplicity.

Applications of the fractal derivative to fractal media have been caught much attention, for examples, it can model heat transfer and water permeation in multi-scale fabric and wool fibers (Fan and He, 2012a-c; Fan and Shang, 2013a, b).

5. Variational iteration method

The variational iteration method (He, 1998, 1999a, b, 2007; He and Wu, 2007) has been shown to solve a large class of nonlinear differential problems effectively, easily and
accurately with the approximations converging rapidly to accurate solutions, and it becomes an effective mathematical tool for various nonlinear problems (Bildik and Konuralp, 2006; Ghanei et al., 2012; Hosseini et al., 2012; Matinfar and Ghasemi, 2013). Though the fractional calculus can trace its history back to Leibniz (July 1, 1646-November 14, 1716), no effective analytical method was available before 1998 for such equations even for linear fractional differential equations.

In 1998, the variational iteration method was first proposed to solve fractional differential equations with greatest success (He, 1998). Following the above idea, Draganescu (2006), Odibat and Momani (2006) applied the variational iteration method to more complex fractional differential equations, showing effectiveness and accuracy of the used method. In 2002 the Adomian method was suggested to solve fractional differential equations (Shawagfeh, 2002). But many researchers found it is very difficult to calculate the Adomian polynomial. Ghorbani and Saberi-Nadjafi (2007) suggested a very simple method for calculation the Adomian polynomial using the homotopy perturbation method (He, 1999a, b, 2007; He and Wu, 2007):}

\[ f_m(s) = f_m(u(s), u'(s), \ldots , u^{(n)}(s)) \]

\[ D^\alpha_t u + f = 0 \]
Its variational iteration formulation can be obtained as:

\[
  u_{n+1}(t) = u_n(t) + \frac{(-1)^x}{\Gamma(x)} \int_0^t (s - t)^{x-1} \left( D^x u_n(s) + f_n(s) \right) ds
\]  

(47)

As an example, we consider heat conduction in silk cocoon. It was found that the cocoon of the silkworm, *Bombyx mori*, while creating a tough barrier offering mechanical protection to the pupa, imposes no barrier to the diffusion of oxygen or water vapor (Blossman-Myer and Burggren, 2010; Chen *et al.*, 2012, 2013; Jiang *et al.*, 2006). The pupa has also a superior ability to survive in an extremely harsh regions from \(-40^\circ C\) to \(50^\circ C\). This phenomenon of excellent thermal protection can be explained by the local fractional calculus.

Silk cocoon of the silkworm, *Bombyx mori*, has special hierarchical microstructures, the fiber diameter reduces greatly from 26 \(\mu m\) in the outer layer to 16 \(\mu m\) in the inner layer (pelade), which enables the cocoon to have superior mechanical properties, excellent protective functions and remarkable transfer efficacy for oxygen and water vapor (Blossman-Myer and Burggren, 2010; Chen *et al.*, 2012; Jiang *et al.*, 2006). It was once thought to be that the cocoon would help conserve water, but Blossman-Myer and Burggren’s (2010) experimental data reveal that the cocoon affects no oxygen and water vapor transfer, as if the cocoon would not exist, silk cocoon is a real “emperor’s new clothes” for pupa (Chen *et al.*, 2013). A theoretical analysis for such fascinating phenomenon is much needed, and the local fractional calculus is adopted for this purpose.

In continuous space, heat transfer follows Fourier’s law. The 1D steady heat flux can be written in the form:

\[
  q = k \frac{dT}{dx}
\]  

(48)

where \(k\) is the heat transfer coefficient.

For the case of heat transfer in fractal media, Fourier’s law is modified as follows (Yang, 2011, 2012):

\[
  K^{2x} \frac{d^x T(x)}{dx^x} = q(x), \quad T(0) = T_0
\]  

(49)

where \(K^{2x}\) denotes the thermal conductivity in the fractal medium, and \(d^x/dx^x\) is local fractional derivative with order \(x\).

Mittag-Leffler type Fourier flux distribution in fractal media can be written as (Yang, 2011, 2012):

\[
  q(x) = E_x(x - t)^x T(t)
\]  

(50)

In order to apply the variational iteration method to fractional calculus, a basic knowledge for local fractional variational principle is needed. Consider the following fractional functional:

\[
  I(y) = a I^b_b f \left( x, y(x), y^{(x)}(x) \right)
\]  

(51)

where \(y^{(x)}(x)\) is taken in local fractional differential operator and \(a \leq x \leq b\).
Local fractional integral of \( f(x) \) of order \( \alpha \) in the interval \([a,b]\) is given by (Yang, 2011, 2012):

\[
a I_a^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t) (dt)^\alpha = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} f(t_j) (\Delta t_j)^\alpha,
\]

where \( \Delta t_j = t_{j+1} - t_j \), \( \Delta t = \max\{\Delta t_1, \Delta t_2, \Delta t_i, \ldots\} \) and \([t_j, t_{j+1}]\), \( j = 0, \ldots, N - 1 \), \( t_0 = a \), \( t_N = b \), is a partition of the interval \([a, b]\).

The stationary condition of Equation (51) reads (Yang, 2011, 2012):

\[
\frac{\partial f}{\partial y} - \frac{d^x}{dx^x} \left( \frac{\partial f}{\partial y(x)} \right) = 0
\]

Equation (53) is useful for identification of the Lagrange multiplier in the local fractional variational iteration method. Consider a general local fractional differential equation:

\[
L_\alpha u_x + N_\alpha u_x = 0
\]

where \( L_\alpha \) and \( N_\alpha \) are linear and nonlinear local fractional operators respectively.

According to the variational iteration method (He, 1998, 1999a, b, 2007; He and Wu, 2007), we can construct a correction local fractional functional in the form:

\[
u_{n+1}(t) = u_n(t) + t_0 I_t^{(\alpha)} \{ \lambda^2 [L_\alpha u_n(s) + N_\alpha u_n(s)] \}
\]

where \( \tilde{u}_n \) is considered as a restricted local fractional variation and \( \lambda^2 \) is a Lagrange multiplier.

Making the functional, Equation (55), stationary, we can identify the Lagrange multiplier easily. After identification of the multiplier, we can obtain the following three local fractional variational iteration algorithms (He and Liu, 2013).

The variational iteration algorithm-I:

\[
u_{n+1}(t) = u_n(t) + t_0 I_t^{(\alpha)} \{ \lambda^2 [L_\alpha u_n(s) + N_\alpha u_n(s)] \}
\]

The variational iteration algorithm-II:

\[
u_{n+1}(t) = u_0(t) + t_0 I_t^{(\alpha)} [\lambda^2 N_\alpha u_n(s)]
\]

The variational iteration algorithm-III:

\[
u_{n+1}(t) = u_n(t) + t_0 I_t^{(\alpha)} [\lambda^2 N_\alpha u_n(s)] - t_0 I_t^{(\alpha)} [\lambda^2 N_\alpha u_{n-1}(s)]
\]

For Equation (49), the Lagrange multiplier can be identified as:

\[
\lambda^2 = - \frac{\lambda^2}{\Gamma(1+\alpha)}
\]
where $\xi = 1/K^2$. As a result the following variational iteration algorithms are obtained.

The variational iteration algorithm-I:

\[
T_{n+1}(x) = T_n(x) - \frac{\xi^2}{\Gamma(1+\alpha)} \int_0^x \left[ K^{2\alpha} \frac{d^2 T_n(s)}{ds^2} - q_n(s) \right] (ds)^\alpha
\]

(60)

The variational iteration algorithm-II:

\[
T_{n+1}(x) = T_0(x) + \frac{\xi^2}{\Gamma(1+\alpha)} \int_0^x q_n(s)(ds)^\alpha
\]

(61)

The variational iteration algorithm-III:

\[
T_{n+1}(x) = T_n(x) + \frac{\xi^2}{\Gamma(1+\alpha)} \int_0^x q_n(s)(ds)^\alpha - \frac{\xi^2}{\Gamma(1+\alpha)} \int_0^x q_{n-1}(s)(ds)^\alpha
\]

(62)

We begin with $T_0(x) = T(0) = T_0$, by Equation (61), we have:

\[
T_1(x) = T_0 + \frac{\xi^2}{\Gamma(1+\alpha)} \int_0^x E_\alpha(x-t)^\alpha T_0(dt)^\alpha.
\]

(63)

Proceeding in this manner, we have the second approximation:

\[
T_2(x) = T_0 + \frac{\xi^2}{\Gamma(1+\alpha)} \int_0^x E_\alpha(x-t)^\alpha T_1(t)(dt)^\alpha.
\]

(64)

and:

\[
T_n(x) = T_0 + \frac{\xi^2}{\Gamma(1+\alpha)} \int_0^x E_\alpha(x-t)^\alpha T_{n-1}(t)(dt)^\alpha
\]

(65)

Taking the limit, we have:

\[
T(x) = \lim_{n \to \infty} T_n(x)
\]

\[
= T_0 + \frac{\xi^2}{\Gamma(1+\alpha)} \int_0^x E_\alpha[(x-t)^\alpha(1+\lambda)^\alpha] T_0(dt)^\alpha
\]

(66)

\[
= T_0 \left\{ 1 + \frac{1}{(1+K^2)^\alpha \Gamma(1+\alpha)} E_\alpha \left[ (x)^\alpha \left( 1 + \frac{1}{1+K^2} \right)^\alpha \right] \right\}
\]

where $1 + (1/(1+K^2))^\alpha$ is a fractal real line number (Yang, 2011, 2012).
Assume that the thickness of the cocoon is \( L \), then we have the body temperature of pupa:

\[
T(L) = T_0 \left\{ 1 + \frac{1}{(1 + K^2)^2 \Gamma(1 + \alpha)} E_\alpha \left[ (L)^\alpha \left( 1 + \frac{1}{1 + K^2} \right)^2 \right] \right\}
\]

The fractal derivative at \( x = L \) is:

\[
\frac{DT}{Dx^\alpha} (x = L) = \frac{T(L) - T_0}{(L)^\alpha} \left\{ \frac{1}{(1 + K^2)^2 \Gamma(1 + \alpha)} E_\alpha \left[ (L)^\alpha \left( 1 + \frac{1}{1 + K^2} \right)^2 \right] \right\}
\]

The ideal case for pupa is:

\[
T(L) = \text{constant} \quad \text{and} \quad \frac{DT}{Dx^\alpha} (x = L) = 0
\]

However, this cannot be satisfied. Actually the fractal derivation at \( x = L \) is much smaller than its continuum model especially for the case when \( L \) tends to micro or nanoscale. A better understanding of the cocoon mechanism could help the further design of cocoon-like space suits or other protective clothes for special applications.

6. Homotopy perturbation method

The homotopy perturbation method (He, 1999a, b, 2000, 2010, 2012a, b) was originally proposed to nonlinear differential equations, and it becomes an effective analytical method for various nonlinear problems (Ganji et al., 2012; Gupta et al., 2012; Madani et al., 2012; Malvandi et al., 2012; Petroudi et al., 2012; Vanani et al., 2013; Yun and Temuer, 2013). In 2007, Momani and Odibat (2007) adopted the method for fractional differential equations with great success, and now it is an effective method for fractional calculus (Ganji et al., 2008; Gupta et al., 2012; Jafari and Momani, 2007; Khan et al., 2012; Liu, 2012; Madani et al., 2012; Odibat and Momani, 2008). In 2010, the Laplace transform and He's polynomials were used in the homotopy perturbation method (Khan and Mohyud-Din, 2010). The Laplace transform is a well-known mathematical tool for linear equations, while it cannot deal with the nonlinear terms. In order to perform the inverse Laplace transform, He's polynomials (Ghorbani, 2007; Khan and Mohyud-Din, 2010; Matinfar and Ghasemi, 2013) are widely applied. Laplace transform can also be used in the variational iteration method to identify the Lagrange multiplier (Wu, 2012a, b).

Consider a following fractional differential equation:

\[
D^\alpha u(x, t) + Ru(x, t) + Nu(x, t) = q(x, t),
\]

where \( D^\alpha \) is the fractional derivative, \( R \) is the linear operator, \( N \) represents the general nonlinear differential operator and \( q(x, t) \) is continuous and exponential order function.

Applying Laplace transform to Equation (70), we have:

\[
L\{D^\alpha u(x, t) + Ru(x, t) + Nu(x, t) - q(x, t)\} = 0
\]

We construct a homotopy equation in the form:

\[
L\{D^\alpha u + R - q\} + \beta L\{Nu\} = 0
\]
In order to make the solution process simple, He's polynomials are included (Ghorbani, 2007):

\[ Nu = \sum_{i=0}^{\infty} \beta^i H_i(u) \]  

(73)

where \( H_i(u) \) are He's polynomials.

The solution is expanded into a series of \( p \) in the form:

\[ u = u_0 + pu_1 + p^2u_2 + \cdots \]  

(74)

The solution process is same as that by the classical perturbation method.

It was Gondal and Khan who walked the first step to couple the homotopy perturbation method and the Laplace transform. Afterwards, the modified homotopy perturbation method coupled with the Laplace transform became a hot topic in analytical methods.

The names for the modification of the homotopy perturbation method include: first, Laplace homotopy perturbation method (Madani et al., 2011); second, homotopy perturbation transform method (Liu, 2012); third, He-Laplace method (Mishra and Nagar, 2012); fourth, a coupled method of homotopy perturbation method and Laplace transform (Khan et al., 2012) and others. All the above names mean a same technology, and it is now widely applied for fractional calculus, and it is recommended to the terminology by Mishra and Nagar (2012) or the modified homotopy perturbation method. The homotopy perturbation method can also couple with other transform technologies, for example Sumudu transform (Elbeleze et al., 2013).

7. Fractional complex transform

The fractional complex transform (Li and He, 2010; He and Li, 2012) was originally suggested to convert a fractional differential equation with Jumarie’s modification of Riemann-Liouville derivative into its classical differential partner.

Consider the following general fractional differential equation:

\[
\begin{align*}
  f\left(u, u_x^{(\alpha)}, u_y^{(\beta)}, u_z^{(\gamma)}, u_t^{(2\alpha)}, u_t^{(2\beta)}, u_t^{(2\gamma)}, \ldots\right) &= 0, 0 < \alpha \leq 1, \ 0 < \beta \leq 1, \\
  &0 < \gamma \leq 1, 0 < \lambda \leq 1
\end{align*}
\]  

(75)

Originally \( u_x^{(\alpha)} = D_x^{\alpha} u = D^\alpha u / D t^{\alpha} \) denotes Jumarie’s fractional derivation, which is a modified Riemann-Liouville derivative defined as (Jumarie, 2006, 2007a, b, 2009). However, an counter-example was found (He et al., 2012), making the method much skeptical. The main problem for its applications is how to define the fractional derivative. The previous demerit can be completely eliminated when the local fractional derivative is used.

In Equation (75), the local fractional derivative is adopted, and the chain rules (Equations (34) and (35)) can be powerfully applied. By the fractional complex transform (Li and He, 2010; He and Li, 2012):

\[
\begin{align*}
  s &= \frac{(p)^{\alpha}}{\Gamma(1 + \alpha)} \\
  X &= \frac{(qx)^{\beta}}{\Gamma(1 + \beta)} \\
  Y &= \frac{(nx)^{\gamma}}{\Gamma(1 + \gamma)} \\
  Y &= \frac{(ny)^{\lambda}}{\Gamma(1 + \lambda)}
\end{align*}
\]  

(76)
Equation (76) can be converted into the following differential equation:
\[ f(u, u_s, u_X, u_Y, u_Z, \ldots) = 0, \quad (77) \]

This equation can be solved by the exp-function method (He and Wu, 2006a, b; Wu and He, 2007, 2008; Bekir and Aksoy, 2013; Mohyud-Din et al. 2012b).

Consider the local fractional differential equation in the form:
\[ \frac{d^\alpha U_1(x)}{dx^\alpha} + \frac{d^\alpha U_2(y)}{dy^\alpha} = 0, \quad 0 < \alpha \leq 1 \quad (78) \]

By the fractional complex transform (He and Li, 2012):
\[
\begin{align*}
X &= \frac{(px)^\alpha}{\Gamma(1+\alpha)} \\
Y &= \frac{(qy)^\alpha}{\Gamma(1+\alpha)}
\end{align*}
\quad (79)
\]

where \( p \) and \( q \) are constants.

Equation (78) turns out to be the following ordinary differential equation:
\[ p^\alpha dU_1(X) + q^\alpha dU_2(Y) = 0 \quad (80) \]

As an example, we consider the local fractional Richards’ equation for ground water flow. The Richards’ equation (Richards, 1931) is the most often used model for water transport in soils (Miller and Miller, 1956; Pachepsky et al., 2003; Ramos et al., 1996; Sadeghi et al., 2012). It was derived by Darcy’s law and the mass conservation law. However, the Richards’ equation can be greatly improved if we consider the soil as a fractal porous media using the local fractal calculus (Yang, 2011, 2012) or the fractal derivative (He, 2011a, b; Fan and He, 2012a, b, c).

Recently fractional models for water transport in soils have been caught much attention (Benson et al., 2013; Di Carlo et al., 2012; Gerolymatou et al., 2006; Tegnander, 2001), in this paper the local fractional theory proposed by Yang (2012) is adopted.

The Darcy’s law and the mass conservation law for fractal porous media can be expressed, respectively, as (Yang, 2011, 2012):
\[ q = -D \frac{\partial^\alpha \theta}{\partial x^\alpha}, \quad (81) \]
and:
\[ \frac{\partial \theta}{\partial t} = -\frac{\partial^\alpha q}{\partial x^\alpha} \quad (82) \]

where \( q \) is flow flux in the soil, \( \theta \) is the volumetric soil water content, \( D \) is the soil water diffusivity in fractal porous media, \( \partial^\alpha /\partial x^\alpha \) is the local fractional derive defined as (Yang, 2011, 2012).

Combining Equations (81) and (82) together results in the local fractional Richards equation, which reads:
\[ \frac{\partial \theta}{\partial t} = \frac{\partial^\alpha}{\partial x^\alpha} \left( D \frac{\partial^\alpha \theta}{\partial x^\alpha} \right) \quad (83) \]

where \( \alpha \) can be explained as the value of the fractional dimensions of the porous media in \( x \) direction (He et al., 2012).
The fractional complex transform (He and Li, 2012; He, 2012a, b) is to convert a fractional differential equation to a partial differential equation. Introducing a fractional complex transform (He and Li, 2012; He, 2012a, b):

\[ s = \frac{x^z}{\Gamma(1+z)} \]  

We can convert Equation (83) to a partial differential equation, which is:

\[ \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial s} \left( \frac{D \partial \theta}{\partial s} \right) \]  

Introduce a generalized Boltzmann variable, \( \lambda \), defined as:

\[ \lambda = s^2 t^{-1} = \left( \frac{s}{t^{1/2}} \right)^2 \]  

Equation (85) becomes an ordinary differential equation:

\[ \frac{\partial}{\partial \lambda} \left( D \frac{\partial \theta}{\partial \lambda} \right) + \frac{1}{4} \frac{\partial \theta}{\partial \lambda} = 0 \]  

The transform, Equation (86), is different from classical Boltzmann transform, which is \( \lambda = s/t^{1/2} \) (Miller and Miller, 1956).

Integrating Equation (86) with respect to \( \lambda \) results in:

\[ D(\theta) \frac{\partial \theta}{\partial \lambda} + \frac{1}{4} \theta = C \]  

where \( C \) is a constant.

Solving Equation (88) results in the following exact solution:

\[ \lambda = \int _0 ^{\theta} \frac{AD(s)}{4C - s} ds \]  

or:

\[ \frac{x^{2z}}{t} = [\Gamma(1+z)]^2 \int _0 ^{\theta} \frac{AD(s)}{4C - s} ds \]  

where \( C \) can be determined using the initial/boundary condition.

Equation (90) means that for a fixed volumetric water content we have:

\[ \frac{x_{1}^{2z}}{t_1} = \frac{x_{2}^{2z}}{t_2} = \frac{x_{3}^{2z}}{t_3} = \ldots \]  

In general for a fixed water content, we have:

\[ x = \frac{A(\theta)}{t^{1/2z}} \]
where the constant $A$ depends only on water content. Equation (92) characterizes the moving wetted front. When $a = 1$, Equation (92) is same as that predicted by the Richards equation (Pachepsky et al., 2003). The value of $a$ for various soil is recommended in Table I according to the experiment (Gardner and Widtsoe, 1921; Rawlins and Gardner, 1963).

Table I Values of the parameter $q$ found from data on horizontal movement of water to soil columns (Pachepsky et al., 2003).

<table>
<thead>
<tr>
<th>Data source</th>
<th>Soil</th>
<th>$q$</th>
<th>Fractal dimensions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gardner and Widtsoe (1921)</td>
<td>Name and texture not reported</td>
<td>0.417 ± 0.006</td>
<td>0.834 ± 0.012</td>
</tr>
<tr>
<td>Nielsen et al. (1962)</td>
<td>Columbia silt loam wet at 50 mb</td>
<td>0.402 ± 0.003</td>
<td>0.804 ± 0.006</td>
</tr>
<tr>
<td></td>
<td>Columbia silt loam wet at 100 mb</td>
<td>0.425 ± 0.006</td>
<td>0.850 ± 0.012</td>
</tr>
<tr>
<td></td>
<td>Columbia silt loam wet with oil at 2 mb</td>
<td>0.480 ± 0.008</td>
<td>0.960 ± 0.016</td>
</tr>
<tr>
<td></td>
<td>Columbia silt loam wet with oil at 38 mb</td>
<td>0.440 ± 0.003</td>
<td>0.880 ± 0.006</td>
</tr>
<tr>
<td></td>
<td>Hesperia sandy loam at 2 mb</td>
<td>0.440 ± 0.004</td>
<td>0.880 ± 0.008</td>
</tr>
<tr>
<td></td>
<td>Hesperia sandy loam at 50 mb</td>
<td>0.384 ± 0.002</td>
<td>0.768 ± 0.004</td>
</tr>
<tr>
<td></td>
<td>Hesperia sandy loam wet at 100 mb</td>
<td>0.344 ± 0.003</td>
<td>0.688 ± 0.006</td>
</tr>
<tr>
<td>Rawlins and Gardner (1963)</td>
<td>Salkum silty clay loam, $\theta = 0.51$</td>
<td>0.439 ± 0.007</td>
<td>0.878 ± 0.014</td>
</tr>
<tr>
<td></td>
<td>Salkum silty clay loam, $\theta = 0.50$</td>
<td>0.430 ± 0.008</td>
<td>0.860 ± 0.016</td>
</tr>
<tr>
<td></td>
<td>Salkum silty clay loam, $\theta = 0.48$</td>
<td>0.437 ± 0.011</td>
<td>0.874 ± 0.022</td>
</tr>
<tr>
<td></td>
<td>Salkum silty clay loam, $\theta = 0.45$</td>
<td>0.467 ± 0.009</td>
<td>0.934 ± 0.018</td>
</tr>
<tr>
<td></td>
<td>Salkum silty clay loam, $\theta = 0.40$</td>
<td>0.479 ± 0.003</td>
<td>0.958 ± 0.006</td>
</tr>
<tr>
<td></td>
<td>Salkum silty clay loam, $\theta = 0.05$</td>
<td>0.461 ± 0.002</td>
<td>0.922 ± 0.004</td>
</tr>
<tr>
<td>Ferguson and Gardner (1963)</td>
<td>Salkum silty clay loam, $\theta = 0.05$</td>
<td>0.454 ± 0.002</td>
<td>0.908 ± 0.004</td>
</tr>
<tr>
<td></td>
<td>Salkum silty clay loam, $\theta = 0.10$</td>
<td>0.453 ± 0.002</td>
<td>0.906 ± 0.004</td>
</tr>
<tr>
<td></td>
<td>Salkum silty clay loam, $\theta = 0.15$</td>
<td>0.452 ± 0.003</td>
<td>0.904 ± 0.006</td>
</tr>
<tr>
<td></td>
<td>Salkum silty clay loam, $\theta = 0.20$</td>
<td>0.452 ± 0.003</td>
<td>0.904 ± 0.006</td>
</tr>
<tr>
<td></td>
<td>Salkum silty clay loam, $\theta = 0.25$</td>
<td>0.452 ± 0.003</td>
<td>0.904 ± 0.006</td>
</tr>
<tr>
<td></td>
<td>Salkum silty clay loam, $\theta = 0.30$</td>
<td>0.454 ± 0.003</td>
<td>0.908 ± 0.006</td>
</tr>
<tr>
<td></td>
<td>Salkum silty clay loam, $\theta = 0.35$</td>
<td>0.458 ± 0.004</td>
<td>0.908 ± 0.008</td>
</tr>
<tr>
<td></td>
<td>Salkum silty clay loam, $\theta = 0.40$</td>
<td>0.465 ± 0.006</td>
<td>0.930 ± 0.012</td>
</tr>
</tbody>
</table>

Source: Pachepsky et al. (2003)

8. Conclusions
This paper is an elementary introduction to fractional calculus and its applications to nanoscale flow and heat transfer. Particular attention is paid to giving an intuitive grasp for how to establish a practical problem, and how to solve a fractional differential equation. A fractional model can be established by fractional Gauss’ divergence theorem, and can be solved analytically. The fractional complex transform (He and Li, 2012; He, 2012a, b) is extremely suitable for beginners, who are not familiar with fractional calculus, to deal with complex problems in fractal media. The understanding of porous cocoon for air/water permeability (Fan and He, 2012a, b, c) and heat conduction can be used for biomimic design of multi-scale fabrics for protective clothes.
References


**Further reading**


**About the authors**

Dr Hong-Yan Liu is a Staff Member in the School of Fashion Technology, Zhongyuan University of Technology, and a Postdoctoral Fellow at the National Engineering Laboratory for Modern Silk, Soochow University. Her recent interests mainly cover in nanotechnology and applied mathematics.

Dr Ji-Huan He is a Chair Professor in the National Engineering Laboratory for Modern Silk, College of Textile and Clothing Engineering, Soochow University. He is the owner of some famous analytical methods, such as the variational iteration method, the homotopy perturbation method and the exp-function method, and patents of bubble electrospinning for fabrication of nanomaterials. He has published over 280 papers including conference presentations, his H-index is as high as 51, his Researcher ID is: www.researcherid.com/rid/K-8504-2013, his personal web is: http://works.bepress.com/ji_huan_he. Dr Ji-Huan He is the corresponding author and can be contacted at: hejihuan@suda.edu.cn

Dr Zheng-Biao Li is a Mathematician working in Fractional Calculus. He has published 13 papers. He is a co-owner of the fractional complex transform, which is especially suitable for the local fractional calculus.

To purchase reprints of this article please e-mail: reprints@emeraldinsight.com
Or visit our web site for further details: www.emeraldinsight.com/reprints