Twenty-six dimensional polytope and high energy spacetime physics

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Abstract

We give the exact geometrical shape and study the combinatorial properties of higher dimensional polytopes for the dimensions from \( n = 4 \) to \( n = 12 \) as well \( n = 26 \) relevant to Heterotic, M and F superstring theories. Connections to E-infinity theory and the holographic principles are also discussed.

1. Introduction

There seems to be at least four roads to quantum gravity, superstring, loop quantum mechanics, twistors and non-commutative geometry [1–7]. Recently a fifth road, E-infinity theory, emerged not only as a new fuzzy-fractal spacetime theory [5], but also as a unifying frame in which the main four theories could be presented as complimentary view points. In fact even theories based on an extension of quantum field theory beyond the standard model such as SU(5) and SO(10) grand unification have been shown recently using E-infinity to be in some sense different homomorphisms of each others [8–11]. Even more recently and based on work by El Naschie on the 120-celles Coxeter polytope, Marek-Crnjac [12,13] has shown that hyperbolic fuzzy polytopes may lead to the spacetime of E-infinity and its instanton content.

It seems therefore that a great deal of conceptual, theoretical and intuitive visual insight may be gained into the structure of all the afore mentioned fundamental theories by simplifying our setting to a non-hyperbolic and crisp space [11–23], but complexifying it by transforming it to a very high dimensionality. In fact a mere glance at Figs. 1–10 reveals that the quantitatively high dimensionality is reflecting for \( n = 8 \) to \( n = 26 \), a qualitatively different outlook approximating fuzziness. This and other related aspects are the subject of the present short paper.

2. Fuzzy dodecahedron and higher dimensional polytopes

Let us consider the conventional view of quantum physics. There we work in a classical 3 + 1 dimensional space with a metric. However, due to Heisenberg uncertainty this metric cannot be a classical metric. It must be a fuzzy metric.
Consequently the null cones must become fuzzy and causality itself becomes fuzzy as argued by Sir R. Penrose and Mohamed El Naschie [23]. It is therefore very difficult to proceed on the basis of conventional quantum mechanics towards a theory of quantum gravity unless by reformulating everything in terms of fuzzy logic, fuzzy topology and fuzzy set theory [5,8,17].

However, although the word fuzzy was replaced by the word fractal and Cantorian geometry [7,23], in our view realizing the intimate relation between weighted infinite dimensional topology, fuzzy set theory, P-Adic number theory and fractals is the key to understanding the methodology of E-infinity theory [2–23].

Let us consider some of the combinational facts expressed in Table 1. Here we consider mainly the 4D cube which unlike the higher dimensional cases satisfies the simple Euler–Schläfli formula.
Thus we have

\[
\begin{align*}
\text{Vertices} &= 16, \\
\text{Edges} &= 32, \\
\text{Faces} &= 24, \\
\text{Cells} &= 8.
\end{align*}
\]

On the other hand, the order of the symmetry group, i.e. the dimension of the symmetries of the 4D cube is given by

\[
\text{Dim}(4D) = 2^4(4)! = 384.
\]

Thus we have

\[
N_0 - N_1 + N_2 - N_3 = 0
\]
It is possible to interpret Dim(4D) after on shell reduction using the corresponding eight real component spinor as the number of fermions in the standard model \[5,12,13\]

\[
N_{\text{fermions}} = \frac{\text{Dim}(4D)}{(2)^{4+1}} = \frac{384}{8} = 48
\]

Adding the 12 gauge bosons of the SU(3) \(\otimes\) SU(2) \(\otimes\) U(1) one finds the 60 experimentally confirmed particles of the standard model.

The situation could be seen in a different way if we use El Naschie’s summing over all elements procedure which he then equates to the number of elementary particles which one can expect to find in a modestly extended standard model. Proceeding in this way one find first that

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Fig. 5. The 8D polytope. This is relevant to the super space of E-infinity. It should be noted that the super symmetric Riemannian tensor have 336 independent component for \(n = 8\) because \(R^8 = 8^4(8^2 - 1)/12 = 336\).

Fig. 6. This is the nearest we can come to see the nine-dimensional spatial picture of 10-dimensional string space time within an Euclidean setting.
$N_0 + N_1 + N_2 + N_3 = (2)^4 + 32 + 24 + 8 = 80$

Consequently subtracting the 16 additional dimension of Heterotic Strings one finds [13,5]

$N_{SU} = 80 - 16 = 64$

There are many observations regarding Table 1 which may or may not be of fundamental importance.

For instance the number of faces for $D = 10$ is 11,520 which is 10-fold the order of the 24 cells Coxeter polytope denoted $\{3,4,3\}$. Also for $D = 6$ the number of vertices plus edges plus faces are exactly equal to 496. This is the order of the fundamental symmetry group of strings and E-infinity namely $E_8 \otimes E_8$. We may mention that only for $n = 8$ which is the super space dimension of E-infinity we have the number of faces 1792 being exactly equal the number of cells. Finally, one notes that the order for 5D is exactly 10 time that of 4D polytope.

Fig. 7. An almost Fuzzy $D = 10$ polytope as space time. Only toward the outer boundary do we see “empty” space.

Fig. 8. This polytope may be of great importance for the 11-dimensional $M$ theory in an Euclidean setting.
3. From here to infinity

In Figs. 1–10 we give the exact geometry of crisp polytopes in $n = 4$ to $n = 12$ as well as $n = 26$. The cases of $n = 4$ and $n = 5$ were considered by many authors from a purely geometrical view point. It seems that El Naschie was the first to realize the intimate relation between the sum of all elements entering the Euler classes and the instanton content of each polytope when taken to be a model for physical micro-spacetime. By contrast we are not aware of any previous computer graphic of the 26-dimensional polytope. What we can say for sure, however, is that $n = 8$ to $n = 12$ was never discussed in the context of high energy physics.

Fig. 9. $F$ theory is 12-dimensional and the number of instantons in this case is given by $n = d(d + 16) = 12(12 + 16) = 336$ exactly as $R^{12} = 336$ and $|SL(2, 7)| = 336$.

Fig. 10. For $N = 26$ we have global space filling deterministic chaos. No empty space is visible at all at this scale. Needless to mention that 26 is the dimension of bosonic string theory.
Pondering Figs. 1–10 the reader must be intrigued by the quasi-fuzziness of the appearance of $n = 11$, $n = 12$ and particularly $n = 26$. This is naturally a consequence of the scale of the picture, but then again for $n \to 1$ we cannot escape an intrinsic fuzziness. This is at least in principles the viewpoint of E-infinity theory. In fact for $n = 26$ Fig. 10 shows a space filling global chaos-like behavior.

4. A topological polytope interpretation of Klein $I(7)$ and El Naschie–’t Hooft’s holographic boundary of E-infinity

As well known the compactified Klein modular curve may be obtained from the modular group $M(7) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(7); \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod 7 \right\}$ by compactification. The original case is genus 3 and since the maximum number of automorphic elements is given by the Hurwitz formula $N(Auto) = 84(g-1)$ then one finds $N(Auto) = 84(3-1) = 168$

Consequently tessellation of the unit disc must give the following number of triangles or degrees of freedom: $|I(7)| = 2)N(Auto) = 2)(168) = 336$

El Naschie called 336 the dimension of $I(7)$. It was shown by him that assuming a discrete Gamma distribution, the values of the compactified curve is given by $N(Auto) = 168 + 8k$

and $\dim I(7) = 336 + 16k \approx 339$

where $K = \phi^2(1 - \phi^2)$ and $\phi = (\sqrt{5} - 1)/2$.

It then turned out that $I(7)$ is the holographic boundary of the $E_8 \otimes E_8$ bulk of E-infinity space–time manifold. This bulk manifold is given by a Fuzzy Kähler manifold for which the Euler characteristic is $\chi = 26 + k = 26.18033989$ as shown by El Naschie. Similarly another Fuzzy Kähler manifold for which $\chi = 18 - 2\phi^6$ was found also by El Naschie for the holographic boundary. It is therefore interesting to see that one could assign a topological polytope to this $I(7)$ holographic boundary. It is easily shown that such a polytope must have $F = 56$ faces, $E = 84$ edges and $V = 24$ vertex. Since the Euler characteristic is given by

Table 1
The combinational facts of n-dimensional cube

<table>
<thead>
<tr>
<th>$n - D$</th>
<th>Vertices</th>
<th>Edges</th>
<th>Faces</th>
<th>Cells</th>
<th>Order</th>
<th>The Euler characteristic</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>16</td>
<td>32</td>
<td>24</td>
<td>8</td>
<td>384</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>32</td>
<td>80</td>
<td>80</td>
<td>40</td>
<td>3840</td>
<td>32</td>
</tr>
<tr>
<td>6</td>
<td>64</td>
<td>192</td>
<td>240</td>
<td>160</td>
<td>46080</td>
<td>112</td>
</tr>
<tr>
<td>7</td>
<td>128</td>
<td>448</td>
<td>672</td>
<td>560</td>
<td>645120</td>
<td>352</td>
</tr>
<tr>
<td>8</td>
<td>256</td>
<td>1024</td>
<td>1792</td>
<td>1792</td>
<td>10321920</td>
<td>1024</td>
</tr>
<tr>
<td>9</td>
<td>512</td>
<td>2304</td>
<td>4608</td>
<td>5376</td>
<td>185794560</td>
<td>2816</td>
</tr>
<tr>
<td>10</td>
<td>1024</td>
<td>5120</td>
<td>11520</td>
<td>15360</td>
<td>3715891200</td>
<td>7424</td>
</tr>
<tr>
<td>11</td>
<td>2048</td>
<td>11264</td>
<td>28160</td>
<td>42240</td>
<td>81749606400</td>
<td>18944</td>
</tr>
<tr>
<td>12</td>
<td>4096</td>
<td>24576</td>
<td>67584</td>
<td>112640</td>
<td>1961990553600</td>
<td>47104</td>
</tr>
<tr>
<td>26</td>
<td>67108864</td>
<td>872415232</td>
<td>5452595200</td>
<td>21810380800</td>
<td>2.70644318171067e + 034</td>
<td>4647288832</td>
</tr>
</tbody>
</table>

In this table, we have used the following formulas:
The number of vertices for $n - D(V_n)$ is $V_n = 2 \times V_{n-1}$ or $V_n = 2^n$.
The number of edges for $n - D(E_n)$ is $E_n = 2 \times E_{n-1} + V_{n-1}$ or $E_n = n2^{n-1}$.
The number of edges for $n - D(F_n)$ is $F_n = 2 \times F_{n-1} + E_{n-1}$ or $F_n = \frac{n(n-1)}{2} 2^{n-2}$.
The number of cells for $n - D(\text{Cell}_n)$ is $\text{Cell}_n = 2 \times \text{Cell}_{n-1} + F_{n-1}$.
The order of the symmetry group for $n - D(\text{Order}_n)$ is $\text{Order}_n = 2^n(n)!$. The Euler characteristic for $n - D$ $(\chi_n)$ is $\chi_n = F_n + V_n - E_n$.


\[ \chi = F + V - E = 2 - 2g \]

Then
\[ \chi = 56 + 24 - 84 = -4 \]

and therefore one finds
\[ g = \frac{-4 - 2}{-2} = 3 \]
as should be.

It should be noted that \( \chi = -4 \) of the polytope leads to the same on shell elementary particles as \( \chi = 24 \) of the classical crisp K3 Kähler manifold.

5. Determining the inverse electromagnetic fine structure constant using El Naschie–t Hooft’s holographic boundary of E-infinit

Using the basic equation of E-infinity one could give an extremely simple and very elegant derivation of \( \bar{\alpha}_0 \approx 137 \).
The argument is ingeniously simple. Since the bulk contains all fundamental interactions, while the holographic boundary has only particle physics then one can translate this situation to the global equation.

All interactions \( \sim \) [particle physics + gravity] = electromagnetism

and then to a symmetry groups dimensional equation
\[ \text{Dim} E_g \otimes E_g - \text{[Dim} f(7) + R^{(4)} \text{]} = \bar{\alpha}_0 \]

That means
\[ 496 - [339 + 20] = \bar{\alpha}_0 \]

Consequently we have \( \bar{\alpha}_0 \approx 137 \) and we note that \( R^{(4)} = (4)^2(4^2 - 1)/12 = 20 \) is the Riemann tensor in \( D = 4 \).

We believe the above derivation, proposed for the first time by Mohamed El Naschie, is the first mathematical derivation of \( \bar{\alpha}_0 \) using a general physical theory. In a sense it is Eddington’s dream which he could not fulfill.

6. Conclusion

As with the Hausdorff dimension of E-infinity
\[ \sim \langle n \rangle = 4 + \frac{1}{4 + \frac{1}{4 + \ldots}} \]
in an infinite dimensional topology any hierarchical space is fractal-like.

In the present short paper, we have given exact graphical images of higher dimensional polytopes. In turn this provided us with a crude but tangible approximate picture of the Fuzzy K3 manifold of E-infinity \([24,25]\). In a sense we have presented in Figs. 1–10 an Euclidean E-infinity theory.

References


El Naschie MS. Estimating the experimental value of the electromagnetic fine structure constant $\alpha_0 = 1/137.036$ using the Leech lattice in conjunction with the monster group and Sphere’s Kissing number in 24 dimensions. Chaos, Solitons & Fractals 2007;32(2):383–7.


