An Improved Amplitude-frequency Formulation for Nonlinear Oscillators

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Abstract
A brief introduction to amplitude-frequency formulae for nonlinear oscillators is given, an improved one is suggested.

Keywords: Nonlinear Oscillation, Duffing equation, period

We consider a generalized nonlinear oscillator in the form

\[ u'' + f(u) = 0, \quad u(0) = A, \quad u'(0) = 0 \]  
(1)

We use two trial functions

\[ u_1(t) = A \cos \omega_1 t \]  
(2)

and

\[ u_2 = A \cos \omega_2 t , \]  
(3)

The residuals are

\[ R_1(t) = - \cos \omega_1 t + f(A \cos \omega_1 t) \]  
(4)

and

\[ R_2(\omega_1 t) = - \omega_1^2 \cos \omega_2 t + f(A \cos \omega_2 t) \]  
(5)

The original frequency-amplitude formulation reads[1-4]

\[ \omega^2 = \frac{\omega_1^2 R_2(\omega_1 t = \pi/3) - \omega_2^2 R_2(\omega_2 t = \pi/3)}{R_2 - R_1} \]  
(6)

In my previous publications[1-4], I just used the following formulation

\[ \omega^2 = \frac{\omega_1^2 R_2(\omega_1 t = 0) - \omega_2^2 R_2(\omega_2 t = 0)}{R_2 - R_1} \]  
(7)

Geng and Cai improved the formulation by choosing another location point[5]:

\[ \omega^2 = \frac{\omega_1^2 R_2(\omega_1 t = \pi / 3) - \omega_2^2 R_2(\omega_2 t = \pi / 3)}{R_2 - R_1} \]  
(8)

Generally we can locate at

\[ \cos \omega_1 t = \cos \omega_2 t = k \]  
(9)

To illustrate this shortcoming, we consider the Duffing equation

\[ u'' + u + \varepsilon u^3 = 0 \]  
(10)

Using trial functions

\[ u_1(t) = A \cos t \]  
(11)

and

\[ u_2 = A \cos 2t \]  
(12)

respectively for Eq.(1), we obtain the following residuals

\[ R_1(t) = \varepsilon A^3 \cos^3 t, \]  
(13)

and

\[ R_2(\omega_1 t) = -3A \cos 2t + \varepsilon A^3 \cos^3 2t . \]  
(14)

Locating at \( \cos t_1 = \cos 2t_2 = k \), we obtain

\[ \omega^2 = \frac{-3Ak + \varepsilon A^3 k^3 - 4\varepsilon A^3 k^3}{-3Ak + \varepsilon A^3 k^3 - \varepsilon A^3 k^3} = 1 + \varepsilon A^3 k^2 \]  
(15)

Its approximate solution reads
In view of the approximate solution, Eq.(16), we re-write Eq.(10) in the form

\[ u'' + (1 + k^2 \varepsilon A^2)u = k^2 \varepsilon A^2 u - \varepsilon u^3 \]  

If, by chance, Eq.(16) is the exact solution, then the right hand side of Eq.(17) is vanishing completely. Since our approach is only an approximation to the exact solution, we set

\[ \int_0^{T/4} \left( k^2 \varepsilon A^2 u - \varepsilon u^3 \right) \cos \omega t dt = 0, \]  

where \( T = 2\pi / \omega \). Substituting (16) in (18), we obtain

\[ k^3 = \frac{3}{4} \]  

Finally the frequency is obtained

\[ \omega = \sqrt{\frac{3}{4 + c^2 A^2}}. \]  

To improve its accuracy, we can use the following trial-functions:

\[ u_i(t) = \sum_{j=1}^{m} A_i \cos \omega_j t \]  

\[ u_j(t) = \sum_{j=1}^{n} A_j \cos \Omega_j t, \]  

or

\[ u_i(t) = \frac{\sum_{j=1}^{m} A_i \cos \omega_j t}{\sum_{j=1}^{n} B_j \cos \omega_j t} \]  

\[ u_j(t) = \frac{\sum_{j=1}^{n} A_j \cos \Omega_j t}{\sum_{j=1}^{n} B_j \cos \Omega_j t}, \]

Most useful trial-functions are

\[ u_i(t) = A \cos t \]  

\[ u_j(t) = a \cos \omega t + (A - a) \cos 3\omega t, \]  

and

\[ u_i(t) = A \cos t \]  

\[ u_j(t) = \frac{A(1 + c) \cos \omega t}{1 + c \cos 2\omega t}, \]

where \( a \) and \( c \) are unknown constants. We can always set \( \cos t = k \) in \( u_i \), and \( \cos \omega t = k \) in \( u_j \).

**Acknowledgement**

This material is based on work supported by the Program for New Century Excellent Talents in University under grand No. NCET-05-0417.

**References**


