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Abstract

The parameter-expansion method is applied to a nonlinear oscillator with discontinuity. One iteration is sufficient to obtain a highly accurate solution, which is valid for the whole solution domain. Comparison of the obtained solution with the exact one shows that the method is very effective and convenient.

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1. Introduction

This paper considers the following nonlinear oscillator with discontinuity [1–3]:

\[ u'' + u|u| = 0, \quad u(0) = A, \quad u'(0) = 0. \quad (1) \]

There exists no small parameter in the equation. Therefore, the traditional perturbation methods cannot be applied directly [3].

Recently, considerable attention has been directed towards analytical solutions for nonlinear equations without small parameters. Many new techniques have appeared in the literature, for example, the homotopy perturbation method [4–10], the variational iteration method [11–14], and the energy balance method [15–17]. A complete review is available in Refs. [3,18]. In this paper, we apply the parameter-expansion method [18–21] to the problem we are discussing.

2. Solution procedure

The parameter-expansion method [18–21] entails the bookkeeping parameter method [18,19] and the modified Lindstedt–Poincare method [18,21–23]. Recently, the method has been applied to various nonlinear oscillators, see Refs. [2,24–28]. In order to use the parameter expansion method, we re-write Eq. (1) in the following form [18,21,29]:

\[ u'' + 0 \cdot u + 1 \cdot u|u| = 0 \quad (2) \]
According to the parameter-expansion method, we may expand the solution, \(u\), the coefficient of \(u\), the zero, and the coefficient of \(u|u|\), 1, in series of \(p\):

\[ u = u_0 + pu_1 + p^2u_2 + \cdots \]  
\[ 0 = \omega^2 + pa_1 + p^2a_2 + \cdots \]  
\[ 1 = pb_1 + p^2b_2 + \cdots \]  

Substituting Eqs. (3)–(5) into Eq. (2) and equating the terms with the identical powers of \(p\), we have

\[ p^0 : u_0'' + \omega^2u_0 = 0 \]  
\[ p^1 : u_1'' + \omega^2u_1 + a_1u_0 + bu_0u_0 = 0 \]  
\[ p^2 : (1 + \omega^2)u_2'' + a_1u_1'' + a_2u_0'' + b_1(|u_0''|u_1'' + u_0'u_1') + b_2u_0''u_0' = 0 \]  

Considering the initial conditions \(u_0(0) = A\) and \(u_1'(0) = 0\), the solution of Eq. (6) is \(u_0 = A\cos\omega t\). Substituting the result into Eq. (7), we have

\[ u_0'' + \omega^2u_1 + a_1A\cos\omega t + b_1A^2\cos\omega t|\cos\omega t| = 0 \]  

It is possible to perform the following Fourier series expansion:

\[ \cos\omega t|\cos\omega t| = \sum_{n=0}^{\infty} c_{2n+1} \cos[(2n+1)\omega t] = c_1 \cos\omega t + c_3 \cos3\omega t + \cdots \]  

where \(c_1\) can be determined by Fourier series, for example

\[ c_1 = \frac{2}{\pi} \int_{0}^{\pi} \cos^2\omega t|\cos\omega t|d(\omega t) = \frac{4}{\pi} \left( \int_{0}^{\pi} \cos^2\tau d\tau - \int_{\frac{\pi}{2}}^{\pi} \cos^2\tau d\tau \right) = \frac{8}{3\pi} \]  

Substitution of Eq. (10) into Eq. (9) gives

\[ u_0'' + \omega^2u_1 + \left(a_1 + b_1A\frac{8}{3\pi}\right)A\cos\omega t + \sum_{n=1}^{\infty} c_{2n+1} \cos[(2n+1)\omega t] = 0 \]  

No secular term in \(u_1\) requires that

\[ a_1 + b_1A\frac{8}{3\pi} = 0 \]  

If the first-order approximation is enough, then, setting \(p = 1\) in Eqs. (4) and (5), we have

\[ 1 = b_1 \]  
\[ 0 = \omega^2 + a_1 \]  

From Eqs. (13)–(15), we obtain

\[ \omega = \sqrt{\frac{8A}{3\pi}} \approx 2.6667\sqrt{\frac{A}{\pi}} \]  

The obtained frequency, Eq. (16), is valid for the whole solution domain, \(0 < A < \infty\). The accuracy of frequency can be improved if we continue the solution procedure to a higher order, however, the amplitude obtained by this method is an asymptotic series, not a convergent one. For conservative oscillator

\[ u'' + f(u)u = 0, \quad f(u) > 0 \]  

where \(f(u)\) is a nonlinear function of \(u\), we always use the zeroth-order approximate solution. Thus we have

\[ u(t) = A\cos\left(t\sqrt{\frac{8A}{3\pi}}\right) \]  

Fig. 1 illustrates various cases with different values of \(A\).
3. Conclusion

The parameter-expansion method is an extremely simple method. One iteration is enough. Furthermore, the obtained frequency is of high accuracy. The method can be applied to many other nonlinear oscillators.
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References