Microarticle

The simplest approach to nonlinear oscillators

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ABSTRACT

This paper gives the simplest approach to the cubic-quintic Duffing equation (M.S.H. Chowdhury et al., Results in Physics 7 (2017) 3962–3967), providing an extremely fast and relatively accurate estimation of the frequency of a nonlinear conservative oscillator.

Recently Chowdhury et al. studied the following cubic-quintic Duffing oscillator

\begin{equation}
 x'' + x + x^3 + x^5 = 0, \quad x(0) = A, \quad x'(0) = 0
\end{equation}

and obtained an approximate solution by the harmonic balance method [1]. This equation can be also solved by the variational iteration method [2], the homotopy perturbation method [3–5] and the Taylor series method [6]. Here we show the frequency of Eq. (1) can be effectively solved by a one-step frequency formulation [7].

Consider the following general nonlinear oscillator

\begin{equation}
 x'' + f(x) = 0, \quad x(0) = A, \quad x'(0) = 0
\end{equation}

The square of its frequency can be expressed as [7]

\begin{equation}
 \omega^2 = \frac{df(x)}{dx} \Big|_{x=A/2}
\end{equation}

For the cubic-quintic Duffing oscillator, \( f(x) = x + x^3 + x^5 \), the frequency can be obtained immediately:

\begin{equation}
 \omega = \sqrt{(1 + 3x^2 + 5x^4)}|_{x=A/2} = \sqrt{1 + \frac{3}{4}A^2 + \frac{5}{16}A^4}
\end{equation}

This approximate frequency gives good accuracy for small \( A \), as shown in Table 1, and its relative error is about 25% when \( A \) tends to

<table>
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<tr>
<th>A</th>
<th>Exact frequency</th>
<th>Eq. (5)</th>
<th>Relative Error of Eq. (5), %</th>
<th>Eq. (4)</th>
<th>Relative Error of Eq. (4), %</th>
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infinite. In order to improve its accuracy, we can use more location points in stead of \( x = 0.5A \), and its average value is used, for example

\[
\omega = \frac{1}{3} \left( \frac{df(x)}{dx} \right)_{x=0.34} + \frac{df(x)}{dx} \bigg|_{x=0.54} + \frac{df(x)}{dx} \bigg|_{x=0.74}
\]

\[
= \sqrt{1 + 0.83A^2 + 0.51783A^4}
\]  

Eq. (5) gives a good accuracy for both small \( A \) and large \( A \), the relative error is less than 4% when \( A = 1000 \), see Table 1.

To be concluded, this paper suggests a simple but effective approach to nonlinear oscillator for fast insight into its basic property. The method can also extend to fractal oscillators [9,10] and the variational approach to nonlinear oscillators [11,12].

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References


