SuperSET: A Mathematical Investigation of a Variation of SET

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SuperSET : A Mathematical Investigation of a Variation of SET

A Senior Project submitted to The Division of Science, Mathematics, and Computing of Bard College

by Jeffrey Pereira

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Aside from being fun, card games have been used as methods to teach elements of probability. One such card game, "SET", has players find groups of three cards called SETs. In this paper, I investigate the mathematics of the game of SET such as the probabilities of finding various SETs in the actual game as well as modelling probabilities of more complicated games identifying the geometry of a SET. I also look at forming a geometric model for identifying SETs and use this as a means of finding the Maximal Cap of the game. In addition to the classic game of SET, I invent a variant of the game called "SuperSET" and use the same methods of analysis used for classic SET and attempt to form a general formula to allow us to compute probabilities of finding various SETs in this new game.
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First and foremost I would like to dedicate this Senior Project to my parents, Jean Daniels and Jose Pereira. They’ve been there for me always and have been the inspiration to my near insane work ethic. Without them, college would have been impossible. I would also like to thank my adviser Lauren Rose for not only supervising this project but for also being one of my biggest supporters here at Bard. Whenever I had serious doubts about my mathematical abilities and my place at Bard it was her advice and motivation that kept me here and gave me the drive I needed to finish despite my challenges. I would also like to thank all my friends here who I’ve met at Bard. You all helped me not lose my head and supported me in ways you’d never imagine.
Acknowledgments
1
Introduction

The game of SET® was invented by Marsha Falco and her husband in 1974 during her research on the genetics of epilepsy in German Shepherds. She classified characteristics of interest with a basic labeling, both she and her husband added three values and four characteristics to the labels and created a game out of it. After years of playing the game they invented with their family, the Falco’s went off to market the game called "SET" in 1991. Currently, SET is played by many people worldwide. Cardless versions exist on various smartphone and tablet platforms including iOS and an online version is hosted both at the SET® homepage and on the New York Times website. The game of SET® is played with a deck of 81 different cards. Figures 1.0.1 - 1.0.3 show what examples of SET cards look like.
1. INTRODUCTION

The game of SET® is played using a deck of SET® cards. At the start of the game, 12 cards are dealt and laid face up as seen in Figure 1.0.1. The goal of the game is to collect groups of 3 cards called SETs and is played either alone, or with others in either a friendly or competitive fashion.

Each card in a game of SET® is distinguished by four characteristics: **color**, **shape**, **number**, and **shading**. For example, the card shown in Figure 1.0.2 has color "**purple**", shape "**diamond**", number "3", and shading "**empty**". Thus, we can view each card as a vector (color, shape, number, shading) or (purple, diamond, 3, empty). Figures 1.0.2, and 1.0.3 also relate a SET card to its a attribute form.

Figure 1.0.1. Layout of SET cards in a typical game

Figure 1.0.2. (purple, diamond, 3, empty)
1. INTRODUCTION

Definition 1.0.1. The different characteristics (color, shape, number, shading) are called attributes. △

Definition 1.0.2. The different values are called the states of an attribute. △

There are three states for each attribute in SET as shown in Table 1.0.1.

<table>
<thead>
<tr>
<th>Attributes</th>
<th>States</th>
</tr>
</thead>
<tbody>
<tr>
<td>Color</td>
<td>red, green, purple</td>
</tr>
<tr>
<td>Shading</td>
<td>Solid, Striped, Empty</td>
</tr>
<tr>
<td>Number</td>
<td>1, 2, 3</td>
</tr>
<tr>
<td>Shape</td>
<td>Ovals, Squiggles, Diamonds</td>
</tr>
</tbody>
</table>

Table 1.0.1. SET attributes and their states

The goal of the game is to find "SETs", which consist of three cards with the properties defined below.

Definition 1.0.3. A SET is a collection of 3 cards, such that for each attribute, the states of that attribute are either all alike or all different. △

Example 1.0.4. The three cards in Figure 1.0.4, (red, solid, 1, oval), (red, striped, 3, oval), and (red, empty, 2, ovals) form a SET because all three cards share the same color, the same shape, different numbers and different shadings. Since the SET has 2
1. INTRODUCTION

attributes with alike states and 2 with different states this is a SET in the form AADD or A²D².

\[ \text{Figure 1.0.4. An example of an A}^2 \text{D}^2 \text{ SET.} \]

**Example 1.0.5.** The three cards in Figure 1.0.4, (red, solid, 1, oval), (red, striped, 3, oval), and (red, empty, 2, ovals) form a SET because all three cards share the same color, the same shape, different numbers and different shadings. Since the SET has 2 attributes with alike states and 2 with different states this is a SET in the form AADD or A²D².

\[ \text{Figure 1.0.5. An example of a D}^4 \text{ SET.} \]

Later, we will see that we can interpret any SET card as a vector.

In general, any set can be labelled in the form A^iD^j where A^i is the number of attributes with all alike states and D^j is the number of attributes with all different states. In fact, there exist 4 kinds of SETs in a classic game: AD³, A²D², A³D, and D⁴. A set in the form A⁴ cannot exist as we will discuss later.

Now that we have a basic understanding of the game, we will provide and outline of this project.
1. *INTRODUCTION*

In Chapter 2 of this paper we break up sets in categories and count how many sets are in each. We do this for set games with $n$ different attributes, not just 4. This will allow us to compute probabilities of finding various sets. We will also describe another representation of cards as squares in a grid.

In Chapter 3, we introduce a variant of SET® which we call SuperSET. We will show how to play the game, introduce half and half sets, a new kind of set, and represent cards as vectors.

In Chapter 4, we analyze SuperSETs in the same manner as we did for classic SET. We classify SuperSETs into categories and find probabilities of different types of SuperSETs. We also find a complete cap for a 2-attribute SuperSET game. A complete cap is the maximal number of cards that can be laid out with no set occurring. We will also discuss the geometry for SuperSET as a method for detecting SETs and complete caps.

Finally, in Chapter 5, I will post some open questions for anyone interested in furthering this study of SET® and SuperSET.
2 Properties of the Game of SET

In this chapter we will discuss the properties of the game of SET®. We will discuss the various properties different types of SETs have and discuss how this relates with the probabilities of finding them. We will introduce a method of counting different SETs using the structure of matrices and use this structure as a foundation for forming a general formula to find the number of different SETs for any number of attributes alike and different. We will also discuss a method of representing SET cards as points on a grid and SETs as geometric figures.

2.1 SETs and Vectors

Previously, we went from a card to a word vector. Since each attribute has 3 states, each state is labelled with a 1, 2, or 3. Table 2.1.1 relates these different values to the states of the attributes. Using this arrangement, we can convert the word vector of the card to a number vector as seen in Figure 2.1.1.
2. PROPERTIES OF THE GAME OF SET

<table>
<thead>
<tr>
<th>State</th>
<th>Color</th>
<th>Shading</th>
<th>Number</th>
<th>Shape</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Red</td>
<td>Filled In</td>
<td>1</td>
<td>Ovals</td>
</tr>
<tr>
<td>2</td>
<td>Green</td>
<td>Striped</td>
<td>2</td>
<td>Squiggles</td>
</tr>
<tr>
<td>3</td>
<td>Purple</td>
<td>Empty</td>
<td>3</td>
<td>Diamonds</td>
</tr>
</tbody>
</table>

Table 2.1.1. Attributes and States

**Example 2.1.1.** Figure 2.1.1 relates a SET card with its word vector and number vector forms.

\[
\begin{align*}
\text{Figure 2.1.1. The card and its vector representations} \\
\text{(purple, diamond, 3, empty), (3, 3, 3, 3)}
\end{align*}
\]

Since a card can be represented a vector, a SET can be represented as a matrix. Each row vector of the matrix represents a different card whereas the column vectors represent the states of each attribute. Each of the states for each attribute can be represented as either 1, 2, or 3. This method of representation will be important in creation of our matrices for SETs.

**Example 2.1.2.** Here we relate a SET in 3 ways: via cards, row vectors in a word matrix, and row vectors of a number matrix.
By relating the sets with matrices it becomes easier to classify them. Since each card is represented as a row vector, then each column vector of the SET matrix can be used to identify whether or not the states are all alike or all different. Example 2.1.2 is an example of a SET with all 4 attribute different which will be denoted as $D^4$.

**Example 2.1.3.** Here we relate two different kinds of SETs with their word matrix and number matrix forms.
2. PROPERTIES OF THE GAME OF SET

Example 2.1.3 highlights two SETs where each SET has a different number of attributes with all alike states. Since SET W has 2 attributes with all alike states and 2 attributes with all different states, SET W can be expressed in the form $A^2D^2$. Since Set V has all different states for all attributes, SET V can be expressed as $A^0D^4 = D^4$. 
2. **PROPERTIES OF THE GAME OF SET**

**Definition 2.1.4.** Let A = alike and D = different, then any SET can be represented as a monomial in the form $A^i D^j$ where $j \neq 0$ where $A^i$ denotes the number of attributes with all alike states and $D^j$ denotes the number of attributes with all different states. \( \triangle \)

We will say $V \in A^i D^j$ if V has $i$ alike and $j$ different attributes.

**Definition 2.1.5.** SETs in the form $A^i D^j$ with particular $i,j$ values are in the class of $A^i D^j$. \( \triangle \)

There are particular classes of SETs. Example 2.0.12 relates the classes of SETs with the physical cards and the representative 4-tuples.

**Example 2.1.6.** Sets and Their Classes

<table>
<thead>
<tr>
<th>Class</th>
<th>4-Tuple Form</th>
<th>Physical Cards</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A^3D$</td>
<td>(1,1,1,1), (1,1,2,1), (1,1,3,1)</td>
<td><img src="image1" alt="Physical Cards" /></td>
</tr>
<tr>
<td>$A^2D^2$</td>
<td>(1,1,1,1), (1,2,2,1), (1,3,3,1)</td>
<td><img src="image2" alt="Physical Cards" /></td>
</tr>
<tr>
<td>$AD^3$</td>
<td>(1,1,1,1), (1,2,2,2), (1,3,3,3)</td>
<td><img src="image3" alt="Physical Cards" /></td>
</tr>
<tr>
<td>$D^4$</td>
<td>(1,1,1,1), (2,2,2,2), (3,3,3,3)</td>
<td><img src="image4" alt="Physical Cards" /></td>
</tr>
</tbody>
</table>

Table 2.1.2. SETs and their vector forms
2. PROPERTIES OF THE GAME OF SET

2.2 Mathematical Properties of SET

Suppose that we have 2 cards and want to find the third card that completes the SET. We will find a 3rd card such that adding the 3rd card to the two cards will form a SET.

Example 2.2.1. Completing the SET

![Figure 2.2.1. 2 SET cards that do not form a SET...YET](image)

By Table 2, the cards in Example 2.2.1 can be expressed as the vectors (1,3,1,1) and (2,3,1,3). We will find a card that completes this SET using the matrix form.

```
Card 1 (1 3 1 1)
Card 2 (2 3 1 3)
Card 3 (w x y z)
```

By Definition 1.0.3 a SET is a collection of three cards such that all the attributes yield either all alike or different states. Since each entry in the vector of a SET card relates to an attribute, the column vectors of the matrix relate to the states. If two of the vertical entries are equal, the third must equal the other two. If not, the third entry must be the remaining element from the set \{1, 2, 3\}. 

2. PROPERTIES OF THE GAME OF SET

For the first attribute, $1 \neq 2$, so $w=3$. For the second attribute, $3=3$, so $x=3$. For the third attribute, $1=1$, so $y=1$. Lastly, for the fourth attribute, $1 \neq 3$ so $z=2$. Therefore, the missing card is $(3,3,1,2)$.

![Figure 2.2.2. The Complete SET](image)

We will prove later that any two cards determine a unique third such that a SET can be completed.

Although classic SET is only a 4-Attribute game, it is possible to construct cards and SETs with more attributes.

**Example 2.2.2. A 5-Attribute SET**

Card 1 \[ \begin{pmatrix} 1 & 2 & 2 & 3 & 2 \end{pmatrix} \]
Card 2 \[ \begin{pmatrix} 1 & 3 & 2 & 3 & 1 \end{pmatrix} \]
Card 3 \[ \begin{pmatrix} 1 & 1 & 2 & 3 & 3 \end{pmatrix} \]

**Example 2.2.3. A 6-Attribute SET**

Card 1 \[ \begin{pmatrix} 1 & 2 & 3 & 2 & 3 & 2 \end{pmatrix} \]
Card 2 \[ \begin{pmatrix} 1 & 3 & 2 & 2 & 3 & 1 \end{pmatrix} \]
Card 3 \[ \begin{pmatrix} 1 & 1 & 1 & 2 & 3 & 3 \end{pmatrix} \]
As Examples 2.2.2 and 2.2.3 show, we can keep making SETs of a classic SET game with a different number of attributes. To verify that Examples 2.2.2 and 2.2.3 are SETs we can utilize the same method used for Example 2.2.1. We can classify every SET as $A^i D^j$ with $i$ being the number of attributes alike and $j$ being the number of attributes with different states. In classic set with 4 attributes, the following are possible types:

$$A^3 D, A^2 D^2, A^1 D^3, D^4$$

By Definition 1.0.3, all of the states for each attribute of the cards in the SET are either all alike or all different. It turns out that any 2 cards uniquely determine a set, as seem below.

Introduce N-Attribute game here.

**Definition 2.2.4.** n-Attribute SET n-Attribute SET is a SET game which yields some positive number of attributes. Typically, greater than 4. △

**Theorem 2.2.5** (Keystone SET Theorem). *Let $S$ denote a SET deck of $n$-attributes. For any two cards $v, w \in S$ there exists a unique third card $z \in S$ such that $(v, w, z)$ is a SET.*

**Proof.** Let $v$ and $w \in S$. Then $v, w$ can be represented as $(v_1, v_2, \ldots, v_n)$ and $(w_1, w_2, \ldots, w_n)$ where $v_i, w_i \in \{1, 2, 3\}$ for $i \in \{1, 2, \ldots, n\}$. We’d like to find a $z \in S$ that makes $(v, w, z)$ a set. Because each card in a SET deck is unique, $v$ and $w$ cannot be the same card. Suppose that $v_1 = w_1$, then we set $z_1 = v_1 = w_1$ and so attribute 1 will be an A. Suppose that $v_1 \neq w_1$. Then we set $z_1$ to be the remaining state $(1, 2, 3)$ that is
not equal to either \( v_1 \) or \( w_1 \), so attribute 1 will be a D. The process for attributes 2 - \( n \) is the same. Then \((v, w, z)\) will be a set of the type \( A^i D^j \) where \( i \) is the number of \( i \) such that \( v_i = w_i = z_i \) and \( D \) is the number such that \( v_i \neq w_i \neq z_i \). Therefore, for any two cards \( v, w \in S \), there exists a unique third card \( z \in S \) such that \((v, w, z)\) is a SET. 

Since any two cards in a SET deck result in a unique third card that will complete the SET, the total number of SETs in a standard deck can be found.

**Theorem 2.2.6.** The general form for the number of SETs in an \( n \)-attribute deck is

\[
\frac{3^n(3^n - 1)}{3!}
\]

for some \( n \in \mathbb{N} \).

**Proof.** Since there are \( n \)-attributes in the deck and 3 states for each attribute this means the number of cards in an \( n \)-attribute SET deck is \( 3^n \). Therefore there are \( 3^n \) ways to pick the first card from the deck. This means that there are \( 3^n - 1 \) ways to choose the second card from our deck. Since the final card in a SET is unique, there is only 1 way to select the final card. Therefore, there are \( 3^n \cdot (3^n - 1) \cdot 1 \) SETs in an \( n \)-attribute deck. Since a SET is consists of 3 cards, and the order of the cards do not matter in a SET, the number of SETs have been overcounted by \( 3! \) since there are \( 3! \) ways to order 3 elements. Therefore there are \( \frac{3^n(3^n - 1)}{3!} \) SETs in an \( n \)-attribute deck.

Since a general form for the number of SETs in an \( n \)-attribute deck is known for some \( n \in \mathbb{N} \) it is now possible to compute the number of SETs in a deck given the number of attributes.

It is important to note that there do not exist SETs in the form \( A^n \).
2. PROPERTIES OF THE GAME OF SET

<table>
<thead>
<tr>
<th>Number of Attributes</th>
<th>Number of SETS</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>9801</td>
</tr>
<tr>
<td>6</td>
<td>88452</td>
</tr>
<tr>
<td>7</td>
<td>796797</td>
</tr>
<tr>
<td>8</td>
<td>7173360</td>
</tr>
</tbody>
</table>

Table 2.2.1. Number of SETs in a 5, 6, 7, and 8 Attribute Game of SET.

Lemma 2.2.7 (Anti Clone SET Property). For any n-dimensional SET game, there are no sets of the form \( A^n \).

Proof. We will prove this via contradiction. Let \( n \in \mathbb{N} \). Suppose that there exists a set of the form \( A^n \). This means that there exists 3 cards that are alike in every possible attribute. Since each card in a SET® deck is different this means that the 3 cards are the same. Therefore, a contradiction. Thus, we cannot have a SET with all n-attributes alike.

Example 2.2.8.

A SET in the form \( A^4 \).

\[
\begin{pmatrix}
1 & 2 & 3 & 1 \\
1 & 2 & 3 & 1 \\
1 & 2 & 3 & 1 \\
1 & 2 & 3 & 1 \\
\end{pmatrix}
\]

Observe that a SET in the form \( A^4 \) contains 4 copies of the same card.

2.3 Counting SETs

In this section, we will talk about counting different kinds of SETs. Since the number of cards in a SET deck is finite, there must be a finite number of SETs. Because there are a finite number of SETs it must be possible to count them. We will look at a 2 attribute SET game and find a pattern that will allow us to derive a general formula to find the number of different SET classes in any n-attribute game. We do not look at 1
2. PROPERTIES OF THE GAME OF SET

According to Theorem 2.2.6, for 2-attribute SET there are \(\frac{3^2(3^2-1)}{3!} = 12\) SETs in our game. In a 2-Attribute SET game, both attributes cannot have all alike states. This means that the SETs are either all different for both attributes, or 1 alike attribute and 1 different attribute.

Example 2.3.1. Pairings of SETs in the form AD.

<table>
<thead>
<tr>
<th>Attribute 1</th>
<th>Attribute 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>1</td>
</tr>
<tr>
<td>a</td>
<td>2</td>
</tr>
<tr>
<td>a</td>
<td>3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Attribute 1</th>
<th>Attribute 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>1</td>
</tr>
<tr>
<td>b</td>
<td>1</td>
</tr>
<tr>
<td>c</td>
<td>1</td>
</tr>
</tbody>
</table>

Lemma 2.3.2. There exist 6 SETs of type AD.

Proof. By Example 2.3.1 there are two ways to have a SET with one attribute yielding all different states.

Case 1: The first attribute yields all alike states and the second attribute yields all different states.
2. PROPERTIES OF THE GAME OF SET

By Table 3, there are three different ways to select the state to remain all alike. Since each state for the second attribute will be different. Given that the first attribute will be all alike, there is only one possible arrangement of the different states of the second attribute with the state of the first. Thus, there are three possible SETs with the states of the first attribute being all alike and the states of the second attribute being all different.

Case 2: The second attribute yields all different states and the first attribute yields all alike states.

The proof is similar to Case 1. Therefore, there are 6 SETs of type AD.

Example 2.3.3. A $D^2$ SET Pairing

<table>
<thead>
<tr>
<th>Attribute 1</th>
<th>Attribute 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>1</td>
</tr>
<tr>
<td>b</td>
<td>2</td>
</tr>
<tr>
<td>c</td>
<td>3</td>
</tr>
</tbody>
</table>

Theorem 2.3.4. There exist 6 sets of type $D^2$ in 2 a attribute game.

Proof. Assume without loss of generality that the Attribute 1 yields all different states. Then, for Attribute 2, there are 3 ways to pair the first state of Attribute 1 with a state of Attribute 2. Having matched a state of Attribute 2 to the first state of Attribute 1, this leaves 2 ways to match a state of Attribute 2 to a the remaining states of Attribute 1. This leaves the final state of Attribute 1 to be matched with a state of Attribute 2. This means that there are 3! ways to match all different states of Attribute 2 with all the different states of Attribute 1. Therefore, there are 6 sets of the type $D^2$. 

\[\Diamond\]
The number of ways to pick some \( k \) attributes to be alike out of a total set of \( n \) attributes is simply \( \binom{n}{k} \) where \( n \) is just the number of attributes of our deck. Knowing this, a general form can be found for the number of SETs with \( k \) attributes alike out of a deck with \( n \) attributes.

**Lemma 2.3.5.** For all classes of SETs \( A^i D^j \) and for \( n \in \mathbb{N} \) where \( n \) is the total number of attributes of a given SET game, \( i + j = n \).

**Proof.** By Definition 2.0.10 a SET class can be expressed as \( A^i D^j \). Since \( A^i \) are the number of attributes with alike states and \( D^j \) are the number of attributes with all different states \( i + j \) must account for the total number of attributes of a SET deck. Let \( n \in \mathbb{N} \) where \( n \) is the number of attributes of a SET deck. Then \( i + j = n \).

**Theorem 2.3.6.** There are \( \binom{n}{k} 3^k 3! (n-k)^{-1} \) sets in the form \( A^k D^{n-k} \).

**Proof.** If a set has form \( A^k D^{n-k} \), then \( 0 \leq k \leq n-1 \) since there are no sets of the form \( A^n \) where \( k \) is the number of alike attributes. Then there are \( \binom{n}{k} \) ways to pick \( k \) attributes out of the \( n \) attributes to be alike. There are 3 ways to pick which state (1, 2, or 3) will be alike for a given attribute and there are \( k \) attributes. This means we have \( 3^k \) ways to pick the alike states for each of the \( k \) attributes. Now having taken care of the alike attributes, we are left with \( n-k \) attributes different.

\[
\begin{bmatrix}
\text{Card 1} \\
\text{Card 2} \\
\text{Card 3}
\end{bmatrix} = \begin{bmatrix}
a & b & c & \cdots & a & a & a \\
a & b & c & \cdots & b & b & b \\
a & b & c & \cdots & c & c & c
\end{bmatrix}
\]

Since there are \( n-k \) attributes with all different states, to pair the first of such attributes with the all alike states yields only one possible arrangement. This leaves \( (n-k)-1 \) attributes with all different states. After matching the first of the attributes with all different states,
there are 3! ways to match each of the remaining attributes with all the different states to the remaining attributes. Therefore, there are $3!(n-k)^{-1}$ ways to match the n-k attributes with all different states with the all alike states. Since there are $\binom{n}{k}$ ways to order the all alike attributes, $3^k$ ways to select the states for all the alike attributes, and $3!(n-k)^{-1}$ to match the all different state attributes with the all alike attributes, this means that there are $\binom{n}{k}3^k3!(n-k)^{-1}$ sets in the form $A^kD^{n-k}$.

Using Theorem 2.3.6 and Theorem 2.2.6, it is now possible to compute the probabilities of finding a certain type of SET in a typical game.

Theorem 2.3.7. The probability of finding a SET with k attributes alike out of an n attribute deck is

$$\frac{\binom{n}{k}3^k3!(n-k)^{-1}}{3^n3^{n-1}}$$

Proof. To find the probabilities of finding a certain SET type with k attributes alike out of an n attribute deck we must know two things: the number of sets of a given number k, the total number of sets in the entire deck. Using Theorem 2.2.6, there are $\binom{n}{k}3^k3!(n-k)^{-1}$ sets with k attributes alike. By Theorem 2.1.2 there are $\frac{3^n3^n-1}{3!}$ sets in a deck with n attributes. Thus, the probability of finding a set given some k attributes alike is

$$\frac{\binom{n}{k}3^k3!(n-k)^{-1}}{3^n3^{n-1}}$$

Table 2.3.1 highlights the probabilities of finding a SET given some number of attributes alike in classic SET.

Since we have a general form for finding the number of SETs of with a given number of attributes alike k in an n attribute deck, it is possible to use said form to find the probabilities of finding different SETs for any sized SET deck.
2. PROPERTIES OF THE GAME OF SET

<table>
<thead>
<tr>
<th>Number of Attributes Alike</th>
<th>Number of SETs</th>
<th>Product Form</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>216</td>
<td>( \binom{4}{0} \times 3!^3 )</td>
<td>.2</td>
</tr>
<tr>
<td>1</td>
<td>432</td>
<td>( 3 \times 3!^2 \times \binom{4}{1} )</td>
<td>.4</td>
</tr>
<tr>
<td>2</td>
<td>324</td>
<td>( 3^2 \times \binom{4}{2} \times 3! )</td>
<td>.3</td>
</tr>
<tr>
<td>3</td>
<td>108</td>
<td>( 3^3 \times \binom{4}{3} )</td>
<td>.1</td>
</tr>
<tr>
<td><strong>Total Number of SETs</strong></td>
<td><strong>1080</strong></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2.3.1. Probabilities of Finding a SET with a Number of Attributes Alike.

2.4 SETs with a POINT (on a Grid)

In calculus and linear algebra, a vector can be related to a coordinate system. The same can be applied to SET cards except we will work on a grid. We will focus on placing SET cards from a 2-Attribute game on a grid.

Definition 2.4.1. The Grid.

For a 2-Attribute game, each of the different attributes can be labeled as the axes of the grid. Each of the different states divides the axis into 3 sections. This gives us 9 boxes in total - the same number of cards in a 2-Attribute SET game.
2. PROPERTIES OF THE GAME OF SET

Since each card in a 2-Attribute game can be represented as a point on the grid, this means that SETs can be represented as a collection of points as well.

Before we talk about placing SETs on a grid, we must know how each card will be placed on a grid.

Since each card can be related to a number vector form if the axes of the grid are labelled with the corresponding entry for each card’s vector then each card can be plotted on the grid.

**Example 2.4.2.** Relating a point on grid, and number vector.

![Diagram of a card with vector form (2,1)](image)

Figure 2.4.1. A card with vector form (2,1)

Plotting a card’s number vector on the grid behaves the same way as plotting on cartesian coordinates. The order of the attributes is the same as in classic SET where an arbitrary card is in the form (color, shading, number, shape).

**Example 2.4.3.** A SET on the grid.

**Example 2.4.4.** Another SET on the grid.
Placing points in each box that correlates to a card in the SET reveals that we have one AD and one DD SET. Figure 2.4.2 refers to a DD SET since each point is in a different row and column from all the others, much like a Sudoku puzzle. We will define this as the Sudoku property of SET and SuperSET.

**Definition 2.4.5** (Sudoku Property). A SET that can be represented by only one point in each row and column of the grid yields the Sudoku Property.
Example 2.4.6. Figure 2.4.2 follows the Sudoku Property since the SET can be represented by only one card in each row and column of the grid. In general, every SET of form DD can be expressed via the Sudoku Property.

Figure 2.4.3 refers to an AD SET. Observe that a line can be drawn that connects all three points in a straight line. Being able to make figures such as lines and as we will see later polyhedra will be key in using geometry to find SETs in a variation known as SuperSET.

Since SET cards can be represented as points on a grid, and SETs are geometric figures on a grid. This means that there can be some number of points such that none of the points form a SET with each other. The invention of geometries in SET will allow us to easily detect when there exists a SET. If there exist a number of cards that do not form a SET, these cards are part of a cap. The maximal number before adding another card yields a SET is called a Complete Cap.

Definition 2.4.7 (Complete Cap). The Complete Cap of a given set deck is the maximal collection of cards such that none of the cards in the collection form a SET.

Theorem 2.4.8 (Complete Cap of a 2-Attribute SET deck). The Complete Cap of a 2 Attribute SET Deck is 4 cards.

Proof. Case 1: Suppose there is a Complete Cap of 5 Cards.

Case 1: Without Loss of Generality, suppose three of those cards are in the same row. Since these three cards are in the same row, they have the form \{(a,0), (a,1), (a,2)\} for some \(a \in \{0,1,2\}\). Since the three cards share one state, the three cards form a horizontal line and thus form a SET. Therefore, if there is a Complete Cap of 5, it cannot involve three cards in the same row.
Case 2: Suppose that there are four cards with two cards placed in two different rows such that no three cards form a SET with each other. Without loss of generality suppose that two cards in one row are adjacent to another. Then the following arrangements are possible:

![Figure 2.4.4. A "5-Attribute" Complete Cap](image)

Suppose that a card is added to these arrangements. If 5 is a Complete Cap for a 2-Attribute SET game, then there must be some arrangement of 5 cards such that no SET if found. We will represent adding a card as placing a triangle on the grid. If the card is placed in the bottom row of both arrangements it forms a vertical or horizontal line, or follows the Sudoku Property. The same is true if a card is placed in the middle or top rows. Figure 2.4.5 uses lines to illustrate the SETs made when a card is placed in any spot on the grid.

![Figure 2.4.5. A "5-Attribute" Complete Cap](image)

Because adding a card in any spot on the grid yields either a vertical line, horizontal line, or a SET following the Sudoku Property, 5 cards cannot be a Complete Cap for a 2-Attribute SET game.
Case 3: The is an example of a complete cap of 4.

Therefore, the complete cap of a 2-Attribute SET Game is 4 cards.

Although somewhat arbitrary, the ability to define geometries in a SET game will be vital for determining Maximal Caps for a variation of SET called SuperSET. These methods are not only limited to 2-Attribute games of SET. However, as the game uses more attributes, this method of representation to find Maximal Caps becomes cumbersome. However, this is not to say that it’s not possible to find a bound for the Maximal Cap before requiring the use of a program. This will lead to some questions I will pose at the end of the paper in Chapter 5.
3
SuperSET

3.1 Introduction to SuperSET

Now that we’ve analyzed the general game of n-attribute SET, we will now analyze a variant of SET called SuperSET. SuperSET is a game similar to SET, but each card now has 4 states instead of 3. Adding another state means that a fourth card must be added. In changing the number of cards in a SET, this yields for surprising changes in terms of the different possible SETs in a game.

Figure 3.1.1. A SuperSET Card
3. **SUPERSET**

In SuperSET each card is associated with the same attributes as classic SET. However, each attribute now has 4 states instead of 3. Each SuperSET card can be represented using the same methods discussed in Chapter 2.

**Example 3.1.1.** The card in Figure is (red, empty, 4, heart) and is an example of a card in 4-Attribute SuperSET.

Since a new state was added, a new type of SET was added as well. In SuperSET this new type of state is called Half and Half.

**Definition 3.1.2** (Half and Half). A Half and Half state is a state that is alike in two different ways.

**Definition 3.1.3** (SuperSET). A SuperSET is a collection of four cards such that for all attributes, the states are either all alike, all different, or half and half.

The addition of a new kind of state is the result of preserving the property of uniqueness present in classic SET.

**Example 3.1.4.** A SuperSET with Playing Cards

![Figure 3.1.2. A Half and Half SuperSET with playing cards](image)

Figure 3.1.2 uses playing cards to model a 3-Attribute SuperSET game with attributes color, shape, and number. The four cards form a SuperSET because they share half and half color, half and half number, and half and half shape.
Definition 3.1.5. Let $A =$ Alike, $D =$ Different, and $H =$ Half and Half. Then any SuperSET can be written in the form $A^iD^jH^k$ where both $i$ and $j \neq 0$ and $j = 0$, $k = 1$.

In SuperSET, the focus will be on 2-attribute and 3-attribute games. For a 2-attribute game, the attributes are number and shape. The states for each attribute remain the same except for an added state as seen in Table 3.1.1.

<table>
<thead>
<tr>
<th>Number</th>
<th>Shape</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Ovals</td>
</tr>
<tr>
<td>2</td>
<td>Squiggles</td>
</tr>
<tr>
<td>3</td>
<td>Diamonds</td>
</tr>
<tr>
<td>4</td>
<td>Hearts</td>
</tr>
</tbody>
</table>

Table 3.1.1. 2-Attribute SuperSET Attributes and Their States.

<table>
<thead>
<tr>
<th>Class</th>
<th>Cards as Vectors</th>
<th>Cards</th>
</tr>
</thead>
<tbody>
<tr>
<td>AD</td>
<td>$(1, 1)(1, 2)(1, 3)(1, 4)$</td>
<td><img src="image" alt="Cards AD" /></td>
</tr>
<tr>
<td>DD</td>
<td>$(1, 1)(2, 2)(3, 3)(4, 4)$</td>
<td><img src="image" alt="Cards DD" /></td>
</tr>
<tr>
<td>DH</td>
<td>$(1, 1)(2, 1)(3, 2)(4, 2)$</td>
<td><img src="image" alt="Cards DH" /></td>
</tr>
<tr>
<td>HH</td>
<td>$(1, 1)(1, 2)(2, 1)(2, 2)$</td>
<td><img src="image" alt="Cards HH" /></td>
</tr>
</tbody>
</table>

Table 3.1.2. Classes of SuperSETs
3. **SUPERSET**

Table 3.1.2 matches the type of SuperSET with its vector representation and physical cards. By now the reader should be familiar with SETs in the form AD and DD from classic SET. Instead, we will introduce the new classes of SuperSETs: DH and HH.

**Example 3.1.6.** A SuperSET of the form DH with its matrix form:

![DH SuperSET matrix form](image)

This is an example of a SuperSET of the form DH because all of the numbers are different and the shapes are half and half.

**Example 3.1.7.** A SuperSET of the form HH:

![HH SuperSET matrix form](image)
Figure 3.1.4. An HH SuperSET and its matrix forms.

\[
\begin{pmatrix}
1 & 1 & \text{Diamond} \\
1 & 1 & \text{Oval} \\
2 & 1 & \text{Diamond} \\
2 & 2 & \text{Oval}
\end{pmatrix}
\]

This is an example of a SuperSET of the form HH because the numbers of shapes are \textbf{half} \quad \text{and} \quad \textbf{half} \quad \text{and} \quad \text{the shapes are} \quad \textbf{half} \quad \text{and} \quad \textbf{half}.

**Theorem 3.1.8** (Half and Half Uniqueness Theorem). \textit{Let} \( U \) \textit{denote the deck of SuperSET cards. For any} \( r, s, t \in U \), \textit{there exists a unique} \( u \) \textit{such that} \((r, s, t, u)\) \textit{form a SuperSET}.

**Proof.** Let \( r, s, \) and \( t \in U \). Then \( r, s, t \) can be represented as \((r_1, r_2, \ldots, r_n), (s_1, s_2, \ldots, s_n), \) and \((t_1, t_2, \ldots, t_n)\) where \( r_i, s_i, t_i \in \{1, 2, 3, 4\} \) \( \forall i \in \{1, 2, \ldots, n\} \). We'd like to find a \( u \in U \) such that makes \((r, s, t, u)\) a SuperSET. This yields 3 cases:

Case 1: Suppose that \( r_1 = s_1 = t_1 \), then we set \( r_1 = s_1 = t_1 = u_1 \).

Case 2: Suppose that \( r_1 \neq s_1 \neq t_1 \), then we set \( r_1 \neq s_1 \neq t_1 \neq u_1 \).

Case 3: Suppose that \( r_1 = s_1 \neq t_1 \), then we set \( u_1 = t_1 \) so that \( r_1 = s_1 \neq t_1 = u_1 \).
The proof for \( i = 2, 3, \cdots, n \) is similar. Since in each case it is possible to identify a fourth card to complete the SuperSET, it must be the case that for any cards \( r, s, t \in U \), there exists \( u \in U \) such that \( (r, s, t, u) \) form a SuperSET.

The proof of Theorem 3.1.8 relies on the existence of Half and Half states to satisfy the definition of a unique fourth card. This was the main reason Half and Half states are considered in SuperSET.

**Lemma 3.1.9.** There are 16 cards in a 2-Attribute SuperSET deck.

**Proof.** In 2-Attribute SuperSET, each card yields 2-Attributes. Since each attribute yield 4 different states, there are \( 4 \cdot 4 \) total possible pairings. Therefore, there are 16 cards in a 2-Attribute SuperSET deck.

**Theorem 3.1.10.** There are 140 SuperSETs in a 2-Attribute SuperSET deck.

**Proof.** By Theorem 3.1.8 every 3 cards imply a unique 4th card to complete a SuperSET. As proven in Theorem 3.1.9, there are 16 cards in a 2-Attribute SuperSET deck. This means there are \( 16 \cdot 15 \cdot 14 \) ways to select 3 cards from a SuperSET deck. Since a SuperSET is a collection of 4 cards, and the order of the cards do not matter, we will divide \( 16 \cdot 15 \cdot 14 \) by \( 4! \), yielding \( \frac{16 \cdot 15 \cdot 14}{4!} = 140 \) SuperSETs. Therefore, there are 140 SuperSETs in a 2-Attribute SuperSET deck.

Much like classic SET, SuperSET has its own subdivisions of SuperSETs. Since for any attribute the states are either all alike, all different, or Half and Half there are 9 pairings that can be made. Table 6 shows which types of 2-Attribute SuperSETs can be made.

**Definition 3.1.11 (SuperSET Pairings).**
Despite the 9 possible pairings that can be made, the number of relevant SuperSETs is much less.

**Lemma 3.1.12.** *SuperSETs in the form AA and AH cannot exist.*

**Proof.** Suppose there exists a SuperSET in the form AA. Then this means the 4 cards all have the same state for both attributes. This implies that there exist 4 copies of the same card, which is a contradiction. Suppose there exists a SuperSET in the form AH. Then this means that the 4 cards all have the same state for one attribute and have half and half for the other. This implies that there are two copies of both cards in the deck and thus yields a contradiction. Therefore, SuperSETs in the form AA and AH cannot exist. \(\Box\)

**Theorem 3.1.13.** *There are 4 types of SuperSETs of a 2-Attribute game: AD, DD, DH, and HH.*

**Proof.** Because SuperSETs in the form AA and AH cannot exist, this means that out of the 9 2-Attribute SuperSET classes, only 6 SuperSETs are valid. Furthermore, the number of SuperSETs in the form AH and HA are the same because they both have one Attribute with all alike states and another with Half and Half states. This being the case, it must be that the number of SuperSETs in the form AD and DH are the same as DA and HD.
Thus, there are really only 4 SuperSET classes of a 2-Attribute game: AD, DD, DH, and HH.

Now that the reader has a basic understanding of the game of SuperSET, we will delve further into the properties of the game. Special interest will be placed on counting the different kinds of SuperSETS, the probabilities of finding certain SuperSETs as we have done in the previous chapter, and the geometries of SuperSET.
Properties of the game SuperSET

<table>
<thead>
<tr>
<th>Type</th>
<th>Number of SuperSETs</th>
<th>Product Form</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>DA</td>
<td>8</td>
<td>$2 \times 4$</td>
<td>.057</td>
</tr>
<tr>
<td>DH</td>
<td>72</td>
<td>$2 \times \left(\frac{4}{2}\right)^2$</td>
<td>.514</td>
</tr>
<tr>
<td>DD</td>
<td>24</td>
<td>$4!$</td>
<td>.171</td>
</tr>
<tr>
<td>HH</td>
<td>36</td>
<td>$\left(\frac{4}{2}\right)^2$</td>
<td>.257</td>
</tr>
<tr>
<td>TOTAL</td>
<td>140</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4.0.1. Number of Different SuperSETs in a 2 - attribute deck

Table 4.0.1 lists the number of different SuperSET types along with their probabilities. Note that DH SuperSETs consist of over half of the entire 2-Attribute deck. Also, compared to classic SET, the probabilities are more skewed to finding some SuperSETs.
For a 3 attribute SuperSET deck, it is tempting to only deal with the SuperSETs of a 2 attribute game. However, as we will see, there exist SuperSETs in a 3 Attribute game that do not exist in a 2 Attribute one.

**Definition 4.0.14.** 3-Attribute SuperSETs are all monomials of degree 3 in variables A, D, and H.

**Proof.** By Table 6, SuperSETs of 2 Attributes have the forms:

\[
AA, AD, AH, DA, DD, DH, HA, HD, HH
\]

Since we are adding a state type, this means that we are adding either an A, D, or an H to the SuperSET types. This means that each 3-Attribute SuperSET type will either feature 3, 2, or all the same SuperSET types. This being the case, each SuperSET Attribute type can be expressed in the form

\[
A^i D^j H^k
\]

for some variation of non-negative integers \(i, j, k\) such that \(i + j + k = 3\). Therefore, each 3-Attribute SuperSET is a monomial of degree 3 in variables A, D, and H.

**Lemma 4.0.15.** SuperSETs of the forms \(A^3\) and \(A^2 H\) cannot exist.

**Proof.** We will prove this in two parts via a contradiction. Part 1, Suppose there exists a SuperSET of the form \(A^3\). Then the SuperSET takes the form

\[
\begin{array}{c|ccc}
\text{Card 1} & 1 & 3 & 4 \\
\text{Card 2} & 1 & 3 & 4 \\
\text{Card 3} & 1 & 3 & 4 \\
\text{Card 4} & 1 & 3 & 4
\end{array}
\]
4. PROPERTIES OF THE GAME SUPERSET

This means that the proof follows Lemma 2.2.7. Part 2, Suppose there exists a SuperSET of the form $A^2H$. Then the SuperSET takes the form

$$
\begin{array}{c}
\text{Card 1} \begin{pmatrix} 1 & 3 & 4 \\ \end{pmatrix} \\
\text{Card 2} \begin{pmatrix} 1 & 3 & 4 \\ \end{pmatrix} \\
\text{Card 3} \begin{pmatrix} 1 & 3 & 2 \\ \end{pmatrix} \\
\text{Card 4} \begin{pmatrix} 1 & 3 & 2 \\ \end{pmatrix}
\end{array}
$$

Observe that Cards 1 and 3 both have duplicates in the SuperSET. Since each card in a SuperSET is unique, this yields a contradiction. Therefore, there cannot be SuperSETs of the form $A^3$ or $A^2H$.

When dealing with the "copies" of these SuperSETs, we must account for how many different ways the different attributes relations can be ordered. This means that for any SuperSET class, there can exist multiples of the same type of SuperSET given the number of different arrangements. This will be important when we start to count the number of SuperSETs in a 3-Attribute game.

To count the number of 3-Attribute SuperSETs we will employ the same methods seen in classic SET. Table 8 highlights the number of different SuperSETs of a 3-Attribute game. However, adding Half and Half states complicates the totals. Not only do we have to account for the states we will use, but we must also take into account how many possible placements there are.
4. PROPERTIES OF THE GAME SUPERSET

<table>
<thead>
<tr>
<th>Type</th>
<th>Number of SuperSETs</th>
<th>Product Form</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAD</td>
<td>48</td>
<td>$4^2 \times 3$</td>
<td>0.005</td>
</tr>
<tr>
<td>DDA</td>
<td>288</td>
<td>$4! \times 4 \times 3$</td>
<td>0.028</td>
</tr>
<tr>
<td>DDH</td>
<td>2592</td>
<td>$4! \times \left(\frac{4}{2}\right)^2 \times 3$</td>
<td>0.249</td>
</tr>
<tr>
<td>HHA</td>
<td>432</td>
<td>$4 \times \left(\frac{4}{2}\right)^2 \times 3$</td>
<td>0.041</td>
</tr>
<tr>
<td>HHD</td>
<td>3888</td>
<td>$\left(\frac{4}{2}\right)^4 \times 3$</td>
<td>0.373</td>
</tr>
<tr>
<td>HHH</td>
<td>1728</td>
<td>$\left(\frac{4}{2}\right)^4 + \left(\frac{4}{2}\right)^3(2)$</td>
<td>0.166</td>
</tr>
<tr>
<td>DDD</td>
<td>576</td>
<td>$4!^2$</td>
<td>0.055</td>
</tr>
<tr>
<td>ADH</td>
<td>864</td>
<td>$4 \times \left(\frac{4}{2}\right)^2 \times 6$</td>
<td>0.083</td>
</tr>
<tr>
<td>Total</td>
<td>10416</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4.0.2. The Number of Different SuperSETs in a 3 - attribute deck

Since we have the number of SuperSETs in both a 2 and 3 Attribute game we will now attempt to find some general form that allows us to compute the number of SuperSETs of any given type. Definition 3.1.5 states that any SuperSET can be written in the form $A^iD^jH^k$. For SuperSET, we’ve been focusing on 2 and 3 attributes. Finding a general form will allow us to compute the number of SuperSETs of a given type easily. For SuperSET, a general form can be found using the data obtained in Tables 4.0.1 and 4.0.2.
Lemma 4.0.16. SuperSETs of the form $A^i H$ do not exist

Proof. Suppose that a SuperSET in the form $A^i H$ exists. By Lemma 3.1.12, SuperSETs of the form $AH$ cannot exist. Suppose that another alike state is added onto this SuperSET giving it the form $A^2 H$. Then the SuperSET takes the form

\[
\begin{pmatrix}
\text{Card 1} & a & b & d \\
\text{Card 2} & a & b & d \\
\text{Card 3} & a & b & c \\
\text{Card 4} & a & b & c \\
\end{pmatrix}
\]

where $a, b, c, \text{and} \ d$ are all different attribute states. Observe that this SuperSET contains 2 copies of the same card. Therefore this SuperSET cannot exist. Adding more alike states will yield the same effect. Therefore, a SuperSET of the form $A^i H$ cannot exist. \hfill \Box

Theorem 4.0.17. There are $(4!)^{j-1}$ SuperSETs of the form $D^j$

Proof. Suppose that there is a SuperSET of form $D^j$ for some positive integer $j$. Then, there is only one way to order all the states of the first attribute such that all of the states are different. For the 2nd attribute there are $4!$ ways to pair a state from the second attribute with that of the first. The same is true for the 3rd to the $j$th attribute. Since there are $4!$ choices for a placement for the 2nd through $j$th attributes, there are $4!(j-1)$ SuperSETs of the form $D^j$.

\hfill \Box

Since we have shown that there are $4!^{j-1}$ SuperSETs of the form $D^j$, we will now show that there are $4!^{j-1}(\binom{4}{2})^{2k}$ SuperSETs of type $D^j H^k$.

Theorem 4.0.18. There are $4!^{j-1}(\binom{4}{2})^{2k}$ SuperSETs of type $D^j H^k$

Proof. It follows from Lemma 4.0.18 that there are $4!^{j-1}$ SuperSETs of the form $D^j$. Suppose a attribute with state type H was added to the SuperSET. Then, there would
be \( \binom{4}{2} \) ways to select the states, and \( \binom{4}{2} \) ways to match the states with the states of the previous attribute. After matching one attribute with state type H with the SuperSET of form D^j, to match another attribute with type H there \( \binom{4}{2} \) ways to select the states, and \( \binom{4}{2} \) ways to match the states with the states of the previous attribute. This means that for any attribute of type H that will be added, there will be \( \binom{4}{2} \) ways to select the states of the attribute, and \( \binom{4}{2} \) ways to match the states with the states of the previous attribute. This means that each additional H to be added onto the D^j will contribute \( \binom{4}{2}^2 \) arrangements to the number of SuperSETs. Therefore, there are \( 4! j^{\binom{4}{2}} \) SuperSETs of type D^jH^k.

**Theorem 4.0.19.** There are \( 4^i 4^{j-1} \) SuperSETs of type A^i D^j

**Proof.** It follows from Lemma 4.0.18 that there are \( 4^{j-1} \) SuperSETs of the form D^j. Suppose that an attribute with state type A was added. Since there are 4 options for each state, this multiplies the number of SuperSETs by 4. If another attribute with state type A is added it will again change the number of SuperSETs by a factor of 4. This means that for any \( i \) attributes with state type A we multiply the number of SuperSETs by \( 4^i \). Therefore, there are \( 4^i 4^{j-1} \) SuperSETs of type A^i D^j.

**Theorem 4.0.20.** There are \( 4^i \binom{4}{2}^{2^k-1} \) SuperSETs of form A^i H^k such that \( k > 2 \).

**Proof.** We will prove this in 2 parts. Part 1, SuperSETs in the form A^iH do not exist, and Part 2, that there are \( 4^i \binom{4}{2}^k \) SuperSETs of form A^iH^k such that \( k > 2 \).

Part 1: Suppose that \( k < 2 \). This means that we have a SuperSET of the form A^iH. By Lemma 4.0.16 a SuperSET of this form cannot exist. This means that \( k > 2 \).

Part 2: Suppose that \( k > 2 \). Suppose that we have \( i \) attributes that share all alike states. This means that there are \( 4^i \) possible arrangements of these elements. Adding on an attribute with state type H yields only one possible placement, but \( \binom{4}{2} \) choices for the
states. This yields a SuperSET of the form $A^iH$. As shown in Lemma 4.0.16, this yields two copies of the same card. However, adding another attribute with state type $H$ implies only one pairing that does not yield copies using the $\binom{4}{2}$ possible pairings. After pairing this $H$ attribute state, this yields a SuperSET of type $A^2H^2$. Adding more $H$ types yields not only $\binom{4}{2}$ possible choices for the states, but also $\binom{4}{2}$ choices for the pairing. Since we obtain $\binom{4}{2}^2$ with each attribute of $H$ type except for the initial one, we gain $\binom{4}{2}^{2k-1}$ possible arrangements. Therefore, there are $4^i\binom{4}{2}^{2k-1}$ SuperSETs of form $A^iH^k$ such that $k > 2$.

**Theorem 4.0.21.** There are $4^i4^j4^k-4^i\binom{4}{2}^{2k}$ SuperSETs of type $A^iD^jH^k$ for $j \neq 0$

**Proof.** Case 1: Suppose that $j=0$. Then $A^iD^jH^k$ behaves the same way as $A^iH^k$.

Case 2: Suppose that $j \neq 0$. From Theorem 4.0.19 there are $4^i4^j4^k-1$ SuperSETs of type $A^iD^j$. Suppose that an attribute with state types $H$ was added. Then, there are $\binom{4}{2}^2$ ways to select and place the states. Adding another attribute with state types $H$ adds contributes an additional $\binom{4}{2}^2$ ways to select and place the states. This being the case, there must be $\binom{4}{2}^{2k}$ ways to add $k$ attributes with $H$ state types to the existing monomial. Therefore, there are $4^i4^j4^k-\binom{4}{2}^{2k}$ SuperSETs of type $A^iD^jH^k$. \qed

Note that if $j=0$ then $A^iD^jH^k$ behaves the same as $A^iH^k$, altering its formula. However, if $i = 0$, then $A^iD^jH^k$ behaves the same way as $D^jH^k$. The theorems previously states lead to the question of what happens with the general formula as any two of the $i, j, k$ are 0. It was already proven that $A^i$ SuperSETs do not exist and that $D^j$ SuperSETs take a form as described in Theorem 4.0.17. However, what about $H^k$?

Looking at the Tables 4.0.1 and ?? reveal something intriguing and surprising about $H^2$ and $H^3$, $H^3$ is not a direct product of some number of SuperSETs as $H^2$. This is strange because when paired with $A^i$, or $D^jH^k$ yields nice patterns. For $H^2$ and $H^3$ a pattern cannot
easily be defined using counting means. This will yield questions I will address later in Chapter 5.

4.1 SuperSETs on a Plane

In addition to changing the different kinds of SETs in the game, another aspect of SET that SuperSET changes is the Maximal Cap of a given number of attribute game. Since we have a working definition of the geometries of a SET, we now have a quicker means to interpret which cards can form a SET as well as having a new means to classify which SETs we have. This chapter will focus on discussing the geometries of SuperSET and their relationships with Maximal Caps.

Since it is possible to represent a 2-Attribute SET game on a grid, it should be the same case for a 2-Attribute SuperSET game.

The points in Figure 4.1.1 represent the cards (1,2), (3,1), and (4,2).

Now that we have demonstrated what a complete cap is in regular SET, we can now define what a complete cap in SuperSET would look like. Since SuperSET allows for Half
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and Half SETs, the geometric figures that are composed by SuperSETs are different then what was seen in SET. Because Half and Half SETs are allowed in SuperSET, for a 2-Attribute game the following SuperSETs exist:

1. SuperSETs with one state alike for one attribute and all different states for the other (AD).
2. SuperSETs with one attribute’s states split half and half and one attribute whose states are all different (HD).
3. SuperSETs with both attribute’s states split half and half (HH).
4. SuperSETs with both attribute’s states being all different (DD).

These SuperSETs can be prepresented as geometric figures created by points of a $4 \times 4$ grid.
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Figure 4.1.2. SuperSETs AD and DA

\[(1, 1), (1, 2), (1, 3), (1, 4)\]

\[(1, 1), (2, 1), (3, 1), (4, 1)\]
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(1, 2), (2, 2), (3, 1), (4, 1)

(3, 1), (3, 2), (4, 3), (4, 4)

Figure 4.1.3. SuperSsets HD and DH
4. PROPERTIES OF THE GAME SUPERSET

Figure 4.1.4. An HH SuperSET

(3, 1), (3, 2), (4, 1), (4, 2)

(3, 1), (3, 4), (4, 1), (4, 4)

Figure 4.1.4. An HH SuperSET
Figure 4.1.5. A DD SuperSET

(1, 1), (2, 2), (3, 3), (4, 4)

(1, 1), (2, 4), (3, 2), (4, 3)
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Observe that SuperSETs have certain geometric figures associated with them. For an AD SuperSET it is a horizontal or vertical line. For a DH SuperSET it is either a rhombus or a trapezoid. For an HH SuperSET it is a square. However, for a DD SuperSET there is no shape that can be associated with them. Despite this, for a DD SuperSET there must only be one point in each row and column of a grid. This behavior is exactly like a Sudoku puzzle where no two alike number occupy a row or column. This geometry will be useful in proving on whether a number of cards form a Complete cap in SuperSET.

**Theorem 4.1.1. Complete Cap of a 2 Attribute SuperSET Deck**
The Complete Cap of a 2-Attribute SuperSET deck is 6 cards

**Proof.** We will prove this in 2 parts. Part 1, we will show there does not exist a Complete Cap of 7 cards. In Part 2, we will give an example of a Complete Cap of 6 cards.

Part 1: Suppose there exists a complete cap of 7 cards.

Case 1: Suppose without loss of generaliy that there are 4 cards in the same row. Then, the four cards have the form \((a,1),(a,2),(a,3),(a,4)\) for \(a \in \{1,2,3,4\}\). Since the four cards form a vertical line, the four cards form a SuperSET of the form AD. Since this arrangement of 4 cards form a SuperSET, if 7 is a Complete Cap for 2-Attribute SuperSET then 4 cards cannot all be in the same row.
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Case 2: Suppose that there is one row with 3 cards and another row with at least 2 cards. Suppose without loss of generality that the three cards are adjacent to each other. Then, the following arrangements are possible:

![Possible Arrangements 1](image)

Figure 4.1.6. Possible Arrangements 1

Since in each of the cases above, either a rectangle, trapezoid, or rhombus can be formed with the points on the grid, all of the possible variations form SuperSETs. Therefore, if 7 is a Complete Cap of SuperSET, then there cannot be a single row of 3 cards and another row with at least 2 cards.
Case 3: Suppose that there are three rows with 2 cards each where none of the cards form a SuperSET. Suppose without loss of generality that the two cards in the top column are adjacent to each other. Then the following arrangements are possible:

![Possible Arrangements 2](image)

Figure 4.1.7. Possible Arrangements 2

Suppose that another card is to be added to any of these arrangements. We will denote adding a card as adding a triangle to the grid.

![Possible Arrangements with added cards](image)

Figure 4.1.8. Possible Arrangements with added cards
4. PROPERTIES OF THE GAME SUPERSET

Since it is possible to create a SuperSET using any triangle on any of the different formations, we cannot add another card to this arrangement to form a Complete Cap. Therefore, it is impossible to add a card to an arrangement of 6 cards with 2 in each row such that none form a SuperSET with another. By these previous cases, it is impossible to form a Complete Cap of 7.
Figure 4.1.9. Complete Cap of 6.

Part 2: This is an example of a Complete Cap of 6.
In this chapter I will pose a few questions for any brave mathematician who wishes to embark on this journey through SuperSET.

Question 1:
Is it possible to use a generating function to find a general formula for the number of SuperSETs of type $H^k$? It seems like there might be a means to describe these patterns using more technical strategies.

Question 2:
We have proved that a Complete Cap of a 2-Attribute SuperSET game is 6 cards, however what of a 3-Attribute game?

**Conjecture 5.0.2.** The Complete Cap of a 3-Attribute SuperSET game is less than 24 cards.

Using the Complete Cap of a 2-Attribute game, copying it over the 4 new states included with the new attribute yields 24 cards. In this configuration there is at least 1 SuperSET so perhaps a Complete Cap of a 3-Attribute game might be less?
5. OPEN QUESTIONS

Figure 5.0.1. A bound of the Complete Cap for a 3-Attribute game

Question 3:

What will happen if we remove the Half and Half states?
Bibliography


