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Internal symmetry in Poincarè gauge gravity

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Abstract

We find a large internal symmetry within 4-dimensional Poincarè gauge theory.

In the Riemann-Cartan geometry of Poincaré gauge theory the field equation and geodesics are invariant under projective transformation, just as in affine geometry. However, in the Riemann-Cartan case the torsion and nonmetricity tensors change. By generalizing the Riemann-Cartan geometry to allow both torsion and nonmetricity while maintaining local Lorentz symmetry the difference of the antisymmetric part of the nonmetricity Q and the torsion T is a projectively invariant linear combination S = T - Q with the same symmetry as torsion. The structure equations may be written entirely in terms of S and the corresponding Riemann-Cartan curvature. The new description of the geometry has manifest projective and Lorentz symmetries, and vanishing nonmetricity.

Torsion, S and Q lie in the vector space of vector-valued 2-forms. Within the extended geometry we define rotations with axis in the direction of S. These rotate both torsion and nonmetricity while leaving S invariant. In n dimensions and (p, q) signature this gives a large internal symmetry. The four dimensional case acquires SO(11,9) or Spin(11,9) internal symmetry, sufficient for the Standard Model.

The most general action up to linearity in second derivatives of the solder form includes combinations quadratic in torsion and nonmetricity, torsion-nonmetricity couplings, and the Einstein-Hilbert action. Imposing projective invariance reduces this to dependence on S and curvature alone. The new internal symmetry decouples from gravity in agreement with the Coleman-Mandula theorem.

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1 Introduction

Geodesics are the preferred paths followed by test particles in general relativity and much of what we know about gravity comes from analyzing these paths. Although relativity gives us proper time as a preferred parameter, even within special relativity we may use any observer's time instead. It is therefore natural to examine the effect of reparameterizing geodesics. The resulting projective transformation [1] of the connection is a well-known symmetry of the curvature of general relativity. However, even in the most restricted form of general relativity, the relationship between projective symmetry and the metric is nontrivial. Fully confronting the conflict between the preferred proper time given by the metric and the obvious freedom to reparameterize suggests the desirability of a reparameterization invariant form of general relativity.

Such compatibility was achieved in a widely known work by Ehlers, Pirani, and Schild [2] and honed in a study by Matveev and Trautman [3] and later Matveev and Scholz [4]. In these papers it is argued that we determine the geometry of spacetime by studying the geodesics of timelike and null geodesics. Concretely, these authors show that we can infer a projective connection from a knowledge of the timelike geodesics of test particles, while the same program for light following null curves determines a conformal connection. Agreement between the two connections in the limit of high velocities leads to an integrable Weyl geometry [5], that is, a geometry with dilatational symmetry which becomes Riemannian with a particular choice of units for proper time.

Given this satisfactory and minimal resolution, we are free to specify the Riemannian gauge and carry out gravitational studies as usual. However, the development of general relativity as a Poincarè gauge theory over the last two-thirds of a century opens some new possibilites. It is these alternative possibilities that we examine here. We explore the combination of projective and Lorentz symmetries starting from a Cartan formulation of gravity. The possibilities include the integrable Weyl form of general relativity, and the integrable Weyl form of the Einstein, Cartan, Sciama, Kibble (ECSK) generalization. The ECSK theory includes field equations driven by both mass and spin. But working in the newer formalism suggests a further, deeper symmetry.

By freeing the connection from the metric, we are led to consider two new fields.

Within Poincarè gauge theory it is natural to include torsion as well as curvature. When fermionic matter is present the torsion becomes the geometric equivalent of spin density in the same way that the Einstein tensor is the geometric equivalent of energy. There is a pleasing justice to this because mass and spin are the Casimir invariants of the Poincarè Lie algebra. But the observation is puzzling because the experimental limits on torsion are strong [6, 7, 8]. This leads to much of the research on Riemann-Cartan geometries being devoted to understanding why torsion effects should be absent or negligible.

The second new field, the nonmetricity, reflects the compatibility of the metric and connection. Since Poincarè gauge theory naturally makes the metric and connection independent, and because the integrable Weyl geometry found in [3, 4] gives the nonmetricity a nonvanishing but removable trace, it is sensible to consider a formulation of gravity in which nonmetricity is free to play a role.

Our goal is to understand the context of general relativity while preserving its overwhelming success as the formulation of gravity. By allowing torsion and nonmetricity within Poincarè gauge theory, we find a surprisingly large internal symmetry. This new symmetry is present even when the torsion and nonmetricity vanish as long as we admit them as possible within the mathematical framework.

The first step of the present investigation is to develop a *non*-minimal class of geometries and variables allowing manifest projective and Lorentz invariance. This is accomplished in Section 4 where we generalize the Einstein-Cartan geometry to explicitly allow both torsion and nonmetricity. Then a linear combination $\mathbf{S}^a = \mathbf{T}^a - \frac{1}{2}\mathbf{e}^b \wedge \mathbf{Q}_{ab}$ of the torsion and antisymmetric part of the nonmetricity is projectively invariant. Remarkably, the structure equations can be expressed entirely in terms of the torsion-like quantity $\mathbf{S}^a$, leading to a Poincarè-equivalent theory with manifest projective and Lorentz symmetries. Equally surprisingly, the new connection based on $\mathbf{S}^a$ is metric compatible.

From these developments, we go on to fully develop the new formulation. Recognizing that $\mathbf{T}^a, \mathbf{Q}^a \equiv \frac{1}{2}\mathbf{e}^b \wedge \mathbf{Q}_{ab}$, and $\mathbf{S}^a$ all lie in the vector space $\mathcal{A}^1_{[2]}$ of vector-valued 2-forms we easily identify the class of rotations of torsion and nonmetricity which leave the sum $\mathbf{S}^a$ invariant. These rotations comprise an internal symmetry of the new system. Because these rotations are fully decoupled from the gravitational variables the Coleman-Mandula theorem is satisfied. While all our calculations are carried out in arbitrary dimension n and signature (p,q), we note that the 4-dim case acquires internal symmetry $SO(10,9)$, $SO(11,8)$ or one of the corresponding spin representations. Any of these cases is sufficient internal symmetry for the Standard Model.

Finally, inclusion of both torsion and nonmetricity prompts reconsideration of the gravitational action. Following a principle often used in general relativity to justify the Einstein-Hilbert action, we write the most general action dependent on no more than second derivatives of the metric, and no more than linear in second derivatives. This motivates a five–parameter addition to the Einstein-Hilbert action which includes quadratic torsion, quadratic nonmetricity and torsion-nonmetricity coupling terms. Imposing projective invariance reduces the additional terms to two kinetic terms. The final action depends on $\mathbf{S}^a$ and curvature alone.

In Section 2 below we lay out basic properties of the Poincarè gauge geometry and ECSK theory, then in Section 3 develop projective symmetry and its effects in Riemann-Cartan geometry. In Section 4 we carry out a revised form of the gauge construction with nonmetricity included from the start, then show how Poincarè symmetry is recovered by introducing the s-torsion $\mathbf{S}^a$. Section 5 describes the new internal rotations of the modified geometry and in Section 6 we find the most general projectively invariant, second order action as described above. We end with a brief summary.

2 Poincarè gauge theory

2.1 General relativity and Poincarè gauge theory

The long history of Poincarè gauging as a gauge version of general relativity is testified by the sequence of researchers–Cartan [9, 10, 11, 12, 13], Einstein [14], Kibble [15], and Sciama [16, 17]–who have lent their initials to the slightly more general ECSK theory of gravity. By adopting the Einstein-Hilbert action, restricting the torsion to zero, and varying the metric, Poincarè gauge theory reproduces general relativity. More generally, leaving the torsion free and varying both the solder form and spin connection enacts the Palatini variation in a systematic way and yields the ECSK theory in Riemann-Cartan geometry.

Nonzero torsion introduces new features beyond general relativity. Dirac fields couple to the totally antisymmetric part of the torsion [18, 19, 20, 21, 22, 23, 24], while Rarita-Schwinger [25, 26] and higher spin fermions give sources to the full torsion [27]. While variation of the Einstein-Hilbert action limits torsion to be nonpropagating and zero in vacuum, some authors add a dynamical term to the action as well [6, 28, 29, 30, 8].

Torsion produces anomalous contributions to parallel transport of any vector in a non-parallel direction. For example the evolution of angular momentum along a timelike curve will depend on the antisymmetric part of the connection, i.e., torsion. While general relativity predicts some effect of gravity on the propagation of spinning objects, the additional change in angular momentum due to torsion conflicts with experiment and places strong limits on the magnitude of torsion.

2.1.1 General relativity as a Lorentz gauge theory

To see these differences clearly, recall the treatment of general relativity as a Lorentz gauge theory, first described by Utiyama [31]. With a Lorentzian spacetime as the base manifold $(\mathcal{M},g)$, we ask for a principal fiber bundle with Lorentz symmetry and symmetric connection. In an orthonormal frame $\mathbf{e}^a$ the spin connection $\boldsymbol{\alpha}^a{}_b$ satisfies

$$\mathbf{d}\mathbf{e}^a = \mathbf{e}^b \wedge \boldsymbol{\alpha}^a{}_b \tag{1}$$

and the Riemann curvature 2-form is given by

$$\mathbf{R}^a{}_b = \mathbf{d}\boldsymbol{\alpha}^a{}_b - \boldsymbol{\alpha}^c{}_b \wedge \boldsymbol{\alpha}^a{}_c \tag{2}$$

In a coordinate basis $\boldsymbol{\alpha}^a{}_b$ is the Christoffel connection. Orthonormality of the basis $\langle \mathbf{e}^a, \mathbf{e}^b \rangle = \eta^{ab}$ leads to the relationship $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}$ between the components $\mathbf{e}^a = e_\mu{}^a \mathbf{d}x^\mu$ and the metric $g_{\mu\nu}$. The field equation for the metric is then determined by the Einstein-Hilbert action plus the action for any matter fields $S = \frac{1}{2}\int \mathbf{R}^{ab} \wedge \mathbf{e}^c \wedge \mathbf{e}^d e_{abcd} + \kappa S_{matter}$ and variation results in the familiar Einstein equation

$$G_{\alpha\beta} = \kappa T_{\alpha\beta}$$

where $G_{\alpha\beta}$ is the Einstein tensor and $T_{\alpha\beta}$ the energy tensor of the matter fields. The metric variation leads to symmetric Einstein and energy tensors.

2.1.2 Poincarè gauge theory of gravity

In its broadest form, Poincarè gauge theory starts with the homogeneous manifold $\mathcal{M}$ formed by the quotient of the Poincarè group by its Lorentz subgroup $\mathcal{P}/\mathcal{L} = \mathcal{M}_0$. The projection mapping from cosets of this quotient to $\mathcal{M}_0$ leads to a principal fiber bundle, effectively a copy of the Lorentz group at each point of $\mathcal{M}_0$. By generalizing the manifold and the Maurer-Cartan form of the spin connection $\boldsymbol{\omega}^a{}_b$ and solder form $\mathbf{e}^a$, while preserving the local Lorentz symmetry of the principal fiber bundle, we arrive at expressions for two 2-form fields.

$$\begin{aligned} \boldsymbol{\mathcal{R}}^a{}_b &= \mathbf{d}\boldsymbol{\omega}^a{}_b - \boldsymbol{\omega}^c{}_b \wedge \boldsymbol{\omega}^a{}_c & (3) \\ \mathbf{T}^a &= \mathbf{d}\mathbf{e}^a - \mathbf{e}^b \wedge \boldsymbol{\omega}^a{}_b & (4) \end{aligned}$$

These are the Riemann-Cartan curvature and the torsion. The action may still be taken as Einstein-Hilbert plus matter, but with Riemann-Cartan curvature scalar. If we then constrain the torsion to zero, $\boldsymbol{\omega}^a{}_b$ reduces to $\boldsymbol{\alpha}^a{}_b$, the curvature reduces to Riemannian and the system reproduces general relativity. Without the torsion constraint the resulting field equations still reduce to the Einstein equation and vanishing torsion in vacuum, but many matter sources lead to nonvanishing torsion [27]. The best known of these sources is the axial current of Dirac fields $\bar{\psi}\gamma^a\gamma_5\psi$ ([18, 19, 20, 21, 22, 23, 24]) which leads to

$$T^a{}_{bc} = \lambda \varepsilon^a{}_{bcd}\bar{\psi}\gamma^d\gamma_5\psi$$

Most research on torsion has focussed on this totally antisymmetric form. However it has been shown that the gravitino field [25], a spin-$\frac{3}{2}$ Rarita-Schwinger field present in supergravity theories (for example, [32, 33, 34, 35, 36]), drives all components of torsion [27]. Without adding a propagaging term for torsion to the theory, torsion still vanishes in vacuum, moving only as its source field moves.

2.2 The structure of Riemann-Cartan geometry

We review the formal features of Poincarè gauge theory. All results below hold in arbitrary dimension $n = p + q$ and signature $s = p - q$ so while we continue to refer to the Poincarè group $ISO(3,1)$ and its Lorentz subgroup $SO(3,1)$ we actually work with $\mathcal{P} = ISO(p,q)$ or $\mathcal{P} = ISpin(p,q)$ with subgroups $\mathcal{L} = SO(p,q)$ or $\mathcal{L} = Spin(p,q)$ respectively. The local Lorentz arena for general relativity in n dimensions follows by setting $q = 1$.

In Appendix A we summarize the formal fiber bundle development of Riemann-Cartan geometry. Here we give only the resulting basic properties.

The most relevant results of this construction are the 2-form expressions for the Riemann-Cartan curvature $\boldsymbol{\mathcal{R}}^a{}_b$ and torsion $\mathbf{T}^a$.

$$\begin{aligned} \mathbf{d}\boldsymbol{\omega}^a{}_b &= \boldsymbol{\omega}^c{}_b \wedge \boldsymbol{\omega}^a{}_c + \boldsymbol{\mathcal{R}}^a{}_b & (5) \\ \mathbf{d}\mathbf{e}^a &= \mathbf{e}^b \wedge \boldsymbol{\omega}^a{}_b + \mathbf{T}^a & (6) \end{aligned}$$

Each of these may be expanded in the orthonormal basis

$$\begin{aligned} \boldsymbol{\mathcal{R}}^a{}_b &= \frac{1}{2}\mathcal{R}^a{}_{bcd}\mathbf{e}^c \wedge \mathbf{e}^d & (7) \\ \mathbf{T}^a &= \frac{1}{2}T^a{}_{bc}\mathbf{e}^b \wedge \mathbf{e}^c & (8) \end{aligned}$$

In a coordinate basis $\mathbf{T}^a$ is given by any antisymmetric part of the connection.

The Bianchi identities generalize to

$$\boldsymbol{\mathcal{D}}\mathbf{T}^a = \mathbf{e}^b \wedge \boldsymbol{\mathcal{R}}^a{}_b \tag{9}$$
$$\boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{R}}^a{}_b = 0 \tag{10}$$

where the covariant exterior derivatives are given by

$$\begin{aligned}\boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{R}}^a{}_b &= \mathbf{d}\boldsymbol{\mathcal{R}}^a{}_b + \boldsymbol{\mathcal{R}}^c{}_b \wedge \boldsymbol{\omega}^a{}_c - \boldsymbol{\omega}^c{}_b \wedge \boldsymbol{\mathcal{R}}^a{}_c \\ \boldsymbol{\mathcal{D}}\mathbf{T}^a &= \mathbf{d}\mathbf{T}^a + \mathbf{T}^b \wedge \boldsymbol{\omega}^a{}_b\end{aligned}$$

The frame field $\mathbf{e}^a$ is (p,q)-orthonormal, $\langle \mathbf{e}^a, \mathbf{e}^b \rangle = \eta^{ab} = diag\,(1,\ldots,1,-1,\ldots,-1)$, with the connection assumed to be metric compatible

$$\mathbf{d}\eta_{ab} - \eta_{cb}\boldsymbol{\omega}^c{}_a - \eta_{ac}\boldsymbol{\omega}^c{}_b = 0 \tag{11}$$

Since $\mathbf{d}\eta^{ab} = 0$, the spin connection is antisymmetric, $\boldsymbol{\omega}_{ab} = -\boldsymbol{\omega}_{ba}$.

Equations (3)-(10) describe Riemann-Cartan geometry in the Cartan formalism. Note that the Riemann-Cartan curvature, $\boldsymbol{\mathcal{R}}^a{}_b$, differs from the Riemann curvature $\mathbf{R}^a{}_b$ by terms dependent on the torsion.

When the torsion vanishes, $\mathbf{T}^a = 0$, the Riemann-Cartan curvature $\boldsymbol{\mathcal{R}}^a{}_b$ reduces to the Riemann curvature $\mathbf{R}^a{}_b$ and Eqs.(3) and (4) exactly reproduce the expressions for the connection and curvature of a general Riemannian geometry. At the same time, Eqs.(9) and (10) reduce to the usual first and second Bianchi identities.

These results are geometric; a physical model follows when we posit an action functional. The action may depend on the bundle tensors $\mathbf{e}^b, \mathbf{T}^a, \boldsymbol{\mathcal{R}}^a{}_b$ and the invariant tensors η_{ab} and $e_{ab\ldots d}$. To this we may add source functionals built from any field representations of the fiber symmetry group $\mathcal{L}$, including scalars, spinors, vector fields, etc.

Constraining the torsion zero, specifying the Einstein-Hilbert form of action, and varying only the solder form, the $q=1$ theory describes general relativity as a gauge theory in n-dimensions. We cannot vary the metric and connection independently because this can introduce nonzero sources for torsion, making the $\mathbf{T}^a = 0$ constraint inconsistent.

Dropping the torsion constraint while retaining the Einstein-Hilbert action gives the Einstein-Cartan-Sciama-Kibble (ECSK) theory of gravity in Riemann-Cartan geometry. The torsion is found to depend on the spin tensor, given by the connection variation of the source $\sigma^\mu{}_{ab} = \frac{\delta L}{\delta \omega^{ab}_\mu}$. Without modifying the action to include dynamical torsion, the resulting torsion survives only within matter.

The structure equations, Eqs.(4) and (3), allow us to derive an explicit form for the connection and a reduced form for the curvature. The result (see Appendix 7) for the spin connection is

$$\boldsymbol{\omega}^a{}_b = \boldsymbol{\alpha}^a{}_b + \mathbf{C}^a{}_b \tag{12}$$

where $\mathbf{C}^a{}_b$ is the *contorsion*,

$$\mathbf{C}^a{}_b = \frac{1}{2}\left(T_c{}^a{}_b + T_b{}^a{}_c - T^a{}_{bc}\right)\mathbf{e}^c \tag{13}$$

Contorsion transforms tensorially so this form is unique. We may recover the torsion by wedging and contracting with $\mathbf{e}^b$.

$$\mathbf{C}^a{}_b \wedge \mathbf{e}^b = \mathbf{T}^a$$

The torsion now enters the curvature through the connection. Expanding the Cartan-Riemann curvature of Eq.(3) using Eq.(12) and identifying the $\boldsymbol{\alpha}$-covariant derivative, $\mathbf{DC}^a{}_b = \mathbf{dC}^a{}_b - \mathbf{C}^c{}_b \wedge \boldsymbol{\alpha}^a{}_c - \boldsymbol{\alpha}^c{}_b \wedge \mathbf{C}^a{}_c$ leads to

$$\boldsymbol{\mathcal{R}}^a{}_b = \mathbf{R}^a{}_b + \mathbf{DC}^a{}_b - \mathbf{C}^c{}_b \wedge \mathbf{C}^a{}_c \tag{14}$$

This is the Riemann-Cartan curvature expressed in terms of the Riemann curvature and the contorsion. If we contract with $\mathbf{e}^b$ we recover the Bianchi identity. This happens because our solution for the connection automatically satisfies the integrability condition for the connection.

3 Projective symmetry

In this Section we review the derivation of projective transformation of the connection in affine (nonmetric) geometry by reparameterization of autoparallels. Then we carry out the derivation of the same transformation in a geometry with both metric and connection, resulting from reparameterization of geodesics. In the third Subsection we show the relationship between projective transformation and the Weyl vector, and how extending to a Weyl geometry creates manifest invariance of induced reparameterizations. In the final Subsection we examine the effect of projective transformation on the torsion in $\mathcal{P}/\mathcal{L}$ gauge theory.

3.1 Projective symmetry in nonmetric geometry

Consider a principal fiber bundle with Lorentz fibers and base manifold $\mathcal{M}$. Given a local Lorentz connection, but no metric, we are able to define the curvature of $\mathcal{M}$ by the single Cartan equation

$$\mathbf{d}\boldsymbol{\omega}^a{}_b = \boldsymbol{\omega}^c{}_b \wedge \boldsymbol{\omega}^a{}_c + \mathbf{R}^a{}_b$$

Here Latin indices refer to a general basis $\mathbf{e}^a = e_\alpha{}^a \mathbf{d}x^\alpha$. The coefficients $e_\alpha{}^a$ must be invertible, but we cannot claim the basis forms $\mathbf{e}^a$ to be orthonormal.

While there are no geodesics without a metric, we may consider autoparallels.

$$v^b D_b v^a = v^b \partial_b v^a + \omega^a{}_{bc} v^b v^c = 0 \tag{15}$$

Projective transformations are changes of the connection that leave the curvature and autoparallels invariant. They arise from reprameterization of the autoparallels.

Let $v^a = e_\alpha{}^a \frac{dx^\alpha}{d\lambda}$ be tangent to the autoparallel curve, and consider a reparameterization $\sigma = \sigma(\lambda)$ to a parallel vector u^α

$$v^a = \frac{d\sigma}{d\lambda}\frac{dx^a}{d\sigma} = \frac{d\sigma}{d\lambda} u^a$$

where $\sigma(\lambda)$ is C^3 and monotonic. Let $f = \frac{d\sigma}{d\lambda}$ and substitute for v^a in Eq.(15) to find

$$u^b \partial_b u^a + \left(\omega^a{}_{bc} + \delta^a_c \partial_b (\ln f)\right) u^b u^c = 0 \tag{16}$$

From this we extract the transformed connection

$$\tilde{\omega}^a{}_{bc} = \omega^a{}_{bc} + \delta^a_c \partial_b (\ln f)$$

Notice that the projective change in the connection could be symmetrized, $\omega^a{}_{bc} + \delta^a_{(c}\partial_{b)} (\ln f)$, when we remove $u^b u^c$ but this does not preserve the curvature.

Setting $\boldsymbol{\xi} = \mathbf{d}\ln f$, the we recover the original form of (15) in terms of $\tilde{\boldsymbol{\omega}}^a{}_b$ if we write

$$\begin{aligned} \tilde{\boldsymbol{\omega}}^a{}_b &= \boldsymbol{\omega}^a{}_b + \delta^a_b \boldsymbol{\xi} \\ \mathbf{d}\boldsymbol{\xi} &= 0 \end{aligned} \tag{17}$$

This is the projective transformation of the connection.

The invariance of the curvature is immediate.

$$\begin{aligned} \tilde{\mathbf{R}}^a{}_b &= \mathbf{d}\tilde{\boldsymbol{\omega}}^a{}_b - \tilde{\boldsymbol{\omega}}^c{}_b \wedge \tilde{\boldsymbol{\omega}}^a{}_c \\ &= \mathbf{d}\left(\boldsymbol{\omega}^a{}_b + \delta^a_b \boldsymbol{\xi}\right) - \left(\boldsymbol{\omega}^c{}_b + \delta^c_b \boldsymbol{\xi}\right) \wedge \left(\boldsymbol{\omega}^a{}_c + \delta^a_c \boldsymbol{\xi}\right) \\ &= \mathbf{d}\boldsymbol{\omega}^a{}_b - \boldsymbol{\omega}^c{}_b \wedge \boldsymbol{\omega}^a{}_c + \delta^a_b \mathbf{d}\boldsymbol{\xi} \end{aligned}$$

Since $\mathbf{d}\boldsymbol{\xi} = 0$, the curvature is unchanged, $\tilde{\mathbf{R}}^a{}_b = \mathbf{R}^a{}_b$ and no new structures are introduced.

If we had symmetrized when stripping the tangent vectors off of Eq.(16) we instead find $\tilde{\boldsymbol{\omega}}^{a}{}_{b} = \boldsymbol{\omega}^{a}{}_{b} + \frac{1}{2}\left(\delta^{a}_{b}\xi_{\alpha} + \delta^{a}_{\alpha}\xi_{b}\right)\mathbf{d}x^{\alpha}$ where $\mathbf{d}x^{\alpha}$ is a coordinate basis on $\mathcal{M}$. With vanishing torsion and $\mathbf{d}\boldsymbol{\xi} = 0$ the curvature now changes to

$$\tilde{\mathbf{R}}^{a}{}_{b} = \mathbf{R}^{a}{}_{b} + \frac{1}{2}\delta^{a}_{\alpha}\left(\mathbf{D}\xi_{b} - \xi_{b}\boldsymbol{\xi}\right)\wedge\mathbf{d}x^{\alpha}$$

In the next subsection we show that when we have a metric the corresponding projective transformation also leaves geodesics invariant.

3.2 Projective symmetry of geodesics

The situation within Poincarè gauge theory is different from the affine case. Here the orthonormal frame fields provide a metric, so we may consider geodesics instead of autoparallels. At the same time, projective symmetry produces additional, non-invariant changes.

The structure equations are now those of Eqs.(3) and (4) with the spin connection appropriate to signature (p,q) symmetry.

Let the proper length of a curve $x^{\alpha}(\lambda)$ be given by $s = \int\sqrt{\kappa\eta_{ab}v^{a}v^{b}}d\lambda$ where λ is an arbitrary parameterization for tangent vectors $v^{a} = e_{\alpha}{}^{a}\frac{dx^{\alpha}}{d\lambda}$. The curve is spacelike or timelike for $\kappa = \pm 1$, respectively. Varying the arclength $s[x]$ with respect to the curve $x^{\alpha}(\lambda)$ with arbitrary parameterization λ leads to

$$\frac{dv^{\nu}}{d\lambda} = -\Gamma^{\nu}{}_{\alpha\beta}v^{\alpha}v^{\beta} + \frac{1}{2}\frac{1}{|v^{2}|}\left(\frac{d}{d\lambda}\left|v^{2}\right|\right)v^{\nu}$$

where $\Gamma^{\nu}{}_{\alpha\beta}$ is the Christoffel connection and $\left|v^{2}\right| = \kappa\eta_{ab}v^{a}v^{b}$.

Since we now have a preferred parameterization by proper time

$$u^{\alpha} = \frac{dx^{\alpha}}{d\tau}$$

we may refer alternate parameterizations to τ.

$$v^{\alpha} = \frac{dx^{\alpha}}{d\lambda} = \frac{1}{f}u^{\alpha}$$

where $f = \frac{1}{c}\frac{d\lambda}{d\tau}$. It follows that $\left|v^{2}\right| = -\frac{\kappa}{f^{2}}$ and the geodesic equation becomes

$$\frac{dv^{\nu}}{d\lambda} = -\left(\Gamma^{\nu}{}_{\alpha\beta} + \delta^{\nu}_{\alpha}\partial_{\beta}\ln f\right)v^{\alpha}v^{\beta}$$

Returning to the spin connection, the projective transformation is

$$\begin{aligned}\tilde{\boldsymbol{\alpha}}^{a}{}_{b} &= \boldsymbol{\alpha}^{a}{}_{b} + \delta^{a}_{b}\boldsymbol{\xi}\\ \boldsymbol{\xi} &= \mathbf{d}\left(\ln f\right)\end{aligned}$$

in agreement with Eq.(17).

It is important to demonstrate that ξ_{ν} is a well-defined field. We know that v^{2} is a function of λ for any curve, but we need to verify that $\lambda(x^{\alpha})$ is a differentiable function. The sketch of a proof follows.

Suppose we start at a fixed point $\mathcal{P}$ and consider curves through $\mathcal{P}$, with parameterizations such that $x^{\alpha}(\lambda = 0) = \mathcal{P}$. Let $\mathcal{Q}$ be a second point and consider curves passing through both $\mathcal{P}$ and $\mathcal{Q}$. Then there is no single value of λ at $\mathcal{Q}$, since the curves will have differing proper length. However, for nearby $\mathcal{P},\mathcal{Q}$ there is a unique geodesic $x^{\alpha}_{0}(\lambda)$ and we may assign the value $\lambda(\mathcal{Q})$ as the parameter value which the geodesic parameter attains at $\mathcal{Q}$, i.e., $x^{\alpha}_{0}(\lambda(\mathcal{Q})) = \mathcal{Q}$. Now suppose $\mathcal{P},\mathcal{Q}$ are points connected by multiple geodesics (e.g., the north and south poles of a sphere). Then these geodesics must yield the same value of λ, or else there is a minimum value of λ (e.g., curves around a cylinder in opposite directions, with $\mathcal{P}, \mathcal{Q}$ nearer in one of the directions). We take this minimum for the value of $\lambda(\mathcal{Q})$. This gives a unique value $\lambda(x^\alpha)$ to each point $\mathcal{Q}$ that can be reached from $\mathcal{P}$. The extremal condition requires small changes in the path to produce small changes in the proper length of any curve, and these only at second order. Therefore, the function is differentiable.

We now have an equivalence class of connections,

$$\tilde{\Gamma}^\alpha{}_{\mu\nu} \in \left\{ \Gamma^\alpha{}_{\mu\nu} - \delta^\alpha_\mu \xi_\nu | \mathbf{d}\boldsymbol{\xi} = 0 \right\} \tag{18}$$

As we have seen, projective transformations preserve the curvature and therefore the action and field equations.

3.3 Minimal compatibility

By minimal compatibility we mean the minimum change in the geometry to achieve manifest reparameterization and local Lorentz invariance. This compatibility is implicit in the Ehlers, Pirani, Schild program [2, 3, 4]. Here we show how extending to an integrable Weyl geometry achieves reparameteriation invariance.

General relativity has both metric and connection, satisfying Eqs.(2) and (1). but Eq.(1)) changes with projective transformation to give

$$\mathbf{d}\mathbf{e}^a = \mathbf{e}^b \wedge \boldsymbol{\alpha}^a{}_b + \mathbf{e}^a \wedge \boldsymbol{\xi} \tag{19}$$

The connection is no longer fully compatible with the metric, but gives it a nonvanishing covariant derivative

$$\begin{aligned} \mathbf{Q}_{ab} &= \mathbf{d}\eta_{ab} - \eta_{cb}\left(\boldsymbol{\alpha}^c{}_a + \delta^c_a \boldsymbol{\xi}\right) - \eta_{ac}\left(\boldsymbol{\alpha}^c{}_b + \delta^c_b \boldsymbol{\xi}\right) \\ &= -2\eta_{ab}\boldsymbol{\xi} \end{aligned} \tag{20}$$

The trace of the nonmetricity is proportional to the Weyl vector, $\boldsymbol{\omega} = \frac{1}{2n}\mathbf{Q}^a{}_a$ where $\boldsymbol{\omega}$ is the gauge vector of dilatations.

Making the substitution $\boldsymbol{\omega} = W_a \mathbf{e}^a = -\boldsymbol{\xi}$ puts the structure equations in the form

$$\begin{aligned} \mathbf{d}\boldsymbol{\omega}^a{}_b &= \boldsymbol{\omega}^c{}_b \wedge \boldsymbol{\omega}^a{}_c + \boldsymbol{\mathcal{R}}^a{}_b \\ \mathbf{d}\mathbf{e}^a &= \mathbf{e}^b \wedge \boldsymbol{\omega}^a{}_b + \boldsymbol{\omega} \wedge \mathbf{e}^a \end{aligned}$$

The connection is now that of a Weyl geometry [5]. This reflects multiple changes. The antisymmetry of the connection is restored and the Weyl connection is metric compatible. However, the connection and curvature now include contributions from the Weyl vector.

$$\begin{aligned} \boldsymbol{\omega}^a{}_b &= \boldsymbol{\alpha}^a{}_b + W_b \mathbf{e}^a - W^a \eta_{bc} \mathbf{e}^c \\ \boldsymbol{\mathcal{R}}^a{}_b &= \mathbf{R}^a{}_b - \left(\delta^a_d \delta^c_b - \eta^{ac}\eta_{bd}\right)\left(\mathbf{D}W_c + W_c \boldsymbol{\omega}\right) \wedge \mathbf{e}^d \end{aligned}$$

These extra contributions make both fields invariant under dilatations,

$$\begin{aligned} \tilde{\boldsymbol{\omega}} &= \boldsymbol{\omega} + \mathbf{d}\phi \\ \mathbf{e}^a &= e^\phi \mathbf{e}^a \end{aligned}$$

Moreover, since the Weyl geometry is integrable there is a choice of e^ϕ with vanishing Weyl vector, restoring the Riemannian form $\mathbf{R}^a{}_b$.

Dilatations induce reparameterizations of curves. Concretely, the line element rescales as

$$d\tilde{s}^2 = e^{2\phi} ds^2$$

so that along a curve $x^\alpha(\tau)$ a tangent vector rescales to

$$\tilde{u}^a = e_\alpha{}^a \frac{dx^\alpha}{d\tau} \to e^\phi u^a$$

If we set $f = \frac{d\lambda}{d\tau} = e^{\phi}$ this is equivalent to a reparameterization

$$\lambda(\tau) = \int_0^\tau e^{\phi} d\tau$$

By realizing $f(\tau) = \frac{d\lambda}{d\tau}$ as a function $f(x)$, the reparameterization of all geodesics is equivalent to a rescaling of the spacetime metric by $e^{2\phi}$, where $\boldsymbol{\xi} = \mathbf{d}\ln f = \mathbf{d}\ln e^{\phi} = \mathbf{d}\phi$.

3.4 Projective symmetry in (p,q) gauge theory

The principal difference between general relativity and Riemann-Cartan geometry is the presence of torsion. Substituting a projective transformation (17) into the equations for the curvature and torsion, the curvature is unchanged but the torsion is altered.

$$\begin{aligned}
\tilde{\boldsymbol{\mathcal{R}}}^a{}_b &= \boldsymbol{\mathcal{R}}^a{}_b \\
\tilde{\mathbf{T}}^a &= \mathbf{d}\mathbf{e}^a - \mathbf{e}^b \wedge \boldsymbol{\omega}^a{}_b - \mathbf{e}^a \wedge \boldsymbol{\xi} \\
&= \mathbf{T}^a - \mathbf{e}^a \wedge \boldsymbol{\xi}
\end{aligned}$$

Just as we found for general relativity, the nonmetricity changes as well

$$\tilde{\mathbf{Q}}_{ab} = \mathbf{Q}_{ab} - 2\eta_{ab}\boldsymbol{\xi} \tag{21}$$

Minimal compatibility, replacing $\boldsymbol{\xi} = -\boldsymbol{\omega}$ to form a Weyl geometry, again leads to simultaneously projective and local Lorentz invariant forms.

However, there is a nonminimal approach that becomes evident when we include both nonmetricity and torsion from the start. Before defining nonminimal compatibility, we explicitly include nonmetricity in (p, q) gauge theory.

4 Nonminimal compatibility

Projective transformations alter both the torsion and the nonmetricity of (p, q) gauge theory, and we showed how minimal compatibility restores invariance of the torsion and curvature while eliminating the nonmetricity. Next, we revise the description of the geometry given in Section (2), explicitly, but in a restricted way, by including nonmetricity[1]. Since nonmetricity is a tensor under local Lorentz transformations we may introduce it while ultimately requiring no modification of the $\mathcal{P}/\mathcal{L}$ fiber bundle.

4.1 Revisiting the Poincarè structure equations

Once again carrying out the Cartan procedure described in Appendix A we develop a principal fiber bundle with $SO(p, q)$ symmetry. This time, we drop the assumption of metric compatibility and elevate the metric compatibility condition to the level of the structure equations.

$$\begin{aligned}
\mathbf{d}\tilde{\boldsymbol{\omega}}^a{}_b &= \tilde{\boldsymbol{\omega}}^c{}_b \wedge \tilde{\boldsymbol{\omega}}^a{}_c \\
\mathbf{d}\tilde{\mathbf{e}}^a &= \tilde{\mathbf{e}}^b \wedge \tilde{\boldsymbol{\omega}}^a{}_b \\
\mathbf{d}\eta_{ab} &= \eta_{cb}\tilde{\boldsymbol{\omega}}^c{}_a + \eta_{ac}\tilde{\boldsymbol{\omega}}^c{}_b
\end{aligned}$$

When we modify the solder form and the spin connection $\left(\tilde{\mathbf{e}}^b, \tilde{\boldsymbol{\omega}}^a{}_b\right) \to \left(\mathbf{e}^b, \boldsymbol{\omega}^a{}_b\right)$, Eqs. (3) and (4) are augmented by a third tensor field, the 1-form nonmetricity $\mathbf{Q}^{ab}$. The presence of $\mathbf{Q}^{ab}$ modifies the curvature $\boldsymbol{\mathcal{R}}^a{}_b$ and torsion $\mathbf{T}^a$, through its effect on the spin connection.

[1] We stress that our approach is *not* that of metric-affine gravity [37], $f(R)$ gravity or any of several other alternative gravity theories based on a $GL(n)$ connection. We retain Lorentz structure.

$$\begin{aligned}\mathbf{d}\boldsymbol{\omega}^a{}_b &= \boldsymbol{\omega}^c{}_b \wedge \boldsymbol{\omega}^a{}_c + \boldsymbol{\mathcal{R}}^a{}_b &(22)\\ \mathbf{d}\mathbf{e}^a &= \mathbf{e}^b \wedge \boldsymbol{\omega}^a{}_b + \mathbf{T}^a &(23)\\ \mathbf{d}\eta_{ab} &= \eta_{cb}\boldsymbol{\omega}^c{}_a + \eta_{ac}\boldsymbol{\omega}^c{}_b + \mathbf{Q}_{ab} &(24)\end{aligned}$$

Each new Lorentz tensor is horizontal

$$\begin{aligned}\boldsymbol{\mathcal{R}}^a{}_b &= \frac{1}{2}\mathcal{R}^a{}_{bcd}\mathbf{e}^c \wedge \mathbf{e}^d\\ \mathbf{T}^a &= \frac{1}{2}T^a{}_{bc}\mathbf{e}^b \wedge \mathbf{e}^c\\ \mathbf{Q}_{ab} &= Q_{abc}\mathbf{e}^c\end{aligned}$$

to preserve the local $SO(p,q)$ symmetry.

The Bianchi identities now take the form

$$\begin{aligned}\boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{R}}^a{}_b &= 0\\ \boldsymbol{\mathcal{D}}\mathbf{T}^a &= \mathbf{e}^b \wedge \boldsymbol{\mathcal{R}}^a{}_b\\ \boldsymbol{\mathcal{D}}\mathbf{Q}_{ab} &= -\boldsymbol{\mathcal{R}}_{ab} - \boldsymbol{\mathcal{R}}_{ba}\end{aligned}$$

with the covariant derivatives defined by

$$\begin{aligned}\boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{R}}^a{}_b &= \mathbf{d}\boldsymbol{\mathcal{R}}^a{}_b + \boldsymbol{\mathcal{R}}^c{}_b \wedge \boldsymbol{\omega}^a{}_c - \boldsymbol{\omega}^c{}_b \wedge \boldsymbol{\mathcal{R}}^a{}_c\\ \boldsymbol{\mathcal{D}}\mathbf{T}^a &= \mathbf{d}\mathbf{T}^a + \mathbf{T}^b \wedge \boldsymbol{\omega}^a{}_b\\ \boldsymbol{\mathcal{D}}\mathbf{Q}_{ab} &= \mathbf{d}\mathbf{Q}_{ab} + \mathbf{Q}_{cb} \wedge \boldsymbol{\omega}^c{}_a + \mathbf{Q}_{ac} \wedge \boldsymbol{\omega}^c{}_b\end{aligned}$$

The plus signs in the derivative of nonmetricity occur because $\mathbf{Q}_{ab}$ is a 1-form.

Equations (22)-(24) describe $SO(p,q)$ covariant tensors $\boldsymbol{\mathcal{R}}^a{}_b, \mathbf{T}^a, \mathbf{Q}_{ab}$. Unlike a full $GL(n)$ connection, the inhomogeneous part of local $SO(p,q)$ transformations of the connection is antisymmetric, hence an element of the Lorentz Lie algebra.

4.2 Solving for the connection

This form of the structure equations is sufficient to lead to the well-known explicit expressions for the connection and curvature. Starting from the Riemann-Cartan connection 12 we add a third term

$$\boldsymbol{\omega}^a{}_b = \boldsymbol{\alpha}^a{}_b + \mathbf{C}^a{}_b + \mathbf{E}^a{}_b \qquad (25)$$

which must satisfy both Eq.(23) and from constancy $\mathbf{d}\eta_{ab} = 0$ of the (p,q) metric

$$\mathbf{Q}_{ab} = -\boldsymbol{\omega}_{ab} - \boldsymbol{\omega}_{ba} = -\mathbf{E}_{ab} - \mathbf{E}_{ba}$$

The torsion equation (23) implies $\mathbf{e}^b \wedge \mathbf{E}^a{}_b = 0$ so the pair of conditions together require

$$\begin{aligned}E_{abc} + E_{bac} &= -Q_{abc}\\ E_{abc} - E_{acb} &= 0\end{aligned}$$

Cycling indices of the first and combining in the usual way $(++-)$ using the second we find

$$E_{abc} = -\frac{1}{2}\left(Q_{abc} + Q_{cab} - Q_{bca}\right) \qquad (26)$$

with the connection given by Eq.(25). We note that $E_{abc} = E_{acb}$, and this insures that $\mathbf{e}^b \wedge \mathbf{E}^a{}_b = 0$. There is no change in the fiber bundle structure.

4.3 Nonminimal compatibility

4.3.1 Hints at a symmetry

A certain symmetry between torsion and nonmetricity is occasionally noted. This generally stems from an ambiguity in the solder form structure equation in Weyl geometry with torsion.

$$\mathbf{d}\mathbf{e}^a = \mathbf{e}^b \wedge \boldsymbol{\omega}^a{}_b + \boldsymbol{\omega} \wedge \mathbf{e}^a + \mathbf{T}^a \tag{27}$$

While the Weyl vector was first introduced as a form of nonmetricity

$$\mathbf{D}g_{\alpha\beta} = \boldsymbol{\omega} g_{\alpha\beta}$$

the extra term $\boldsymbol{\omega} \wedge \mathbf{e}^a$ in Eq.(27) can also be absorbed into the torsion

$$\tilde{\mathbf{T}}^a = \mathbf{T}^a + \boldsymbol{\omega} \wedge \mathbf{e}^a$$

Of course, since dilatational gauging induces projective transformations, this duality is seen for projective transformations as well (for a recent example, see [38]).

This dual role is also evident in the basis dependence of torsion and nonmetricity.

The torsion is algebraic in a coordinate basis. The inhomogeneous change in the Christoffel connection under diffeomorphisms is symmetric, so any antisymmetric part of the connection is a tensor

$$T^\alpha{}_{\mu\nu} = \Gamma^\alpha{}_{\nu\mu} - \Gamma^\alpha{}_{\mu\nu}$$

The nonmetricity in the same basis is differential, given by the covariant derivative of the metric

$$Q_{\alpha\beta\mu} = D_\mu g_{\alpha\beta}$$

This situation is reversed in an orthonormal basis, $\mathbf{e}^a$. The torsion becomes the covariant exterior derivative of the solder form

$$\mathbf{T}^a = \mathbf{D}\mathbf{e}^a$$

while the nonmetricity is algebraic

$$\mathbf{Q}_{ab} = \mathbf{D}\eta_{ab} = -\boldsymbol{\omega}_{ab} - \boldsymbol{\omega}_{ba}$$

Since an infinitesimal local Lorentz transformation is antisymmetric, the inhomogeneous change in the spin connection is antisymmetric and the symmetric part is a tensor. Torsion and nonmetricity have exchanged roles.

4.3.2 A new projective invariant

We find that the simplicity of these somewhat vague observations stems from a much broader overlap between nonmetricity and torsion. Under projective transformation (17) the tensors of Eqs.(22)-(24) become

$$\tilde{\boldsymbol{\mathcal{R}}}^a{}_b = \boldsymbol{\mathcal{R}}^a{}_b \tag{28}$$

$$\tilde{\mathbf{T}}^a = \mathbf{T}^a - \mathbf{e}^a \wedge \boldsymbol{\xi} \tag{29}$$

$$\tilde{\mathbf{Q}}_{ab} = \mathbf{Q}_{ab} - 2\eta_{ab}\boldsymbol{\xi} \tag{30}$$

The changes produced by projective transformations in both torsion (29) and nonmetricity (30) allow us to construct a projectively invariant tensor. The irreducible parts of nonmetricity are totally symmetric $Q_{(abc)}$ and mixed symmetry $Q_{a[bc]}$. The vector space spanned by $Q_{a[bc]}$ includes the non-totally-symmetric pieces $Q_{a(bc)} - Q_{(abc)}$. We separate the mixed symmetry part by writing the 2-form

$$\begin{aligned}\mathbf{Q}^a &\equiv \frac{1}{2}\mathbf{e}^c \wedge \mathbf{Q}^a{}_c \\ &= \frac{1}{2}\eta^{ab}\mathbf{e}^c \wedge \mathbf{Q}_{bc}\end{aligned}$$

From Eq.(30) transforms as

$$\tilde{\mathbf{Q}}^a = \mathbf{Q}^a - \mathbf{e}^a \wedge \boldsymbol{\xi}$$

It follows that the combination

$$\mathbf{S}^a \equiv \mathbf{T}^a - \mathbf{Q}^a \tag{31}$$

is projectively invariant.

4.4 The spin connection in terms of $\mathbf{S}^a$

The component expansion

$$S_{abc} = T_{abc} - \frac{1}{2}(Q_{abc} - Q_{acb})$$

allows us to solve for the torsion in terms of S_{abc} and the mixed symmetry part of Q_{abc}. Substituting to eliminate the torsion

$$\begin{aligned}
\mathbf{C}_{ab} + \mathbf{E}_{ab} &= \frac{1}{2}(T_{cab} + T_{bac} - T_{abc})\,\mathbf{e}^c - \frac{1}{2}(Q_{abc} + Q_{cab} - Q_{bca})\,\mathbf{e}^c \\
&= \frac{1}{2}\left(S_{cab} + \frac{1}{2}(Q_{cab} - Q_{cba}) + S_{bac} + \frac{1}{2}(Q_{bac} - Q_{bca}) - S_{abc} - \frac{1}{2}(Q_{abc} - Q_{acb})\right)\mathbf{e}^c \\
&\quad -\frac{1}{2}(Q_{abc} + Q_{cab} - Q_{bca})\,\mathbf{e}^c \\
&= \frac{1}{2}(S_{cab} + S_{bac} - S_{abc})\,\mathbf{e}^c - \frac{1}{2}Q_{abc}\mathbf{e}^c
\end{aligned}$$

where somewhat surprisingly all but one of the nonmetricity terms cancel. Define

$$\mathbf{C}^{(S)}_{ab} \equiv \frac{1}{2}(S_{cab} + S_{bac} - S_{abc})\,\mathbf{e}^c \tag{32}$$

This is the contorsion tensor of S_{cab}. Finally, with

$$\begin{aligned}
\boldsymbol{\omega}^a{}_b &= \boldsymbol{\alpha}^a{}_b + \mathbf{C}^a{}_b + \mathbf{E}^a{}_b \\
&= \boldsymbol{\alpha}^a{}_b + \mathbf{C}^{(S)a}{}_b - \frac{1}{2}Q^a{}_{bc}\mathbf{e}^c
\end{aligned} \tag{33}$$

the structure equation Eq.(23) becomes

$$\begin{aligned}
\mathbf{d}\mathbf{e}^a &= \mathbf{e}^b \wedge \boldsymbol{\alpha}^a{}_b + \mathbf{e}^b \wedge \mathbf{C}^{(S)a}{}_b + \mathbf{T}^a - \frac{1}{2}\mathbf{e}^b \wedge \mathbf{Q}_{ab} \\
&= \mathbf{e}^b \wedge \boldsymbol{\alpha}^a{}_b + \mathbf{e}^b \wedge \mathbf{C}^{(S)a}{}_b + \mathbf{S}^a
\end{aligned}$$

The connection from this form of the structure equation will now be

$$\boldsymbol{\omega}^a_{(S)\ b} = \boldsymbol{\alpha}^a{}_b + \mathbf{C}^{(S)a}{}_b \tag{34}$$

and we recover the form of the original (p,q) structure equations with the projectively invariant s-torsion

$$\mathbf{d}\mathbf{e}^a = \mathbf{e}^b \wedge \boldsymbol{\omega}^a_{(S)\ b} + \mathbf{S}^a \tag{35}$$

Comparing the original connection in Eq.(33) to $\boldsymbol{\omega}^{a}_{(S)\ b}$ in Eq.(34),

$$\begin{aligned}\boldsymbol{\omega}^{(S)}_{ab} &= \boldsymbol{\omega}_{ab}+\frac{1}{2}Q^{a}{}_{bc}\mathbf{e}^{c}\\ &= \frac{1}{2}\left(\boldsymbol{\omega}_{ab}-\boldsymbol{\omega}_{ba}\right)\end{aligned}$$

so that $\boldsymbol{\omega}^{(S)}_{ab}$ is simply the antisymmetric part of the original connection. When the original connection changes by a projective transformation $\tilde{\boldsymbol{\omega}}_{ab}=\boldsymbol{\omega}_{ab}+\eta_{ab}\boldsymbol{\xi}$ the new connection is unchanged.

$$\begin{aligned}\tilde{\boldsymbol{\omega}}^{(S)}_{ab} &= \tilde{\boldsymbol{\omega}}_{ab}+\frac{1}{2}\tilde{Q}^{a}{}_{bc}\mathbf{e}^{c}\\ &= \boldsymbol{\omega}_{ab}+\eta_{ab}\boldsymbol{\xi}+\frac{1}{2}\left(Q_{abc}-2\eta_{ab}\boldsymbol{\xi}\right)\mathbf{e}^{c}\\ &= \boldsymbol{\omega}^{(S)}_{ab}\end{aligned}$$

Therefore the s-nonmetricity vanishes $Q^{(S)}_{ab}=-\boldsymbol{\omega}^{(S)}_{ab}-\boldsymbol{\omega}^{(S)}_{ba}=0$, and $\boldsymbol{\omega}_{(S)ab}=-\boldsymbol{\omega}_{(S)ba}$ is an $SO\left(p,q\right)$ connection. We have returned to the usual form of the (p,q) or Poincarè structure equations, but now with both manifest projective invariance and local $SO\left(p,q\right)$ invariance.

$$\begin{aligned}\mathbf{d}\boldsymbol{\omega}^{a}{}_{b} &= \boldsymbol{\omega}^{\ c}_{(S)\ b}\wedge\boldsymbol{\omega}^{\ a}_{(S)\ c}+\boldsymbol{\mathcal{R}}^{a}{}_{b}\\ \mathbf{d}\mathbf{e}^{a} &= \mathbf{e}^{b}\wedge\boldsymbol{\omega}^{\ a}_{(S)\ b}+\mathbf{S}^{a}\\ \mathbf{d}\eta_{ab} &= \eta_{cb}\boldsymbol{\omega}^{\ c}_{(S)\ a}+\eta_{ac}\boldsymbol{\omega}^{\ c}_{(S)\ b}\end{aligned}$$

The complete merging of the mixed symmetry subspace of the nonmetricity tensor $\mathbf{Q}^{a}$ with the torsion is a much stronger relationship than simple overlap for projective symmetry or dilatations. In the next Section we show that we may fully rotate $\mathbf{Q}^{a}$ and $\mathbf{T}^{a}$ into one another without changing the revised structure equation, Eq.(35). Regardless of the values of $\mathbf{Q}^{a}$ and $\mathbf{T}^{a}$ separately, their combination into $\mathbf{S}^{a}$ gives a metric compatible connection.

In a straightforward yet nonminimal way, we have generalized the form of the Cartan equations of the Poincarè group to produce manifest invariance under both Lorentz and projective transformations while reproducing the usual form of the Poincarè structure equations.

4.5 Why does this work?

The surprising reduction of mixed nonmetricity and torsion into the single, torsion-like tensor $\mathbf{S}^{a}$ forces us to as whether there is some deeper symmetry at work. This appears to be the case. We added Eq.(24) to include the nonmetricity from the start but it does not have the form of the other structure equations. It is natural to ask whether the new 2-form $\mathbf{Q}_{a}$ arises from some symmetry.

We wedge with $\frac{1}{2}\mathbf{e}^{b}$ into Eq.(24) to form an equation for $\mathbf{Q}_{a}$ alone.

$$\begin{aligned}\frac{1}{2}\mathbf{e}^{b}\wedge\mathbf{d}\eta_{ab} &= \eta_{cb}\frac{1}{2}\mathbf{e}^{b}\wedge\boldsymbol{\omega}^{c}{}_{a}+\eta_{ac}\frac{1}{2}\mathbf{e}^{b}\wedge\boldsymbol{\omega}^{c}{}_{b}+\frac{1}{2}\mathbf{e}^{b}\wedge\mathbf{Q}_{ab}\\ -\mathbf{d}\left(\frac{1}{2}\eta_{ab}\mathbf{e}^{b}\right)+\frac{1}{2}\eta_{ab}\mathbf{d}\mathbf{e}^{b} &= \frac{1}{2}\eta_{cb}\mathbf{e}^{b}\wedge\boldsymbol{\omega}^{c}{}_{a}+\frac{1}{2}\eta_{ac}\mathbf{e}^{b}\wedge\boldsymbol{\omega}^{c}{}_{b}+\mathbf{Q}_{a}\end{aligned}$$

Using the Eq.(23) to replace $\mathbf{d}\mathbf{e}^{b}$ and rearanging, this becomes

$$\mathbf{d}\left(\frac{1}{2}\eta_{ab}\mathbf{e}^{b}\right) = \boldsymbol{\omega}^{c}{}_{a}\wedge\left(\frac{1}{2}\eta_{cb}\mathbf{e}^{b}\right)+\frac{1}{2}\eta_{ab}\mathbf{T}^{b}-\mathbf{Q}_{a}$$

Defining $\mathbf{f}_{a}\equiv\frac{1}{2}\eta_{ab}\mathbf{e}^{b}$ and $\mathbf{U}_{a}\equiv\frac{1}{2}\eta_{ab}\mathbf{T}^{b}+\mathbf{Q}_{a}$ this takes the simple form

$$\mathbf{d}\mathbf{f}_{a}=\boldsymbol{\omega}^{d}{}_{a}\wedge\mathbf{f}_{d}+\mathbf{U}_{a} \tag{36}$$

This is recognizable as the Cartan structure equation of special conformal transformations, which like dilatations induce a reparameterization on curves. A moment's reflection reveals the necessity for a transformation that reparameterizes curves to be related to nonmetricity.

This suggests an alternative decomposition of a general connection. Rather than separating the connection into compatible, torsion, and nonmetricity parts, we might consider irreducible representations and the corresponding vector spaces. Viewed in this way, Young tableau reduce the n^3 degrees of freedom of a general connection into four irreducible subsets. The two mixed symmetry subsets form bases for the same vector space, so the general connection is spanned by three vector subspaces.

- V_A, the $\frac{1}{6}n(n-1)(n-2)$ dimensional vector space of the totally antisymmetric part of the connection
- V_M, a single vector space of dimension $\frac{1}{3}n\left(n^2-1\right)$ formed with either set of mixed symmetry components as basis
- V_S, the $\frac{1}{6}n(n+1)(n+2)$ dimensional totally totally symmetric part.

This breakdown further suggests repeating the present work with a top-down approach starting with the auxiliary [39] or biconformal gauging [40, 41, 42] of the conformal group. This investigation is in progress [43].

5 Further invariance of $\mathbf{S}^a$

Projective symmetry is not the only invariance of nonminimal compatibility. Having identified $\mathbf{S}^a$ as the sum of two terms it becomes possible to introduce a larger symmetry. Since the tensors $\mathbf{T}^a, \mathbf{Q}^a, \mathbf{S}^a$ lie within the vector space $\mathcal{A}^1_{[2]}$ of vector-valued 2-forms, may consider rotations within that subspace that leave $\mathbf{S}^a$ invariant while mixing $\mathbf{T}^a$ and $\mathbf{Q}^a$. Clearly, these will be rotations about $\mathbf{S}^a \in \mathcal{A}^1_{[2]}$.

The vector space of vector-valued 2-forms is large, and any linear transformation ϕ that maps

$$\begin{aligned}
\phi &: \mathbf{T}^a \rightarrow \tilde{\mathbf{T}}^a \\
\phi &: \mathbf{Q}^a = \frac{1}{2}\mathbf{e}^b \wedge \mathbf{Q}^a{}_b \rightarrow \tilde{\mathbf{Q}}^a \\
\phi &: \mathbf{S}^a \rightarrow \mathbf{S}^a
\end{aligned}$$

also preserves the connection $\boldsymbol{\omega}^{\ ab}_{(S)}$. Any such transformation ϕ therefore preserves the revised Poincarè structure equations and results in an internal symmetry of general relativistic spacetimes.

Let $\Sigma : \mathcal{A}^1_{[2]} \leftrightarrow \mathcal{V}$: be any convenient 1-1 onto linear mapping, $\mathcal{V} = \left\{V^A = \Sigma^A{}_a{}^{bc} V^a{}_{bc}, A = 1, \ldots, N\right\}$ with norm g induced from the underlying (p,q) metric η_{ab}.

$$g_{AB} \equiv \eta_{ad}\eta_{be}\eta_{cf}\Sigma_A{}^{abc}\Sigma_B{}^{def}$$

The signature (P,Q) of g_{AB} is determined by (p,q). Note that $\dim \mathcal{V} = \frac{1}{2}n^2(n-1)$.

There are three constraints on $\mathbf{S}^a$-preserving mappings ϕ of vectors $V^A \in \mathcal{V}$.

1. ϕ must preserve the the (P,Q) metric g_{AB} induced by the underlying (p,q) metric.

$$\left|V^A\right|^2 = g_{AB}V^A V^B = \frac{1}{2}\eta_{ab}\left(\eta^{ce}\eta^{df} - \eta^{cf}\eta^{de}\right)V^a{}_{cd}V^b{}_{ef}$$

2. ϕ must affect only the mixed symmetry part of V_{abc}. The S-torsion decomposes into independent mixed and totally antisymmetric parts either of which may or may not enter the field equations. It is well-known [18, 19, 20, 21, 22, 23, 24] that the antisymmetric part of the contorsion $C^{(S)}_{[cab]} = -\frac{1}{2}T_{[cab]}$ is driven by Dirac fields. The transformation ϕ must not affect this part. Couplings to fields such as the spin-$\frac{3}{2}$ Rarita-Schwinger field–which couples strongly to torsion [27]–may need modification to be compatible with ϕ.

3. ϕ must preserve $\mathbf{S}^a$.

In Appendix 7 we find the dependence of P and Q on (p,q), and the reduction of the full $SO(P,Q)$ which preserves $\mathbf{S}^a$ and the mixed symmetry subspace.

We find that the proper rotation group preserving the metric on $\mathcal{V}$ is

$$SO\left(\frac{1}{2}\left(p^2(p-1)+pq(3q-1)\right),\frac{1}{2}\left(q^2(q-1)+pq(3p-1)\right)\right)$$

In 4-dim $(3,1)$ spacetime this is the split orthogonal form

$$SO(12,12)$$

This split form $P=Q$ (reminiscent of Kähler, biconformal, and double field manifolds) occurs if and only if $s=0$ or $n=s^2$.

Eliminating totally antisymmetric combinations to affect only the mixed symmetry subspace reduces this group to

$$SO\left(\frac{1}{3}p\left(p^2+3q^2-1\right),\frac{1}{3}q\left(q^2+3p^2-1\right)\right)$$

Finally, holding $\mathbf{S}^a$ constant reduces the total dimension N by one. The resulting symmetry group depends on whether $\mathbf{S}^a$ is timelike or spacelike. The group is either:

$$SO\left(\frac{1}{3}p\left(p^2+3q^2-1\right)-1,\frac{1}{3}q\left(q^2+3p^2-1\right)\right)$$

or

$$SO\left(\frac{1}{3}p\left(p^2+3q^2-1\right),\frac{1}{3}q\left(q^2+3p^2-1\right)-1\right)$$

In $(3,1)$ spacetime the two possibilities are

$$SO(10,9),SO(11,8)$$

In either 4-dimensional case the internal symmetry is large enough to contain the Standard Model, with spacelike $\mathbf{S}^a$ leading directly to the $SO(10)$ of grand unification. Since the internal symmetry leaves $\mathbf{S}^a$ unchanged, gravity has decoupled and the Coleman-Mandula theorem is satisfied.

5.1 2-form subgroup

Because $\mathcal{V}$ has the internal structure of $\mathcal{A}^1_{[2]}$, there are some natural subgroups. For example, it may be useful to transform the vector and 2-form characters of $\mathcal{V}$ separately. The vector part of the space is, of course, n-dimensional with signature (p,q). For the 2-forms we have three cases with multiplicities

$$\begin{array}{cc} \omega_{a_1a_2} & \frac{1}{2}p(p-1) \\ \omega_{a_1b_2} & pq \\ \omega_{b_1b_2} & \frac{1}{2}q(q-1) \end{array}$$

leading to signature

$$(P_2,Q_2) = \left(\frac{1}{2}p(p-1)+\frac{1}{2}q(q-1),pq\right)$$

For $n>3$ the antisymmetry constriant imposes more than n restrictions, which therefore cannot be implemented within the vector part alone, so the (P_2,Q_2) symmetry will reduce further.

The 4-dim case permits an interesting conjecture. There are 2 positive norm and 2 negative norm antisymmetry constraints. Three of these can be imposed on the vector part of the full $A^1_{[2]}$ symmetry. Using the spinor representation $SU(2) \times SU(2)$ of the 2-form subgroup $SO(P_2, Q_2) = SO(3,3)$ one constraint remains. This must break one of the $SU(2)$ subgroups, forcing a reduction of an initially left-right symmetric electroweak model to the actual $SU(2) \times U(1)$.

5.2 Restrictions on Q

For $\mathbf{S}_a$ to be fully general we must have arbitrary $\mathbf{Q}_a = \frac{1}{2}\mathbf{e}^b \wedge \mathbf{Q}_{ab} = \frac{1}{2}Q_{a[bc]}\mathbf{e}^b \wedge \mathbf{e}^c$. We ask whether nonmetricities of the form $Q_{a[bc]}$ form an invariant vector subspace. The Young tableaux for $\mathcal{A}_{(2)1}$ tensors includes a totally symmetric part and a mixed symmetry part

$$n \otimes \left(\frac{n(n+1)}{2}\right) = \frac{1}{6}n(n+1)(n+2) \oplus \frac{1}{3}n\left(n^2-1\right)$$

This means that the partially symmetrized piece $Q_{a(bc)} - Q_{(abc)}$ must be dependent upon $Q_{a[bc]}$. Checking by adding and subtracting from a sum of two vectors we find

$$\frac{1}{3}\left(Q_{b[ac]} + Q_{c[ab]}\right) = Q_{a(bc)} - Q_{(abc)}$$

Therefore, the vector subspaces $\left\{Q_{a[bc]}\right\}$ and $\left\{Q_{(abc)}\right\}$ are disjoint and span the full space of nonmetricities. Since the most general form of S_{abc} requires general $\tilde{Q}_{a[bc]}$ no further reduction possible. The necessary and sufficient condition we seek is to transform all nonmetricities $\tilde{Q}_{abc}$ with vanishing totally symmetric part $Q_{(abc)} = 0$. Neither the totally symmetric part of Q_{abc} nor the totally antisymmetric part of T_{abc} will be altered by the internal symmetry.

6 The action

The inclusion of torsion and nonmetricity in the description of gravity motivates a fresh look at the form of the action.

One motivation for choosing the Einstein-Hilbert action (beyond, of course, that it works spectacularly) is that when we include the cosmological constant it is the most general action with field equations of no more than second order in derivatives of the metric, and linear in those second derivatives. With torsion and nonmetricity the Einstein-Hilbert action is no longer the most general action satisfying these conditions. These tensors depend only on first derivatives of the metric, and under these criteria may enter quadratically.

In addition to the Einstein-Hilbert term, we consider the most general action up to quadratic order built from the torsion and non-metricity as well as the 2- and 3-forms

$$\mathbf{T} = \frac{2}{3}\mathbf{e}^a \wedge \mathbf{T}_a$$
$$\mathbf{Q}_a = \frac{1}{2}\mathbf{e}^b \wedge \mathbf{Q}_{ab}$$

It is therefore natural to include the general quadratic combination[2]

$$S_{Q,T} = \int \left(\alpha \mathbf{Q}_{ab} \wedge {}^*\mathbf{Q}^{ab} + \beta \mathbf{Q}_a \wedge {}^*\mathbf{Q}^a + \mu \mathbf{T}_a \wedge {}^*\mathbf{T}^a + \nu \mathbf{Q}_a \wedge {}^*\mathbf{T}^a + \rho \mathbf{T} \wedge {}^*\mathbf{T}\right) \tag{37}$$

[2]If we allow combinations involving the three traces $S_a = S^b{}_{ba}, Q_a \equiv Q^b{}_{ba}, \bar{Q}_a \equiv Q_{ab}{}^b$ there are two projectively invariant scalars. We may write the most general projectively invariant action quadratic in the scalars as

$$S_{additional} = \int \left(\mu S_a S^a + \nu S^3_a S^{3a}\right) \mathbf{\Phi}$$

where $S^3_c \equiv Q_c - n\bar{Q}_c$.

It is interesting to note that there is now coupling between the torsion and nonmetricity.

The form may now be restricted by requiring projective invariance of $S_{Q,T}$. Substituting the projective changes of the torsion and nonmetricity given in Eqs.(29) and (30), together with

$$\begin{aligned} \tilde{\mathbf{T}} &= \mathbf{T} \\ \tilde{\mathbf{Q}}_a &= \mathbf{Q}_a - \eta_{ab}\mathbf{e}^b \wedge \boldsymbol{\xi} \end{aligned}$$

into Eq.(37) and collecting terms, $S_{Q,T}$ is invariant if and only if $\alpha = 0, \beta = -\frac{\nu}{2}, \mu = -\frac{\nu}{2}$, while ρ remains arbitrary. Including these values the most general action up to linear in second derivatives built from $\mathbf{T}^a, \mathbf{T}, \mathbf{Q}_{ab}$ and $\mathbf{Q}_a$ is

$$\begin{aligned} S_{Q,T} &= \int \beta \left(\mathbf{T}_a - \mathbf{Q}_a\right) \wedge {}^* \left(\mathbf{T}^a - \mathbf{Q}^a\right) + \rho \int \mathbf{T} \wedge {}^*\mathbf{T} \\ &= \beta \int \mathbf{S}_a \wedge {}^*\mathbf{S}^a + \rho \int \mathbf{T} \wedge {}^*\mathbf{T} \end{aligned}$$

Since $\mathbf{T} = \frac{2}{3}\mathbf{e}^a \wedge \mathbf{T}_a = \frac{2}{3}\mathbf{e}^a \wedge \mathbf{S}_a \equiv \mathbf{S}$ we may write the full gravitational action as

$$\begin{aligned} S_S\left[\mathbf{e}^a, \boldsymbol{\omega}_{(S)\ \ b}^{\ \ a}\right] &= \frac{\kappa}{(n-2)!} \int \boldsymbol{\mathcal{R}}^{ab} \wedge \mathbf{e}^c \wedge \ldots \wedge \mathbf{e}^d e_{abc\ldots d} \\ &\quad + \beta \int \mathbf{S}_a \wedge {}^*\mathbf{S}^a + \rho \int \mathbf{S} \wedge {}^*\mathbf{S} \end{aligned}$$

The functional S_S is Lorentz, projective, and $SO(P-1,Q)$ (or $SO(P,Q-1)$) invariant.

Now, in addition to the usual sources we may ask that matter fields be representations of the internal symmetry, for example, spinor fields ψ^A transforming under the internal rotations, e.g. $Spin(P-1,Q)$. Then after gauging we may add

$$S_{Matter}\left[\mathbf{A}^B, \psi^C\right] = \int \alpha g_{AB} \bar{\psi}^A \left(i\gamma^a D_a - m\right)\psi^B \boldsymbol{\Phi} + \frac{1}{4}\lambda g_{AB}\mathbf{F}^A \wedge^* \mathbf{F}^B$$

where $\mathbf{F}^B = \mathbf{d}\mathbf{A}^B + \frac{1}{2}c^B_{\ \ CD}\mathbf{A}^C \wedge \mathbf{A}^D$. Even in 4-dimensions either the $SO(10,9)$ or $SO(11,8)$ symmetry is large enough to describe the known interactions.

We vary $S_S\left[\mathbf{e}^a, \boldsymbol{\omega}_{(S)\ \ b}^{\ \ a}\right]$ a la Palatini. We may vary the antisymmetric and symmetric parts of $S_S\left[\mathbf{e}^a, \boldsymbol{\omega}^a_{\ \ b}, \mathbf{Q}^a\right]$ independently, with the symmetric variation is equivalent to varying $\mathbf{Q}_{ab}$. Alternatively, we may disregard the symmetric part altogether since the added structure equation

$$\cancel{\mathbf{d}\eta^{ab}} = -\eta^{cb}\boldsymbol{\omega}_{(S)\ \ c}^{\ \ a} - \eta^{ac}\boldsymbol{\omega}_{(S)\ \ c}^{\ \ b} + \cancel{\mathbf{Q}^{ab}_{(S)}}$$

now implies metric compatibility while the remaining structure equations

$$\begin{aligned} \mathbf{d}\boldsymbol{\omega}_{(S)\ \ b}^{\ \ a} &= \boldsymbol{\omega}_{(S)\ \ b}^{\ \ c} \wedge \boldsymbol{\omega}_{(S)\ \ c}^{\ \ a} + \boldsymbol{\mathcal{R}}^a_{(C)\ \ b} \\ \mathbf{d}\mathbf{e}^a &= \mathbf{e}^b \wedge \boldsymbol{\omega}_{(S)\ \ b}^{\ \ a} + \mathbf{S}^a \end{aligned}$$

have returned to the Cartan equations of the Poincarè group.

While the rotations of $A^1_{[2]}$ explicitly exclude the totally antisymmetric part of the torsion, $\mathbf{S}$, it must be included because Dirac fields provide a source for it.

7 Conclusions

We find a large symmetry within Poincarè (or $ISO(p,q)$) gauge theory by explicitly allowing both torsion and nonmetricity. The resulting gravity theory is still a metric compatible Riemann-Cartan theory of gravity with the internal symmetry decoupled from gravity. We summarize the steps leading to this conclusion.

We begin with Poincarè-type gauge theory. Poincarè gauge theory gives a natural arena for several developments in the theory of gravity. When the torsion is constrained to zero, it provides a gauge theory of general relativity, and an arena in which the Palatini variation is natural. Dropping the constraint on torsion but retaining the Einstein-Hilbert action gives the well-known ECSK theory of gravity. Even with the addition of a kinetic term for torsion the theory is consistent with experiment in certain scenarios. We gave a condensed description of these geometries in arbitrary dimension n and signature (p, q).

The next step is an examination of projective transformations. In affine geometries the connection possesses projective symmetry. This symmetry arises from reparameterizing autoparallels and preserves both the autoparallels and the curvature. While Poincarè gauge theory has both metric and connection, the Palatini variation makes the connection independent of the metric and we can again consider projective transformations of the connection. Here the situation is different, for while the Poincarè geodesics agree with affine autoparallels and the transformations still preserve both geodesics and the curvature, there are other structures–the torsion and nonmetricity tensors–which are not projectively invariant.

This sets up a conflict between the (p, q) gauge theory on one hand and our ability to reparameterize geodesics on the other. We reviewed the well-known Ehlers, Pirani, and Shild resolution to this dissonance. Extending to an integrable Weyl geometry absorbs reparameterizations in a manifestly local Lorentz and reparameterization invariant formalism. This is the minimal modification of the geometry to achieve the dual invariance.

To achieve manifest local Lorentz and projective invariance we extended Riemann-Cartan geometry by explicitly including nonmetricity. Within this geometry we defined a new 2-form tensor given by the difference of the torsion and the antisymmetric part of the nonmetricity $\mathbf{Q}_a \equiv \frac{1}{2}\mathbf{e}^b \wedge \mathbf{Q}_{ab}$.

$$\mathbf{S}^a = \mathbf{T}^a - \mathbf{Q}^a$$

This new s-torsion is projectively invariant and has the same vector-valued 2-form symmetry as the original torsion. Surprisingly, we find that the extended geometry may be written as a metric-compatible geometry with $\mathbf{S}^a$ replacing the torsion. In the new variable the structure equations reduce to their original Poincarè form even though the theory is formulated with a fully general connection. This provides a nonminimal means of including reparameterization invariance within Poincarè gauge theory. The new approach appears to be the result of using special conformal transformations rather than dilatations to absorb reparameterizations, but this remains to be confirmed.

The large internal symmetry arises when considering the space of vector-valued 2-forms, which contains $\mathbf{T}^a, \mathbf{Q}^a$ and $\mathbf{S}^a$. Simultaneous rotations of the torsion and nonmetricity can leave $\mathbf{S}^a$ and the connection–invariant. By using $\mathbf{S}^a$ as the axis of rotations, a subgroup of rotations preserving the induced metric decouple from gravity giving an internal symmetry in agreement with the Coleman-Mandula theorem. This internal symmetry depends only on the presence of $\mathbf{S}^a$, not on its magnitude or the strength of its couplings to other fields.

Precisely, within the extended geometry we may perform rotations in the vector space $v^A \in \left\{\mathcal{A}_{1[2]} - \mathcal{A}_{[3]}\right\}$ of mixed symmetry, co-vector-valued 2-forms, where $v^A = \Sigma^A{}_a{}^{bc} v^a{}_{[bc]}$ and $\Sigma^A{}_a{}^{bc}$ is a fixed bijection. The subgroup of rotations leaving $\mathbf{S}^a$ and the induced metric $g_{AB} = \Sigma_A{}^{abc} \Sigma_B{}^{def} \eta_{ad}\eta_{be}\eta_{cf}$ invariant leaves the connection invariant as well.

All calculations are valid in any dimension $n > 2$ and any signature (p, q). This induces a signature (P, Q) on g_{AB} and (P', Q') on the reduced rotation subgroup. We show that the resulting symmetry of the extended Poincarè theory is $SO(P', Q')$ or $Spin(P', Q')$ where

$$\begin{aligned} P' &= \frac{1}{3}p\left(p^2 + 3q^2 - 1\right) - 1 \\ Q' &= \frac{1}{3}q\left(q^2 + 3p^2 - 1\right) \end{aligned}$$

or

$$P' = \frac{1}{3}p\left(p^2 + 3q^2 - 1\right)$$

$$Q' = \frac{1}{3}q\left(q^2+3p^2-1\right)-1$$

depending on whether $\mathbf{S}^a$ is spacelike or timelike with respect to g.

In 4-dimensional spacetime $SO(12,12)$ preserves the induced metric, and the reduced internal symmetry group is $SO(10,9)$, $Spin(10,9)$, $SO(11,8)$ or $Spin(11,8)$. This opens the possibility of a fully unified theory without the need for higher dimensions.

The most general action up to linearity in second derivatives of the solder form contains six terms. These include combinations quadratic in torsion and nonmetricity as well as torsion-nonmetricity couplings, in addition to the Einstein-Hilbert action. Imposing projective invariance reduces this to three terms dependent on S and curvature only.

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Appendix A: Formal development of SO(p,q) geometry

We develop the arena for local $SO(p,q)$ symmetric physical models based on the unrestricted Cartan gauge theory of the Lie group $\mathcal{P} = ISO(p,q)$. Additional discussion may be found in [27].

Starting with the Maurer-Cartan equations of $\mathcal{P}$

$$\begin{aligned} \mathbf{d}\tilde{\boldsymbol{\omega}}^a{}_b &= \tilde{\boldsymbol{\omega}}^c{}_b \wedge \tilde{\boldsymbol{\omega}}^a{}_c \\ \mathbf{d}\tilde{\mathbf{e}}^a &= \tilde{\mathbf{e}}^b \wedge \tilde{\boldsymbol{\omega}}^a{}_b \end{aligned}$$

we take the quotient by the Lie subgroup $\mathcal{L} = SO(p,q)$. The projection from cosets $\mathcal{L}_g$ to the homogeneous quotient manifold $\mathcal{M}^n = \mathcal{P}/\mathcal{L}$ allows us to develop a principal fiber bundle with $SO(p,q)$ symmetry. By modifying the solder form and the spin connection 1-forms $\left(\tilde{\mathbf{e}}^b, \tilde{\boldsymbol{\omega}}^a{}_b\right) \rightarrow \left(\mathbf{e}^b, \boldsymbol{\omega}^a{}_b\right)$ we introduce a $\mathcal{P}$-covariant curvature 2-form with two $\mathcal{L}$-covariant components: the *curvature* $\boldsymbol{\mathcal{R}}^a{}_b$ and the *torsion* $\mathbf{T}^a$

$$\begin{aligned} \mathbf{d}\boldsymbol{\omega}^a{}_b &= \boldsymbol{\omega}^c{}_b \wedge \boldsymbol{\omega}^a{}_c + \boldsymbol{\mathcal{R}}^a{}_b \\ \mathbf{d}\mathbf{e}^a &= \mathbf{e}^b \wedge \boldsymbol{\omega}^a{}_b + \mathbf{T}^a \end{aligned} \tag{38}$$

We require the $\boldsymbol{\mathcal{R}}^a{}_b$ and $\mathbf{T}^a$ to be horizontal,

$$\begin{aligned} \boldsymbol{\mathcal{R}}^a{}_b &= \frac{1}{2}\mathcal{R}^a{}_{bcd}\mathbf{e}^c \wedge \mathbf{e}^d \\ \mathbf{T}^a &= \frac{1}{2}T^a{}_{bc}\mathbf{e}^b \wedge \mathbf{e}^c \end{aligned}$$

thereby preserving the bundle structure by making integrals of the connection independent of lifting. Integrability of the Cartan equations Eqs.(38) is insured by $\mathbf{d}^2\boldsymbol{\omega}^a{}_b \equiv 0$ and $\mathbf{d}^2\mathbf{e}^a \equiv 0$, which lead to the Bianchi identities,

$$\begin{aligned} \boldsymbol{\mathcal{D}}\mathbf{T}^a &= \mathbf{e}^b \wedge \boldsymbol{\mathcal{R}}^a{}_b \\ \boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{R}}^a{}_b &= 0 \end{aligned} \qquad (39)$$

The covariant exterior derivatives are given by

$$\begin{aligned} \boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{R}}^a{}_b &= \mathbf{d}\boldsymbol{\mathcal{R}}^a{}_b + \boldsymbol{\mathcal{R}}^c{}_b \wedge \boldsymbol{\omega}^a{}_c - \boldsymbol{\omega}^c{}_b \wedge \boldsymbol{\mathcal{R}}^a{}_c \\ \boldsymbol{\mathcal{D}}\mathbf{T}^a &= \mathbf{d}\mathbf{T}^a + \mathbf{T}^b \wedge \boldsymbol{\omega}^a{}_b \end{aligned}$$

The frame field $\mathbf{e}^a$ is taken (p,q)-orthonormal $\left\langle \mathbf{e}^a, \mathbf{e}^b \right\rangle = \eta^{ab} = diag\,(1,\ldots,1,-1,\ldots,-1)$ with the connection assumed to be metric compatible

$$\mathbf{d}\eta^{ab} + \eta^{cb}\boldsymbol{\omega}^a{}_c + \eta^{ac}\boldsymbol{\omega}^b{}_c = 0$$

Since $\mathbf{d}\eta^{ab} = 0$, the spin connection is antisymmetric, $\boldsymbol{\omega}_{ab} = -\boldsymbol{\omega}_{ba}$.

The equations above describe Riemann-Cartan geometry in the Cartan formalism. Note that the Riemann-Cartan curvature, $\boldsymbol{\mathcal{R}}^a{}_b$, differs from the Riemann curvature $\mathbf{R}^a{}_b$ by terms dependent on the torsion.

When the torsion vanishes, $\mathbf{T}^a = 0$, the Riemann-Cartan curvature $\boldsymbol{\mathcal{R}}^a{}_b$ reduces to the Riemann curvature $\mathbf{R}^a{}_b$ and Eqs.(38) exactly reproduce the expressions for the connection and curvature of a general Riemannian geometry. At the same time, Eqs.(39) reduce to the usual first and second Bianchi identities. We may constrain $\mathbf{T}^a = 0$ in the Cartan equations of Riemann-Cartan geometry, reducing the structure equations to those of Riemannian geometry with its known consistency.

These results are geometric; a physical model follows when we posit an action functional. The action may depend on the bundle tensors $\mathbf{e}^b, \mathbf{T}^a, \boldsymbol{\mathcal{R}}^a{}_b$ and the invariant tensors η_{ab} and $e_{ab\ldots d}$. To this we may add source functionals built from any field representations of the fiber symmetry group $\mathcal{L}$, including scalars, spinors, vector fields, etc.

Constraining the torsion zero, specifying the Einstein-Hilbert form of action, and varying only the solder form, the $q=1$ theory describes general relativity as a gauge theory in n-dimensions. We cannot vary the metric and connection independently because this can introduce nonzero sources for torsion, making the $\mathbf{T}^a = 0$ constraint inconsistent.

Dropping the torsion constraint while retaining the Einstein-Hilbert action gives the Einstein-Cartan-Sciama-Kibble (ECSK) theory of gravity in Riemann-Cartan geometry. The torsion is found to depend on the spin tensor, given by the connection variation of the source $\sigma^\mu{}_{ab} = \frac{\delta L}{\delta \omega^{ab}_{\ \mu}}$. Without modifying the action to include dynamical torsion, the resulting torsion survives only within matter.

Solving for the connection

Contorsion

The Cartan structure equations (38), allow us to derive an explicit form for the connection and reduced form for the curvature. Starting from the equation for the torsion

$$\mathbf{d}\mathbf{e}^a = \mathbf{e}^b \wedge \boldsymbol{\omega}^a{}_b + \mathbf{T}^a$$

write the spin connection as the sum of two terms

$$\boldsymbol{\omega}^a{}_b = \boldsymbol{\alpha}^a{}_b + \boldsymbol{\beta}^a{}_b$$

where $\boldsymbol{\alpha}_{ab} = -\boldsymbol{\alpha}_{ba}$ is the torsion-free connection, $\mathbf{d}\mathbf{e}^a = \mathbf{e}^b \wedge \boldsymbol{\alpha}^a{}_b$ and $\boldsymbol{\beta}_{ab} = -\boldsymbol{\beta}_{ba}$. Then $\boldsymbol{\beta}^a{}_b$ must satisfy

$$0 = \mathbf{e}^b \wedge \boldsymbol{\beta}^a{}_b + \mathbf{T}^a$$

To solve this the 1-form $\boldsymbol{\beta}_{ab}$ will be linear in the torsion and antisymmetric. These conditions dictate the ansatz

$$\boldsymbol{\beta}_{ab} = (aT_{cab} + b(T_{acb} - T_{bca}))\,\mathbf{e}^c$$

for some constants a, b. Substitution quickly leads to $a = b = \frac{1}{2}$, and the spin connection is

$$\boldsymbol{\omega}^a{}_b = \boldsymbol{\alpha}^a{}_b + \mathbf{C}^a{}_b \tag{40}$$

where $\mathbf{C}^a{}_b$ is the *contorsion*,

$$\mathbf{C}^a{}_b = \frac{1}{2}\left(T_c{}^a{}_b + T_b{}^a{}_c - T^a{}_{bc}\right)\mathbf{e}^c$$

Contorsion transforms tensorially so this form is unique. We may recover the torsion by wedging and contracting with $\mathbf{e}^b$.

$$\mathbf{C}^a{}_b \wedge \mathbf{e}^b = \mathbf{T}^a$$

The torsion now enters the curvature through the connection. Expanding the Cartan-Riemann curvature $\boldsymbol{\mathcal{R}}^a{}_b$ using Eq.(40) and identifying the $\boldsymbol{\alpha}$-covariant derivative, $\mathbf{DC}^a{}_b = \mathbf{dC}^a{}_b - \mathbf{C}^c{}_b \wedge \boldsymbol{\alpha}^a{}_c - \boldsymbol{\alpha}^c{}_b \wedge \mathbf{C}^a{}_c$ leads to

$$\boldsymbol{\mathcal{R}}^a{}_b = \mathbf{R}^a{}_b + \mathbf{DC}^a{}_b - \mathbf{C}^c{}_b \wedge \mathbf{C}^a{}_c \tag{41}$$

This is the Riemann-Cartan curvature expressed in terms of the Riemann curvature and the contorsion. If we contract with $\mathbf{e}^b$ we recover the Bianchi identity. This happens because our solution for the connection automatically satisfies the integrability condition for the connection.

Bianchi identities

Given Eq.(41) for the Riemann-Cartan curvature, we may also expand the generalized Bianchi identities (39). The first Bianchi becomes

$$\mathbf{dT}^a + \mathbf{T}^b \wedge (\boldsymbol{\alpha}^a{}_b + \mathbf{C}^a{}_b) = \mathbf{e}^b \wedge \mathbf{R}^a{}_b + \mathbf{e}^b \wedge \mathbf{DC}^a{}_b - \mathbf{e}^b \wedge \mathbf{C}^c{}_b \wedge \mathbf{C}^a{}_c$$

Using $\mathbf{C}^c{}_b \wedge \mathbf{e}^b = \mathbf{T}^c$ and $\mathbf{De}^a = 0$ the torsion terms cancel and we may write $\mathbf{e}^b \wedge \mathbf{DC}^a{}_b = \mathbf{D}\left(\mathbf{C}^a{}_b \wedge \mathbf{e}^b\right) = \mathbf{DT}^a$. The Riemannian Bianchi $\mathbf{e}^b \wedge \mathbf{R}^a{}_b = 0$ follows immediately. Similarly, expanding the derivative in the second Bianchi gives

$$0 = \mathbf{D}\boldsymbol{\mathcal{R}}^a{}_b + \boldsymbol{\mathcal{R}}^c{}_b \wedge \mathbf{C}^a{}_c - \mathbf{C}^c{}_b \wedge \boldsymbol{\mathcal{R}}^a{}_c$$

and replacing $\boldsymbol{\mathcal{R}}^a{}_b = \mathbf{R}^a{}_b + \mathbf{DC}^a{}_b - \mathbf{C}^c{}_b \wedge \mathbf{C}^a{}_c$ throughout then using $\mathbf{C}^c{}_b \wedge \mathbf{e}^b = \mathbf{T}^c$ and the Ricci identity $\mathbf{D}^2\mathbf{C}^a{}_b = \mathbf{C}^c{}_b \wedge \mathbf{R}^a{}_c - \mathbf{C}^a{}_c \wedge \mathbf{R}^c{}_b$ lead to cancellations down to the second Riemannian Bianchi identity

$$\mathbf{DR}^a{}_b = 0$$

Therefore, the Cartan-Riemann Bianchi identities hold if and only if the Riemann Bianchi identities hold.

Because the curvature is a 2-form, and the spin connection is antisymmetric, $\mathcal{R}_{abcd} = \mathcal{R}_{ab[cd]} = \mathcal{R}_{[ab]cd}$ and there is still only one independent contraction of the curvature. The first Bianchi identity then shows that the curvature tensor $\boldsymbol{\mathcal{R}}^a{}_b$ has nonvanishing triply antisymmetric part. Expanding both sides of the first Bianchi identity in components and antisymmetrizing we take a single contraction to show an antisymmetric part

$$\mathcal{R}_{ba} - \mathcal{R}_{ab} = \mathcal{D}_c \mathscr{T}^c{}_{ab} \tag{42}$$

where $\mathscr{T}^c{}_{ab} \equiv T^c{}_{ab} - \delta^c_a T^d{}_{db} + \delta^c_b T^d{}_{da}$, so the Ricci tensor of the Cartan-Riemann curvature acquires an antisymmetric part dependent on derivatives of the torsion.

Appendix B: Constraints on rotations

We impose three conditions on linear mappings ϕ on vectors $V^A \in \mathcal{V}$.

1. ϕ must preserve the the (P,Q) metric g_{AB} induced by the underlying (p,q) metric.

$$\left|V^A\right|^2 = g_{AB}V^A V^B = \frac{1}{2}\eta_{ab}\left(\eta^{ce}\eta^{df} - \eta^{cf}\eta^{de}\right) V^a{}_{cd} V^b{}_{ef}$$

2. ϕ must affect only the mixed symmetry part of V_{abc}.

3. ϕ must preserve $\mathbf{S}^a$.

We look at each condition in turn.

Metric

We begin by finding the metric signature (P,Q). Let the spacelike and timelike components of any vector V^a be separated as $V^a = \left(V^{a_i}, V^{b_j}\right)$ with $a_i; i = 1,\ldots,p$ and $b_j; j = 1,\ldots,q$. Then the norm of V_{abc} is

$$\begin{aligned}\left|V^A\right|^2 &= \sum_{a_1;a_2<a_3}\left(V_{a_1a_2a_3}\right)^2 - \sum_{b_1;a_1<a_2}\left(V_{b_1a_1a_2}\right)^2 - \sum_{a_1,b_1,a_2}\left(V_{a_1b_1a_2}\right)^2 \\ &\quad + \sum_{a_1,b_1,b_2}\left(V_{b_1b_2a_1}\right)^2 + \sum_{a_1,b_1<b_2}\left(V_{a_1b_1b_2}\right)^2 - \sum_{b_1,b_2<b_3}\left(V_{b_1b_2b_3}\right)^2\end{aligned}$$

The multiplicities of the positive sums are, respectively: $\frac{1}{2}p^2(p-1), q^2p, \frac{1}{2}pq(q-1)$. For the negative sums multiplicities are: $\frac{1}{2}qp(p-1), p^2q, \frac{1}{2}q^2(q-1)$. Combining

$$\begin{aligned} P &= \frac{1}{2}\left(p^2(p-1) + pq(3q-1)\right) \\ Q &= \frac{1}{2}\left(q^2(q-1) + pq(3p-1)\right)\end{aligned}$$

with $P + Q = N = \frac{1}{2}n^2(n-1)$ and $S = P - Q = \frac{1}{2}s\left(s^2 - n\right)$.

The proper rotation group preserving the metric on $\mathcal{V}$ is therefore

$$SO\left(\frac{1}{2}\left(p^2(p-1) + pq(3q-1)\right), \frac{1}{2}\left(q^2(q-1) + pq(3p-1)\right)\right)$$

For example, in 4-dim $(3,1)$ spacetime this is the split orthogonal form

$$SO(12,12)$$

This split form $P = Q$ (reminiscent of Kähler, biconformal, and double field manifolds) occurs if and only if $s = 0$ or $n = s^2$ is a perfect square.

Antisymmetry constraint

The antisymmetry constraints fall into four types, each of definite causality type in the induced norm:

$$\begin{aligned} X_1 = V_{a_1a_2a_3} + V_{a_3a_1a_2} + V_{a_2a_3a_1} &= 0 \quad (\textit{spacelike}) \\ X_2 = V_{b_1a_2a_3} + V_{a_3b_1a_2} + V_{a_2a_3b_1} &= 0 \quad (\textit{timelike}) \\ X_3 = V_{b_1b_2a_3} + V_{a_3b_1b_2} + V_{b_2a_3b_1} &= 0 \quad (\textit{spacelike}) \\ X_4 = V_{b_1b_2b_3} + V_{b_3b_1b_2} + V_{b_2b_3b_1} &= 0 \quad (\textit{timelike})\end{aligned}$$

It is straightforward to count the multiplicities, bearing in mind that the three indices must all differ:

$$\begin{array}{ll} X_1 & \frac{p(p-1)(p-2)}{3!} \\ X_2 & \frac{qp(p-1)}{2!} \\ X_3 & \frac{pq(q-1)}{2!} \\ X_4 & \frac{q(q-1)(q-2)}{3!} \end{array}$$

and we check that these sum to the required $\frac{1}{3!}n(n-1)(n-2)$ components of $V_{[abc]}$.

Setting each set of components $X_i = 0$ reduces P and Q for the symmetry to

$$\begin{aligned} P &= \frac{1}{2}\left(p^3 + 3q^2 p - p^2 - pq\right) - \frac{p(p-1)(p-2)}{3!} - \frac{pq(q-1)}{2!} \\ &= \frac{1}{3} p\left(p^2 + 3q^2 - 1\right) \\ Q &= \frac{1}{2}\left(-qp + 3p^2 q + q^3 - q^2\right) - \frac{qp(p-1)}{2!} - \frac{q(q-1)(q-2)}{3!} \\ &= \frac{1}{3}\left(q^3 + 3p^2 q - q\right) \end{aligned}$$

The reduced group is now

$$SO\left(\frac{1}{3}p\left(p^2 + 3q^2 - 1\right), \frac{1}{3}q\left(q^2 + 3p^2 - 1\right)\right)$$

The representation is split if and only if $p - q = 0, \pm 1$. In $(3,1)$ spacetime this reduces the internal symmetry from $SO(12,12)$ to $SO(11,9)$.

Constant $\mathbf{S}^a$

Holding $\mathbf{S}^a$ constant reduces the total dimension N by one. The resulting symmetry group depends on whether $\mathbf{S}^a$ is timelike or spacelike. The group is either:

$$SO\left(\frac{1}{3}p\left(p^2 + 3q^2 - 1\right) - 1, \frac{1}{3}q\left(q^2 + 3p^2 - 1\right)\right)$$

or

$$SO\left(\frac{1}{3}p\left(p^2 + 3q^2 - 1\right), \frac{1}{3}q\left(q^2 + 3p^2 - 1\right) - 1\right)$$

In $(3,1)$ spacetime the two possibilities are

$$SO(10,9), SO(11,8)$$

In either case the internal symmetry is large enough to contain the Standard Model, with spacelike $\mathbf{S}^a$ leading directly to the $SO(10)$ of grand unification.