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Sources of torsion in Poincare gauge theory

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Abstract

We give a concise geometric development of Poincarè gauge theory in any dimension and signature, and trace the difference between the canonical and Belinfante-Rosenfield energy tensors to different choices of independent variables. Then we give extensive attention to sources for torsion, finding that symmetric kinetic terms for non-Yang-Mills bosonic fields of arbitrary rank drive torsion. Our detailed discussion of spin-3/2 Rarita-Schwinger fields shows that they source all independent parts of the torsion. We develop systematic notation for spin-(2k+1)/2 fields and find the spin tensor for arbitrary k in n $\geq$ 2k+1 dimensions. For $k > 0$ there is a novel direct coupling between torsion and spinor fields.

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1 Introduction

1.1 General relavity as a gauge theory

The Standard Model emerged as a gauge theory over a period of half a century. Early developments [1] coupling the electromagnetic interaction to quantized matter as $U(1)$ gauge theory evolved into to our current understanding of the electroweak and strong interactions as arising from local symmetries. Gauge theories' success motivated the later but parallel development of general relativity as a Poincarè gauge theory.

Utiyama [2] gave the first treatment of general relativity as a gauge theory, choosing the Lorentz group as the local symmetry. Later, Sciama [3] developed the Lorentz gauge theory further, while Kibble [4] generalized to the full Poincarè group by identifying translational gauge fields with the co-tangent basis. With the use of Cartan's quotient method for constructing homogeneous manifolds and generalizing them to curved geometries [5, 6], Ne'eman and Regge [7, 8] applied the gauging to supergravity. These methods still provide a powerful tool for the study of general relativity within broader symmeties. Shortly afterward Ivanov and Niederle used the techniques to study gravity theories based on the Poincarè, de Sitter, anti-de Sitter, and conformal groups[9, 10].

While a number of symmetry groups lead to general relativity or equivalent gravity theories with additional structure [9, 10, 11], Poincarè gauge theory employs the smallest group yielding the essential features and therefore enjoys the most consistent attention as a gauge theory of gravity. Yet even this modest extention of general relativity introduces new features, most notably the torsion.

Our version of Poincarè gauging using Cartan's methods is described in Sections (2) and (3). The principle fields are the curvature and torsion 2-forms, given in terms of the solder form and spin connection. The inclusion of torsion produces a Riemann-Cartan geometry rather than Riemannian.

To reproduce general relativity from Poincarè gauging in Riemannian geometry we can disregard the torsion and vary only the metric. The resulting Riemannian geometry is known to be consistent and metric variation leads to a symmetric energy tensor. Exploration of the unconstrained Riemann-Cartan geometry is the purview of ECSK theory and its generalizations to dynamical torsion.

1.2 Palatini variation and ECSK

The original formulation of general relativity assumed the metric compatible Christoffel connection, with the metric as the independent variable so the Einstein-Hilbert action is a functional of the metric g alone $S_{EH}[g]$. It was soon shown by Palatini [12] that if the action is regarded as a functional of the metric and an arbitrary symmetric connection $S_{EH}[g,\Gamma]$, we find the usual field equation along with the condition of metric compatibility. With this Palatini variation, the use of the Christoffel connection is derived. However, the assumption of a symmetric connection rules out any role for torsion.

In a gravitational gauge theory built from Poincarè symmetry the connection forms are dual to the generators of the original symmetry and it is natural to vary all of them independently. This means varying both the solder form $\mathbf{e}^a$ and the spin connection $\boldsymbol{\omega}^a{}_b$ in the style of Palatini. When the $(\mathbf{e}^a, \boldsymbol{\omega}^a{}_b)$ variation is carried out with vanishing torsion the usual Einstein theory of gravity results. However, when the full Riemann-Cartan geometry including torsion is allowed, the spin tensor of matter sources will lead to nonvanishing torsion in the same way that the energy tensor drives curvature. In this sense Poincarè gauge theory can make preditions beyond those of general relativity.

The development of Riemann-Cartan geometry using the Einstein-Hilbert action is now known as the Einstein-Cartan-Sciama-Kibble (ECSK) model of gravity. Its long history begins with Cartan's generalization of Riemannian geometry [13, 14, 15, 16]. A few years later Einstein used torsionful geometry to discuss teleparallel model [17] though this theory is not cast in the same terms as general relativity. Originally, the evolving ECSK theory was the study of the metric variation of the Einstein-Hilbert action $S_{EH}[g]$ in a Riemann-Cartan geometry. The gauge theory approach was more fully developed starting with Utiyama and continuing as outlined above [2, 3, 4, 7, 8, 9, 10]. A detailed review is given in [18]. With the advent of modern gauge theory it has become natural to vary both metric and connection $S_{EH}[g,\Gamma]$ or both solder form and spin connection $S_{EH}[e,\omega]$.

Basing gravity theory on the Einstein-Hilbert action with source fields, torsion is found to be non-propagating and vanishing away from material sources. This is perhaps a benefit, since the geometric understanding of torsion implies non-integrability of functions around closed curves, in much the same way as vectors are rotated under parallel transport around loops in Riemannian geometry. Since there is no experimental evidence in favor of torsion, and limits on torsion coupling to matter are strong (see Donald E. Neville[1] [19]) much study of ECSK has focussed on showing that torsion does not persist in physical situations (e.g., [20]). It is natural that the seemingly pathological non-integrability, the anomolous effect on angular momentum, and in general the extreme success of general relativity should have this effect. Nonetheless, the study of ECSK theory has drawn considerable attention over the last century, including generalizations to propagating torsion [19, 21, 20, 22, 23]. The latter have been criticized as incapable of simultaneous unitarity and normalizability [24].

On the other hand, sometimes a deeper understanding of geometry and general relativity is to be gained by fully exploring nearby theories. This is the goal of the present work: to describe broad classes of sources for torsion in Poincarè gauge theory. Our results hold in any dimension n and any signature (p, q). The exercise includes some important physical predictions, since some of the sources we discuss, notably the spin-$\frac{3}{2}$ Rarita-Schwinger field, are predicted by string and other supergravity theories.

In the next Section we present the basic properties of Poincarè gauge theory using Cartan methods. We include the structure equations, Bianchi identities, the solution for the spin connection in terms of the compatible connection and the contorsion, and the decomposition of the torsion into invariant parts. These results are geometrrical.

The ECSK action is introduced in Section (3), where we discuss two distinct methods of variation. For the first method the action is taken as a functional of the solder form and the full spin connection, $S\left[\mathbf{e}^a, \boldsymbol{\omega}^a{}_b\right]$, in the spirit of Palatini but allowing torsion. The second method uses the decomposition of the spin connection into a compatible piece and the contorsion tensor $\boldsymbol{\omega}^a{}_b = \boldsymbol{\alpha}^a{}_b + \mathbf{C}^a{}_b$. This allows us to respect the Lorentz fiber structure of the bundle by varying only the Lorentz tensors–the solder form and the contorsion, while treating the compatible part of the spin connection as a functional of the solder form $\boldsymbol{\alpha}^a{}_b = \boldsymbol{\alpha}^a{}_b\left(\mathbf{e}^a\right)$.

The effect of generic matter fields is studied in Section (4), where the contrast between the two variational approaches of the previous Section become important: different choices of independent variables give different energy tensors. We show that this leads to the difference between the canonical energy tensor and the Belinfante-Rosenfield tensor. Additionally, we show that while the solder form variation leads to an antisymmetric piece of the Einstein equation, Lorentz invariance restores symmetry.

The bulk of our investigation, presented in Section (5), concerns the effects of various types of fundamental fields on torsion. The exceptional cases of Klein-Gordon and Yang-Mills fields are treated first. The actions for these fields do not depend on the spin connection and therefore do not provide sources for torsion. Next, we study a class of bosonic fields of arbitrary spin with actions quadratic and symmetric in covariant derivatives. Except for scalars, these drive torsion. In Subsection (5.3) we derive the well-known axial current source for totally antisymmetric torsion arising from Dirac fields. We also check the effect of nonvanishing spin tensor in the limit of general relativity where the torsion vanishes.

The effect of the less thoroughly studied Rarita-Schwinger field on torsion is examined in Subsection (5.4). While the axial source for Dirac fields arises from the anticommutator of a γ-matrix with the spin connection, the Rarita-Schwinger field couples through a similar anticommutator but with the product of three γ-matrices. In addition, we find a new direct coupling of the spin-$\frac{3}{2}$ field to torsion. Unlike the Dirac field with only an axial current source, the Rarita-Schwinger field drives all three independent pieces of the torsion. Except in dimensions $5, 7$ and 9, spin-$\frac{3}{2}$ fields have enough degrees of freedom to drive all components of the torsion independently.

Finally, we introduce new compact notation for spin-$\frac{2k+1}{2}$ spinor-valued p-form fields in Subsection (5). This enables us to write actions for arbitrary k and find the general form of the spin tensor. The physical properties appear to echo those of the Rarita-Schwinger field.

[1] "The torsion must couple to spins with coupling constants much smaller than the electromagnetic fine-structure constant, or the force between two macroscopic ferromagnets, due to torsion exchange, would be huge, far greater than the familiar magnetic force due to photon exchange."

We conclude with a brief summary of our results.

2 Poincare gauge theory

All results below hold in arbitrary dimension $n = p + q$ and signature $s = p - q$. The group we gauge is then $SO(p, q)$ or $Spin(p, q)$ with the familiar spacetime case having $p = 3, q = 1$.

There are two stages to building the Poincarè gauge theory: First, we apply Cartan's construction to develop a fiber bundle and second, we specify an action functional.

The construction of the geometry is described in Section (2). Briefly, we use structure constants of Poincarè Lie algebra to write the Maurer-Cartan equations, a set of first order differential equations. These equations are equivalent to the Lie algebra. Next, we form the quotient of the Poincarè group by its Lorentz subgroup and the Lorentz equivalence classes (cosets) form a manifold. Defining a projection from the cosets to this manifold gives a principal fiber bundle. The manifold is homogeneous and the fibers are Lorentz. The final step is to change the connection forms to give horizontal curvatures and to (perhaps) change the manifold.

2.1 Geometric relations of Riemann-Cartan geometry

By Poincarè gauge theory, we mean physical models based on the unrestricted Cartan gauge theory of the Poincarè group. Starting with the generators $M^a{}_b$ and P_a of the Poincarè Lie algebra, we define 1-forms $\boldsymbol{\omega}^a{}_b$ and $\mathbf{e}^b$

$$
\begin{aligned}
\langle M^c{}_d, \boldsymbol{\omega}^a{}_b \rangle &= \eta^{ac}\eta_{bd} - \delta^c_b \delta^a_d \\
\left\langle P_a, \mathbf{e}^b \right\rangle &= \delta^b_a
\end{aligned}
$$

The Maurer-Cartan equations for dual forms for any Lie algebra $\left\langle G_A, \boldsymbol{\omega}^B \right\rangle = \delta^B_A$ are given by $\mathbf{d}\tilde{\boldsymbol{\omega}}^A = -\frac{1}{2} c^A{}_{BC} \tilde{\boldsymbol{\omega}}^B \wedge \tilde{\boldsymbol{\omega}}^C$ where $c^A{}_{BC}$ are the structure constants. For the Poincarè group $\mathcal{P}$ this gives

$$
\begin{aligned}
\mathbf{d}\tilde{\boldsymbol{\omega}}^a{}_b &= \tilde{\boldsymbol{\omega}}^c{}_b \wedge \tilde{\boldsymbol{\omega}}^a{}_c \\
\mathbf{d}\tilde{\mathbf{e}}^a &= \tilde{\mathbf{e}}^b \wedge \tilde{\boldsymbol{\omega}}^a{}_b
\end{aligned}
$$

and we take the quotient by the Lorentz subgroup $\mathcal{L}$ allows us to develop a principal fiber bundle with Lorentz symmetry over a homogeneous n-dimensional manifold $\mathcal{M}^{(n)}$.

By modifying the solder form and the spin connection 1-forms $\left(\tilde{\mathbf{e}}^b, \tilde{\boldsymbol{\omega}}^a{}_b\right) \rightarrow \left(\mathbf{e}^b, \boldsymbol{\omega}^a{}_b\right)$ we introduce a Poincarè covariant tensors with two Lorentz covariant components: the *curvature* $\boldsymbol{\mathcal{R}}^a{}_b$ and the *torsion* $\mathbf{T}^a$

$$
\begin{aligned}
\mathbf{d}\boldsymbol{\omega}^a{}_b &= \boldsymbol{\omega}^c{}_b \wedge \boldsymbol{\omega}^a{}_c + \boldsymbol{\mathcal{R}}^a{}_b &\quad (1) \\
\mathbf{d}\mathbf{e}^a &= \mathbf{e}^b \wedge \boldsymbol{\omega}^a{}_b + \mathbf{T}^a &\quad (2)
\end{aligned}
$$

We require the $\boldsymbol{\mathcal{R}}^a{}_b$ and $\mathbf{T}^a$ to be horizontal,

$$
\begin{aligned}
\boldsymbol{\mathcal{R}}^a{}_b &= \frac{1}{2} \mathcal{R}^a{}_{bcd} \mathbf{e}^c \wedge \mathbf{e}^d &\quad (3) \\
\mathbf{T}^a &= \frac{1}{2} T^a{}_{bc} \mathbf{e}^b \wedge \mathbf{e}^c &\quad (4)
\end{aligned}
$$

thereby preserving the bundle structure. Integrability of the Cartan equations Eqs.(1) and (2) is insured by $\mathbf{d}^2\boldsymbol{\omega}^a{}_b \equiv 0$ and $\mathbf{d}^2\mathbf{e}^a \equiv 0$, which require the Bianchi identities,

$$
\begin{aligned}
\boldsymbol{\mathcal{D}}\mathbf{T}^a &= \mathbf{e}^b \wedge \boldsymbol{\mathcal{R}}^a{}_b &\quad (5) \\
\boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{R}}^a{}_b &= 0 &\quad (6)
\end{aligned}
$$

where the covariant exterior derivatives are given by

$$\begin{aligned}
\boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{R}}^{a}{}_{b} &= \mathbf{d}\boldsymbol{\mathcal{R}}^{a}{}_{b} + \boldsymbol{\mathcal{R}}^{c}{}_{b} \wedge \boldsymbol{\omega}^{a}{}_{c} - \boldsymbol{\omega}^{c}{}_{b} \wedge \boldsymbol{\mathcal{R}}^{a}{}_{c} \\
\boldsymbol{\mathcal{D}}\mathbf{T}^{a} &= \mathbf{d}\mathbf{T}^{a} + \mathbf{T}^{b} \wedge \boldsymbol{\omega}^{a}{}_{b}
\end{aligned}$$

When the connection is assumed to be compatible with the metric, Eqs.(1)-(6) describe *Riemann-Cartan geometry* in the Cartan formalism. Note that the *Cartan-Riemann* curvature, $\boldsymbol{\mathcal{R}}^{a}{}_{b}$, differs from the Riemann curvature $\mathbf{R}^{a}{}_{b}$ by terms dependent on the torsion. When the torsion vanishes, $\mathbf{T}^{a} = 0$, the Riemann-Cartan curvature $\boldsymbol{\mathcal{R}}^{a}{}_{b}$ reduces to the Riemann curvature $\mathbf{R}^{a}{}_{b}$ and Eqs.(1) and (2) exactly reproduce the expressions for the connection and curvature of a general Riemannian geometry. At the same time, Eqs.(5) and (6) reduce to the usual first and second Bianchi identities.

The orthonormal frame fields $\mathbf{e}^{a}$ satisfy

$$\left\langle \mathbf{e}^{a}, \mathbf{e}^{b} \right\rangle = \eta^{ab}$$

In ECSK theory, the connection is assumed compatible with the Lorentz (or $SO\left(p,q\right)$, $Spin\left(p,q\right)$) metric η^{ab}. This implies antisymmetry of the spin connection.

$$\begin{aligned}
0 &= \boldsymbol{\mathcal{D}}\eta_{ab} \\
&= \mathbf{d}\eta_{ab} - \eta_{cb}\boldsymbol{\omega}^{c}{}_{a} - \eta_{ac}\boldsymbol{\omega}^{c}{}_{b} \\
&= -\left(\boldsymbol{\omega}_{ba} + \boldsymbol{\omega}_{ab}\right)
\end{aligned}$$

Antisymmetry together with Eq.(2) fully determines the spin connection up to local Lorentz transformations.

These results are geometric; a physical model follows when we posit an action functional. The action may depend on the bundle tensors $\mathbf{e}^{b}, \mathbf{T}^{a}, \boldsymbol{\mathcal{R}}^{a}{}_{b}$ and the invariant tensors η_{ab} and $e_{ab\ldots d}$. To this we may add action functionals built from any field representations of the fiber symmetry group (Lorentz, $SO\left(p,q\right), Spin\left(p,q\right)$)–scalars, spinors, vector fields, etc.

The relation between the Riemann-Cartan curvature $\boldsymbol{\mathcal{R}}^{a}{}_{b}$ and the Riemann curvature $\mathbf{R}^{a}{}_{b}$ is developed below.

From the known consistency of Riemannian geometry, we know we may set $\mathbf{T}^{a} = 0$ in the Cartan equations of Riemann-Cartan geometry. However, this does not mean that a Poincarè theory of gravity following from an action based on Poincarè symmetry leads to the same restriction. Vanishing torsion must also be a satisfactory solution to the field equations, including sources.

We continue to develop geometric properties in the remainder of this Section. We first solve for the spin connection in the presence of torsion. This allows us to express the Riemann-Cartan curvature in terms of the torsion and Riemann curvature. For use in some subsequent calculations we also find these results in a coordinate basis. We conclude the Section with the decomposition of the torsion into invariant subspaces before moving on to the ECSK action in Section 3.

2.2 Solving for the connection

The structure equations, Eqs.(1) and (2), allow us to derive explicit forms for the connection and curvature. Starting from the Cartan structure equation, Eq.(2), write the spin connection as the sum of two terms

$$\boldsymbol{\omega}^{a}{}_{b} = \boldsymbol{\alpha}^{a}{}_{b} + \boldsymbol{\beta}^{a}{}_{b}$$

where $\boldsymbol{\alpha}^{a}{}_{b}$ is defined to be the torsion-free connection, $\mathbf{d}\mathbf{e}^{a} = \mathbf{e}^{b} \wedge \boldsymbol{\alpha}^{a}{}_{b}$. Then $\boldsymbol{\beta}^{a}{}_{b}$ must satisfy

$$0 = \mathbf{e}^{b} \wedge \boldsymbol{\beta}^{a}{}_{b} + \mathbf{T}^{a} \tag{7}$$

To solve this the 1-form $\boldsymbol{\beta}_{ab}$ must linear in the torsion and antisymmetric. These conditions dictate the ansatz

$$\boldsymbol{\beta}_{ab} = a\mathbf{e}^{c}T_{cab} + b\mathbf{e}^{c}\left(T_{acb} - T_{bca}\right)$$

for some constants a, b. Substitution into Eq.(7) quickly leads to $a = b = \frac{1}{2}$, and the spin connection is

$$\begin{aligned} \boldsymbol{\omega}^a{}_b &= \boldsymbol{\alpha}^a{}_b + \frac{1}{2}\left(T_c{}^a{}_b + T^a{}_{cb} - T_{bc}{}^a\right)\mathbf{e}^c \\ &= \boldsymbol{\alpha}^a{}_b + \mathbf{C}^a{}_b \end{aligned} \tag{8}$$

where $\mathbf{C}^a{}_b$ is the *contorsion*,

$$\mathbf{C}^a{}_b = \frac{1}{2}\left(T_c{}^a{}_b + T^a{}_{cb} - T_{bc}{}^a\right)\mathbf{e}^c \tag{9}$$

The decomposition of the connection is unique. Local Lorentz transformations transform $\boldsymbol{\alpha}^a{}_b$ inhomogeneously in the familiar way while torsion and contorsion are tensors. The form of contorsion (9) in terms of torsion is unique and invertible.

We may recover the torsion by wedging and contracting with $\mathbf{e}^b$.

$$\mathbf{C}^a{}_b \wedge \mathbf{e}^b = \mathbf{T}^a$$

Conversely, we can write the contorsion in terms of the torsion 2-form. First, write the contorsion as

$$\mathbf{C}_{ab} = \left(\frac{3}{2}T_{[abc]} + T_{bac} - T_{abc}\right)\mathbf{e}^c$$

Now convert the 2-form $\mathbf{T}^b$ and the 3-form $\mathbf{e}^c \wedge \mathbf{T}_c$ to 1-forms ${}^*\left(\mathbf{e}^a \wedge {}^*\mathbf{T}^b\right)$ and ${}^*\mathbf{e}^a \wedge \mathbf{e}^b \wedge {}^*\left(\mathbf{e}^c \wedge \mathbf{T}_c\right)$ respectively, leading to the somewhat daunting form

$$\mathbf{C}^{ab} = (-1)^p\, {}^*\left(\mathbf{e}^a \wedge {}^*\mathbf{T}^b - \mathbf{e}^b \wedge {}^*\mathbf{T}^a - \frac{1}{2}\mathbf{e}^a \wedge \mathbf{e}^b \wedge {}^*\left(\mathbf{e}^c \wedge \mathbf{T}_c\right)\right)$$

Clearly, for some calculations, the component notation is simpler.

The torsion now enters the curvature through the connection. Expanding the Cartan-Riemann curvature of Eq.(1) using Eq.(8) then identifying the $\boldsymbol{\alpha}$-covariant derivative, $\mathbf{DC}^a{}_b = \mathbf{dC}^a{}_b - \mathbf{C}^c{}_b \wedge \boldsymbol{\alpha}^a{}_c - \boldsymbol{\alpha}^c{}_b \wedge \mathbf{C}^a{}_c$ leads to

$$\boldsymbol{\mathcal{R}}^a{}_b = \mathbf{R}^a{}_b + \mathbf{DC}^a{}_b - \mathbf{C}^c{}_b \wedge \mathbf{C}^a{}_c \tag{10}$$

This is the Riemann-Cartan curvature expressed in terms of the Riemann curvature and the contorsion. Note that the $\boldsymbol{\alpha}$-covariant derivative is compatible with the solder form, $\mathbf{De}^a = \mathbf{de}^a - \mathbf{e}^b \wedge \boldsymbol{\alpha}^a{}_b = 0$.

Given Eq.(10) for the Cartan-Riemann curvature in terms of the Riemannian curvature and connection, we may also expand the generalized Bianchi identities of Eqs.(5) and (6). The first Bianchi becomes

$$\mathbf{dT}^a + \mathbf{T}^b \wedge \left(\boldsymbol{\alpha}^a{}_b + \mathbf{C}^a{}_b\right) = \mathbf{e}^b \wedge \mathbf{R}^a{}_b + \mathbf{e}^b \wedge \mathbf{DC}^a{}_b - \mathbf{e}^b \wedge \mathbf{C}^c{}_b \wedge \mathbf{C}^a{}_c$$

Using $\mathbf{De}^a = 0$ and replacing $\mathbf{C}^c{}_b \wedge \mathbf{e}^b = \mathbf{T}^c$ leads to the Riemannian Bianchi $\mathbf{e}^b \wedge \mathbf{R}^a{}_b = 0$.

Similarly, expanding the derivative in the second Bianchi gives

$$0 = \mathbf{D}\boldsymbol{\mathcal{R}}^a{}_b + \boldsymbol{\mathcal{R}}^c{}_b \wedge \mathbf{C}^a{}_c - \mathbf{C}^c{}_b \wedge \boldsymbol{\mathcal{R}}^a{}_c$$

Replacing $\boldsymbol{\mathcal{R}}^a{}_b = \mathbf{R}^a{}_b + \mathbf{DC}^a{}_b - \mathbf{C}^c{}_b \wedge \mathbf{C}^a{}_c$ throughout then using $\mathbf{e}^b \wedge \mathbf{C}^c{}_b = \mathbf{T}^c$ and $\mathbf{D}^2\mathbf{C}^a{}_b = \mathbf{C}^c{}_b \wedge \mathbf{R}^a{}_c - \mathbf{C}^a{}_c \wedge \mathbf{R}^c{}_b$ leads to several cancellations and finally

$$\mathbf{DR}^a{}_b = 0$$

so that the Cartan-Riemann Bianchi identities hold if and only if the Riemann Bianchi identities hold.

The first Bianchi identity relates the triply antisymmetric part of the curvature tensor $\boldsymbol{\mathcal{R}}^a{}_b$ to the exterior derivative of the torsion. Expanding both sides of Eq.(5), antisymmetrizing, then stripping the basis,

$$\mathcal{R}^a{}_{bcd} + \mathcal{R}^a{}_{cdb} + \mathcal{R}^a{}_{dbc} = \mathcal{D}_d T^a{}_{bc} + \mathcal{D}_b T^a{}_{cd} + \mathcal{D}_c T^a{}_{db}$$

Contracting ad and using $\mathcal{R}^{c}{}_{cdb} = 0$ (by the structure equation Eq.(1) and the antisymmetry of the spin connection) we have

$$\mathcal{R}_{cb} - \mathcal{R}_{bc} = \mathcal{D}_a \mathscr{T}^{a}{}_{bc}$$

where we define

$$\mathscr{T}^{a}{}_{bc} = T^{a}{}_{bc} - \delta^a_b T^{e}{}_{ec} + \delta^a_c T^{e}{}_{eb}$$

For all $n > 2$ this is invertible, $T^{a}{}_{bc} = \mathscr{T}^{a}{}_{bc} + \frac{1}{n-2}\left(\delta^a_c \mathscr{T}^{e}{}_{eb} - \delta^a_b \mathscr{T}^{e}{}_{ec}\right)$. Then the antisymmetric part of the Ricci-Cartan tensor is simply minus the divergence

$$\mathcal{R}_{ab} - \mathcal{R}_{ba} = -\mathcal{D}_c \mathscr{T}^{c}{}_{ab} \tag{11}$$

Therefore the Ricci tensor of the Cartan-Riemann curvature acquires an antisymmetric part dependent on derivatives of the torsion.

Because the curvature is a 2-form, and the spin connection is antisymmetric, the curvature satisfies $\mathcal{R}_{abcd} = \mathcal{R}_{ab[cd]} = \mathcal{R}_{[ab]cd}$ and there is still only one independent contraction.

2.2.1 Coordinate expressions

The solder form equation (2) may be solved algebraically for the either the spin connection or the general linear connection. Here we solve for the general linear case. The combined components of the vanishing 2-form $\mathbf{de}^a - \mathbf{e}^b \wedge \boldsymbol{\omega}^{a}{}_{b} - \mathbf{T}^a = 0$ must be symmetric

$$\partial_\mu e_\nu{}^a + e_\nu{}^b \omega^{a}{}_{b\mu} + \frac{1}{2} T^{a}{}_{\nu\mu} = \Lambda^{a}{}_{\nu\mu} \tag{12}$$

where lower case Latin indices refer to the pseudo-orthonormal frames $\mathbf{e}^a$ while lower case Greek indices refer to a coordinate basis, $\mathbf{d}x^\mu$. We recognize Eq.(12) as a vanishing covariant derivative

$$\mathcal{D}_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a + e_\nu{}^b \omega^{a}{}_{b\mu} - e_\sigma{}^a \Sigma^{\sigma}{}_{\nu\mu} = 0$$

where $\Sigma^{\beta}{}_{\nu\mu} = \Lambda^{\beta}{}_{\nu\mu} - \frac{1}{2} T^{\beta}{}_{\nu\mu}$. Contracting Eq.(12) with $\eta_{ac} e_\beta{}^c$ we symmetrize on $\beta\nu$. The spin connection terms cancel and the derivatives combine into a single covariant derivative of the metric.

$$0 = \partial_\mu g_{\beta\nu} - g_{\beta\sigma} \Sigma^{\sigma}{}_{\nu\mu} - g_{\nu\sigma} \Sigma^{\sigma}{}_{\beta\mu} = \mathcal{D}_\mu g_{\beta\nu}$$

We solve this familiar form of metric compatibility in the usual way by cycling indices then adding two permutations and subtracting the third, but using $\Sigma_{\beta\nu\mu} - \Sigma_{\beta\mu\nu} = T_{\beta\mu\nu}$ to rearrange index order. Restoring the usual index positions the result is

$$\Sigma^{\nu}{}_{\beta\mu} = \Gamma^{\nu}{}_{\mu\beta} - C^{\nu}{}_{\beta\mu}$$

wherewhere $\Gamma^{\alpha}{}_{\mu\nu}$ is the Christoffel connection and we recognize the contorsion tensor,

$$C_{\beta\nu\mu} = -C_{\nu\beta\mu} = \frac{1}{2}\left(T_{\beta\nu\mu} + T_{\nu\mu\beta} - T_{\mu\beta\nu}\right)$$

The vanishing covariant derivative of the vielbein takes the form

$$0 = \mathcal{D}_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a + e_\nu{}^b \omega^{a}{}_{b\mu} - e_\sigma{}^a \Gamma^{\sigma}{}_{\nu\mu} + e_\sigma{}^a C^{\sigma}{}_{\nu\mu}$$

2.3 Decompostion of the torsion

We identify well-known invariant pieces of the torsion. The torsion includes a totally antisymmetric piece

$$\mathbf{T} \equiv \frac{1}{3} \mathbf{e}^a \wedge \mathbf{T}_a = \frac{1}{3!} T_{abc} \mathbf{e}^a \wedge \mathbf{e}^b \wedge \mathbf{e}^c \tag{13}$$

with $\frac{1}{6}n\left(n-1\right)\left(n-2\right)$ degrees of freedom. Note that in 4 or 5 dimensions the dual of $\mathbf{T}$ is a lower rank object.

$$\begin{aligned} {}^{*}\mathbf{T} &= \frac{1}{3!}T^{abc}e_{abcd}\mathbf{e}^{d} \\ {}^{*}\mathbf{T} &= \frac{1}{3!2!}T^{abc}e_{abcde}\mathbf{e}^{d}\wedge\mathbf{e}^{e} \end{aligned}$$

in particular giving the well-known axial vector in 4-dimensions. There is also a single vectorial contraction.

$$T^{b}{}_{ba}\mathbf{e}^{a} = (-1)^{p}\,{}^{*}\left(\mathbf{e}^{b}\wedge{}^{*}\mathbf{T}_{b}\right) \tag{14}$$

Writing Eqs.(13) and (14) as 2-forms

$$\begin{aligned} \frac{1}{2}\eta^{ab}T_{[bcd]}\mathbf{e}^{c}\wedge\mathbf{e}^{d} &= (-1)^{q}\,3!\,{}^{*}\left(\mathbf{e}^{a}\wedge{}^{*}\mathbf{T}\right) \\ \mathbf{e}^{b}\wedge\left(T^{c}{}_{ca}\mathbf{e}^{a}\right) &= (-1)^{p}\,\mathbf{e}^{b}\wedge{}^{*}\left(\mathbf{e}^{c}\wedge{}^{*}\mathbf{T}_{c}\right) \end{aligned}$$

we may decompose the full torsion in $n=p+q$ dimensions as

$$\mathbf{T}^{a} = \boldsymbol{\tau}^{a}+\frac{1}{n-1}(-1)^{p}\,\mathbf{e}^{b}\wedge{}^{*}\left(\mathbf{e}^{c}\wedge{}^{*}\mathbf{T}_{c}\right)+(-1)^{q}\,3!\,{}^{*}\left(\mathbf{e}^{a}\wedge{}^{*}\mathbf{T}\right) \tag{15}$$

where $\boldsymbol{\tau}^{a}$ is a traceless, mixed symmetry 2-form with $N=\frac{n}{3}\left(n^{2}-4\right)$ degrees of freedom. This remaining piece may be further decomposed into symmetric $\tau_{(ab)c}$ and antisymmetric $\tau_{[ab]c}$ parts.

In components the decomposition is simply

$$T^{a}{}_{bc} = \tau^{a}{}_{bc}+\frac{1}{n-1}\left(\delta^{a}_{b}T^{e}{}_{ec}-\delta^{a}_{c}T^{e}{}_{eb}\right)+\eta^{ae}T_{[ebc]} \tag{16}$$

While the vector and pseudovector each have 4 degrees of freedom in 4-dimensions, the situation is very different in higher dimensions. In general the torsion has a total of $\frac{n^{2}(n-1)}{2}$ degrees of freedom. Therefore, while the trace contains only n degrees of freedom for a fraction $\frac{2}{n(n-1)}\sim\frac{1}{n^{2}}$ of the total, the antisymmetric part includes $\frac{1}{3!}n\left(n-1\right)\left(n-2\right)$ or roughly

$$\frac{n-2}{3n}\sim\frac{1}{3}$$

The residual tensor $\boldsymbol{\tau}^{a}$ includes the remaining $\frac{2\left(n^{2}-4\right)}{3n(n-1)}\sim\frac{2}{3}$. Thus, the antisymmetric part is a major contributor in higher dimensions.

3 Vacuum ECSK theory

The physical constent of the Einstein-Cartan-Sciama-Kibble theory enters through use of the Einstein-Hilbert action in Riemann-Cartan geometry. The physical content also depends on making one of several possible choices of independent variables: the metric $g_{\alpha\beta}$ alone, the metric and connection $\left(g_{\alpha\beta},\Gamma^{\mu}{}_{\alpha\beta}\right)$, the solder form and spin connection $\left(\mathbf{e}^{a},\boldsymbol{\omega}^{a}{}_{b}\right)$ or the solder form and contorsion $\left(\mathbf{e}^{a},\mathbf{C}^{a}{}_{b}\right)$. We carry out two forms of the variation, $\left(\mathbf{e}^{a},\boldsymbol{\omega}^{a}{}_{b}\right)$ and $\left(\mathbf{e}^{a},\mathbf{C}^{a}{}_{b}\right)$.

Two differences from general relativity arise with these choices. First, the asymmetry of the solder form means that the Einstein tensor and energy tensor acquire antisymmetric parts [25]. We show in general in Section (3) and explicitly for the Dirac field in Subsection (5.3.3), that the antisymmetric parts vanish as a consequence of Lorentz invariance. The second issue is that varying the spin connection in a Riemann-Cartan geometry gives nonvanishing sources for torsion. We explore the nature of these sources for a variety of types of field.

For the gravity action, we restrict attention to the Einstein-Hilbert form but with the Riemann-Cartan scalar curvature. Alternatives with propagating torsion are considered in [19, 21, 20, 22], and with additional modification in [26].

3.1 Gravity action

The Einstein-Hilbert form of the action with the Riemann-Cartan curvature scalar, in n-dimensions is

$$S_{ECSK}\left[\mathbf{e}^{a},\boldsymbol{\omega}^{a}{}_{b}\right]=\frac{\kappa}{2\left(n-2\right)!}\int\boldsymbol{\mathcal{R}}^{ab}\wedge\mathbf{e}^{c}\wedge\ldots\wedge\mathbf{e}^{d}e_{abc\ldots d} \tag{17}$$

This action, plus arbitrary source terms, is our definition of the ECSK theory of gravity.

We define a volume form as the Hodge dual of unity, $\mathbf{\Phi}={}^{*}1=\frac{1}{n!}e_{ab\ldots c}\mathbf{e}^{a}\wedge\mathbf{e}^{b}\wedge\ldots\wedge\mathbf{e}^{c}$ and therefore, ${}^{*}\mathbf{\Phi}=(-1)^{q}$ in signature (p,q). It follows that

$$\underbrace{\mathbf{e}^{a}\wedge\mathbf{e}^{b}\wedge\ldots\wedge\mathbf{e}^{c}}_{n\ terms} = (-1)^{q}\,e^{ab\ldots c}\mathbf{\Phi}$$

where $e_{ab\ldots c}$ is the Levi-Civita tensor. Let $\varepsilon_{ab\ldots c}$ be the totally antisymmetric symbol with $\varepsilon_{12\ldots n}=1$ and $e=\det\left(e_{\mu}{}^{a}\right)=\sqrt{|g|}$, so that $e_{12\ldots n}=e\varepsilon_{12\ldots n}$ and $e^{12\ldots n}=(-1)^{q}\,\frac{1}{e}\varepsilon_{abcd}$. Expanding the curvature 2-form,

$$\begin{aligned}\boldsymbol{\mathcal{R}}^{ab}\wedge\mathbf{e}^{c}\wedge\ldots\wedge\mathbf{e}^{d}e_{abc\ldots d} &= \frac{1}{2}\mathcal{R}^{ab}{}_{ef}\mathbf{e}^{e}\wedge\mathbf{e}^{f}\wedge\mathbf{e}^{c}\wedge\ldots\wedge\mathbf{e}^{d}e_{abc\ldots d}\\ &= \frac{1}{2}\mathcal{R}^{ab}{}_{ef}\,(-1)^{q}\,e^{efc\ldots d}e_{abc\ldots d}\mathbf{\Phi}\\ &= (n-2)!\mathcal{R}^{ab}{}_{ab}\mathbf{\Phi}\end{aligned}$$

shows the equivalence to the scalar curvature and we may write $S\left[\mathbf{e}^{a},\boldsymbol{\omega}^{a}{}_{b}\right]=\frac{1}{2}\kappa\int\mathcal{R}\mathbf{\Phi}$.

We first vary the solder form and the spin connection. As noted above, some differences arise from the metric $S\left[g\right]$ or metric/connection $S\left[g,\Gamma\right]$ variations because the solder form is not symmetric.

3.1.1 Two considerations

There are two subtle points regarding the independent variation of the solder form and connection.

First, we require the Gibbons-Hawking-York surface term [[27, 28, 29, 30]] because fixing both $\delta\mathbf{e}^{a}=0$ and $\delta\boldsymbol{\omega}^{a}{}_{b}=0$ overdetermines the solution in the bulk. This can be seen from the conditions for the initial value problem–specifying the metric and the intrinsic curvature of an initial Cauchy surface is enough to propagate a unique solution as the time evolution. It is straightforward to check that adding the Gibbons-Hawking-York surface term resolves the issue, while leaving the expected field equations in the bulk.

The second point is that the decomposition of the connnection $\boldsymbol{\omega}^{ab}=\boldsymbol{\alpha}^{ab}+\mathbf{C}^{ab}$ makes it possible treat the action as either a functional $S\left[\mathbf{e}^{a},\boldsymbol{\omega}^{a}{}_{b}\right]$ or as $S\left[\mathbf{e}^{a},\mathbf{C}^{a}{}_{b}\right]$. In the latter case the remainder of the connection is taken as $\boldsymbol{\alpha}^{ab}=\boldsymbol{\alpha}^{ab}\left(\mathbf{e}^{c}\right)$ where the form of $\delta_{e}\boldsymbol{\alpha}^{ab}\left(\mathbf{e}^{c}\right)$ follows from the structure equation. Varying $\mathbf{de}^{a}=\mathbf{e}^{b}\wedge\boldsymbol{\alpha}^{a}{}_{b}$ we find

$$\boldsymbol{\mathcal{D}}\left(\delta\mathbf{e}^{a}\right) = \mathbf{e}^{b}\wedge\delta\boldsymbol{\alpha}^{a}{}_{b}$$

Then expanding in components $\delta\alpha^{a}{}_{bc}-\delta\alpha^{a}{}_{cb}=e_{c}{}^{\nu}\mathcal{D}_{b}\left(\delta e_{\nu}{}^{a}\right)-e_{b}{}^{\nu}\mathcal{D}_{c}\left(\delta e_{\nu}{}^{a}\right)$ and solving by cycling indices yields

$$\delta\boldsymbol{\alpha}^{a}{}_{b} = \frac{1}{2}\left(\delta_{d}^{a}\delta_{b}^{c}-\eta_{bd}\eta^{ac}\right)\left[D_{c}\left(\delta\mathbf{e}^{d}\right)-e_{c}{}^{\mu}\eta_{gh}\mathbf{e}^{g}D^{d}\left(\delta e_{\mu}{}^{h}\right)-e_{c}{}^{\alpha}\mathbf{D}\left(\delta e_{\alpha}{}^{d}\right)\right] \tag{18}$$

If the action includes no explicit torsion dependence, the linear relation between $\boldsymbol{\omega}^{ab}$ and $\mathbf{C}^{ab}$ means varying either gives the same result, but the solder form variations give different results for the energy tensor.

The conceptual difference between the variations is seen from the fiber bundle structure. While the first variation $S\left[\mathbf{e}^{a},\boldsymbol{\omega}^{a}{}_{b}\right]$ embodies the Palatini principle fully, varying the Lorentz gauge symmetry gives a different combination of the field equations. The second form of variation, $S\left[\mathbf{e}^{a},\mathbf{C}^{a}{}_{b}\right]$, gauge transformations are all included in the solder form variation. The difference shows up physically in the source for the Einstein equation, producing the difference between the canonical energy tensor and the Belinfante-Rosenfield energy tensor [31, 32]. We examine this in detail, carrying out both methods.

3.2 Palatini variation

We vary $\mathbf{e}^a$ and $\boldsymbol{\omega}^{ab}$ independently. The connection variation of the gravity action is

$$\begin{aligned}\delta S_{ECSK}\left[\mathbf{e}^a, \boldsymbol{\omega}^a{}_b\right] &= \frac{\kappa}{2(n-2)!}\int \delta\boldsymbol{\mathcal{R}}^{ab}\wedge\mathbf{e}^c\wedge\ldots\wedge\mathbf{e}^d e_{abc\ldots d}+\delta S_{GHY}\\ &= \frac{\kappa}{2(n-2)!}\int \boldsymbol{\mathcal{D}}\left(\delta\boldsymbol{\omega}^{ab}\right)\wedge\mathbf{e}^c\wedge\ldots\wedge\mathbf{e}^d e_{abc\ldots d}+\delta S_{GHY}\end{aligned}$$

where $\boldsymbol{\mathcal{D}}\left(\delta\boldsymbol{\omega}^{ab}\right)=\mathbf{d}\left(\delta\boldsymbol{\omega}^{ab}\right)-\left(\delta\boldsymbol{\omega}^{eb}\right)\wedge\boldsymbol{\omega}^a{}_e-\left(\delta\boldsymbol{\omega}^{ae}\right)\wedge\boldsymbol{\omega}^b{}_e$. We integrate only the exterior derivative by parts, using Lorentz invariance of the Levi-Civita tensor to redistribute the spin connections.

As mentioned above, the normal derivative of the connection must be allowed to vary on the boundary, so the surface term does not vanish. This contribution is cancelled by including the Gibbons-Hawking-York surface term, δS_{GHY}, which depends only on the induced metric and the extrinsic curvature of the boundary. Here we assume S_{GHY} is used and focus on the variation in the interior.

Disregarding surface terms the variation becomes

$$\begin{aligned}\delta\mathcal{S}_{ECSK}=I_1+I_2 &= \frac{\kappa}{2(n-2)!}\int\delta\boldsymbol{\omega}^{ab}\wedge\left(\mathbf{d}\mathbf{e}^c\wedge\ldots\wedge\mathbf{e}^d+\ldots+(-1)^{n-3}\,\mathbf{e}^c\wedge\ldots\wedge\mathbf{d}\mathbf{e}^d\right)e_{abc\ldots d}\\ &\quad-\frac{\kappa}{2(n-2)!}\int\left(\left(\delta\boldsymbol{\omega}^{eb}\right)\wedge\boldsymbol{\omega}^a{}_e+\left(\delta\boldsymbol{\omega}^{ae}\right)\wedge\boldsymbol{\omega}^b{}_e\right)\wedge\mathbf{e}^c\wedge\ldots\wedge\mathbf{e}^d e_{abc\ldots d}\end{aligned}$$

Now use the invariance of $e_{abc\ldots d}$ under infinitesimal $SO(p,q)$ to write

$$0 = \boldsymbol{\omega}^e{}_a e_{ebc\ldots d}+\boldsymbol{\omega}^e{}_b e_{aec\ldots d}+\ldots+\boldsymbol{\omega}^e{}_d e_{abc\ldots e}$$

so that the second integral becomes may be rearranged to give

$$I_2 = -\frac{\kappa}{2(n-2)!}\int\delta\boldsymbol{\omega}^{ab}\wedge\left(\mathbf{e}^c\wedge\boldsymbol{\omega}^e{}_c\wedge\mathbf{e}^f\ldots\wedge\mathbf{e}^d e_{abe\ldots d}-\ldots+(-1)^{n-3}\,\mathbf{e}^c\ldots\wedge\mathbf{e}^f\wedge\mathbf{e}^d\wedge\boldsymbol{\omega}^e{}_d e_{abc\ldots fe}\right)$$

The $\mathbf{d}\mathbf{e}^a$ and $\mathbf{e}^c\wedge\boldsymbol{\omega}^a{}_c$ terms recombine as $n-2$ factors of the torsion, $\mathbf{T}^a=\mathbf{d}\mathbf{e}^a-\mathbf{e}^c\wedge\boldsymbol{\omega}^a{}_c$ so

$$\delta\mathcal{S}_{ECSK} = \frac{\kappa}{2(n-3)!}\int\delta\boldsymbol{\omega}^{ab}\wedge\mathbf{T}^c\wedge\mathbf{e}^d\ldots\wedge\mathbf{e}^e e_{abcd\ldots e} \tag{19}$$

Setting $\delta\boldsymbol{\omega}^{ab}=\delta A^{ab}{}_c\mathbf{e}^c$ and resolving the product of solder forms into a volume element, the vacuum field equation is the vanishing of

$$\frac{\kappa}{2(n-3)!}\mathbf{e}^c\wedge\mathbf{T}^d\wedge\mathbf{e}^e\ldots\wedge\mathbf{e}^f e_{abde\ldots f} = \frac{\kappa}{2}\left(T^c{}_{ab}+\delta^c_a T^d{}_{bd}-\delta^c_b T^d{}_{ad}\right)\boldsymbol{\Phi}=\frac{\kappa}{2}\mathscr{T}^c{}_{ab}\boldsymbol{\Phi}$$

Notice that $\mathscr{T}^c{}_{ab}$ is the same combination found for the Bianchi identity. Here it arises here from the connection variation.

Varying the solder form now involves only the explicit solder forms. The result is the usual Einstein tensor, but with the Riemann-Cartan curvature.

$$\begin{aligned}\frac{\kappa}{2(n-2)!}\delta_e\int\boldsymbol{\mathcal{R}}^{ab}\wedge\mathbf{e}^c\wedge\ldots\wedge\mathbf{e}^d e_{abc\ldots d} &= \frac{\kappa}{2(n-3)!}\int\delta A^a{}_b\mathbf{e}^b\wedge\boldsymbol{\mathcal{R}}^{cd}\wedge\mathbf{e}^e\wedge\ldots\wedge\mathbf{e}^f e_{acde\ldots f}\\ &= -\frac{\kappa}{2}\int\delta A^a{}_b\left(\mathcal{R}^{cb}{}_{ca}+\mathcal{R}^{bc}{}_{ac}-\delta^b_a\mathcal{R}^{cd}{}_{cd}\right)\boldsymbol{\Phi}\end{aligned}$$

taking care to keep indices in the correct order. Since the first and second pairs of $\mathcal{R}^{ab}{}_{cd}$ retain their antisymmetry, $\mathcal{R}^{cb}{}_{ca}=\mathcal{R}^{bc}{}_{ac}$ the vacuum field equations are

$$-\kappa\left(\mathcal{R}_{ab}-\frac{1}{2}\eta_{ab}\mathcal{R}\right) = 0 \tag{20}$$

$$\frac{\kappa}{2}\mathscr{T}^c{}_{ab} = 0 \tag{21}$$

For all $n > 2$ Eq.(21) immediately leads to vanishing torsion and therefore vanishing contorsion, $\mathbf{C}^a{}_c = 0$. Using Eq.(10) to separate the usual Einstein tensor from the contorsion contributions

$$\boldsymbol{\mathcal{R}}^a{}_b = \mathbf{R}^a{}_b + \mathbf{D}\mathbf{C}^a{}_b - \mathbf{C}^c{}_b \wedge \mathbf{C}^a{}_c$$

and setting $\mathbf{C}^a{}_c = 0$ reduces $\boldsymbol{\mathcal{R}}^a{}_b$ to the usual Einstein equation of Riemannian geometry, $R_{ab} - \frac{1}{2}\eta_{ab}R = 0$. Therefore vacuum Poincarè gauge theory reproduces vacuum general relativity. The theories typically differ when matter fields other than Yang-Mills or Klein-Gordon type are included.

Notice a crucial difference between the solder form variation and the metric variation. The metric variation takes the form

$$\delta S = \delta \int \delta g^{\alpha\beta} \left(\mathcal{R}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \mathcal{R} \right) \sqrt{|g|} d^n x$$

so the symmetry of the metric is induced upon the Einstein tensor to give

$$\mathcal{G}_{(\alpha\beta)} = \mathcal{R}_{(\alpha\beta)} - \frac{1}{2} g_{\alpha\beta} \mathcal{R} = 0$$

This gives ten equations that determine the ten components of the metric. By contrast, the coefficient $\delta A^a{}_b$ of the solder form variation $\delta_e \mathbf{e}^c = \delta A^a{}_b \mathbf{e}^b$ is asymmetric. This results in the vanishing of the entire asymmetric Einstein tensor

$$\mathcal{G}_{\alpha\beta} = 0$$

Accordingly, this determines the sixteen components of the solder form. While an additive term [31, 32] is known to symmetrize the energy tensor–thereby forcing the antisymmetric part of the Einstein tensor to zero–we retain asymmetry on both sides of the gravity equation and find a systematic approach to the antisymmetric part. With the alternate form of the variation, variation of the Lorentz gauge affects only the Einstein equation, accounting for the different number of degrees of freedom due to differing symmetry.

3.3 Fiber preserving variation

While the $(\mathbf{e}^a, \boldsymbol{\omega}^a{}_b)$ form of variation gets us quickly to general relativity, a significant issue arises.

The variables $(\mathbf{e}^a, \boldsymbol{\omega}^a{}_b)$ do not transform independently under the fiber symmetry. Specifically, under Lorentz transformation $\Lambda^a{}_b$ the solder form transforms as a tensor, $\tilde{\mathbf{e}}^a = \Lambda^a{}_b \mathbf{e}^b$ but the spin connection *also* transforms as a local Lorentz connection, $\tilde{\boldsymbol{\omega}} = \Lambda \boldsymbol{\omega} \Lambda^{-1} - d\Lambda\, \Lambda^{-1}$. This means that while the field equations arising from separate $\mathbf{e}^a$ and $\boldsymbol{\omega}^a{}_b$ variations are correct, they will be shuffled by the fiber symmetry. This is most evident with matter sources, where it leads to the difference between the asymmetric canonical energy tensor and the symmetric Belinfante-Rosenfield energy tensor. We show this explicitly with our discussion of sources in the next Section.

We now consider the variation of two Lorentz tensors, $(\mathbf{e}^a, \mathbf{C}^a{}_b)$. Writing $\boldsymbol{\alpha}^a{}_b = \boldsymbol{\alpha}^a{}_b(e)$ places the effect of a lifting in the bundle entirely within the solder form variation. Explicitly separating the compatible and torsion pieces leads in a straightforward way to the Belinfante-Rosenfield energy tensor.

Before we begin, note that when we separate the contorsion parts of the curvature

$$\begin{aligned} S_{ECSK} &= \frac{\kappa}{2(n-2)!} \delta_{e,\alpha(e)} \int \boldsymbol{\mathcal{R}}^{ab} \wedge \mathbf{e}^c \wedge \ldots \wedge \mathbf{e}^d e_{abc\ldots d} + \delta S_{GHY} \\ &= \frac{\kappa}{2(n-2)!} \delta_{\alpha(e)} \int \left(\mathbf{R}^{ab} + \mathbf{D}\mathbf{C}^{ab} - \mathbf{C}^{eb} \wedge \mathbf{C}^a{}_e \right) \wedge \mathbf{e}^c \wedge \ldots \wedge \mathbf{e}^d e_{abc\ldots d} + \delta S_{GHY} \end{aligned}$$

it is tempting to integrate the derivative term by parts and use $\mathbf{D}\mathbf{e}^c = 0$ to set it to zero

$$\int \mathbf{D}\mathbf{C}^{ab} \wedge \mathbf{e}^c \wedge \ldots \wedge \mathbf{e}^d e_{abc\ldots d} = \int \mathbf{C}^{ab} \wedge \mathbf{D}\mathbf{e}^c \wedge \mathbf{e}^e \wedge \ldots \wedge \mathbf{e}^e e_{abcd\ldots e} = 0$$

However, this is inconsistent with the solder form variation

$$(n-2)\int \mathbf{DC}^{ab}\wedge\delta\mathbf{e}^{c}\wedge\ldots\wedge\mathbf{e}^{d}e_{abc\ldots d}=\int \mathbf{C}^{ab}\wedge\mathbf{D}\left(\delta\mathbf{e}^{c}\right)\wedge\mathbf{e}^{e}\wedge\ldots\wedge\mathbf{e}^{e}e_{abcd\ldots e}\neq 0$$

For this reason it is important to vary the action before integrating.

3.3.1 Varying the contorsion

The contorsion variation is straightforward. After variation of the contorsion the compatible derivative term is integrated by parts, where $\mathbf{De}^c$ vanishes.

$$\frac{\kappa}{2\left(n-2\right)!}\int \mathbf{D}\left(\delta\mathbf{C}^{ab}\right)\wedge\mathbf{e}^{c}\wedge\ldots\wedge\mathbf{e}^{d}e_{abc\ldots d}=\frac{\kappa}{2\left(n-3\right)!}\int \delta\mathbf{C}^{ab}\wedge\cancelto{0}{\mathbf{De}^{e}}\wedge\mathbf{e}^{d}\wedge\ldots\wedge\mathbf{e}^{e}e_{abcd\ldots e}=0$$

For the remaining contorsion term

$$\begin{aligned}\delta S_{C} &= -\frac{\kappa}{2\left(n-2\right)!}\int 2\delta\mathbf{C}^{cb}\wedge\mathbf{C}^{a}{}_{c}\wedge\mathbf{e}^{c}\wedge\ldots\wedge\mathbf{e}^{d}e_{abc\ldots d}\\ &= \kappa\int \delta C^{bc}{}_{e}\left(C^{e}{}_{[cb]}-\delta^{e}_{[b}C^{a}{}_{c]a}\right)\mathbf{\Phi}\end{aligned}$$

Substituting $C^{a}{}_{bc}=\frac{1}{2}\left(T_{c}{}^{a}{}_{b}+T^{a}{}_{cb}-T_{bc}{}^{a}\right)$ to express the resulting field equation in terms of the torsion yields

$$\frac{\kappa}{2}\mathscr{T}^{a}{}_{bc} = 0$$

This is the same result as from the original Palatini variation.

3.3.2 Varying the solder form

The solder form variation is now more involved. After setting $\boldsymbol{\mathcal{R}}^{ab}=\mathbf{d}\boldsymbol{\omega}^{ab}-\boldsymbol{\omega}^{cb}\wedge\boldsymbol{\omega}^{a}{}_{c}$ and substituting $\boldsymbol{\omega}^{ab}=\boldsymbol{\alpha}^{ab}+\mathbf{C}^{ab}$ we vary both $\mathbf{e}^a$ and $\boldsymbol{\alpha}^{ab}$ to find

$$\begin{aligned}\delta_{e,\alpha(e)}S_{ECSK} &= \frac{\kappa}{2\left(n-2\right)!}\delta_{e,\alpha(e)}\int \boldsymbol{\mathcal{R}}^{ab}\wedge\mathbf{e}^{c}\wedge\ldots\wedge\mathbf{e}^{d}e_{abc\ldots d}+\delta S_{GHY}\\ &= \frac{\kappa}{2\left(n-2\right)!}\int \boldsymbol{\mathcal{D}}\left(\delta\boldsymbol{\alpha}^{ab}\right)\wedge\mathbf{e}^{c}\wedge\ldots\wedge\mathbf{e}^{d}e_{abc\ldots d}+\delta S_{GHY}\\ &\quad -\kappa\int \delta A^{c}{}_{k}\left(\mathcal{R}^{k}{}_{c}-\frac{1}{2}\mathcal{R}\delta^{k}_{c}\right)\mathbf{\Phi}\end{aligned}$$

From here the handling of the first integral is parallel to that leading up to Eq.(19) but with the compatible connection instead. The result is

$$\delta\mathcal{S}_{ECSK} = \frac{\kappa}{2\left(n-3\right)!}\int \delta\boldsymbol{\alpha}^{ab}\wedge\mathbf{T}^{e}\wedge\mathbf{e}^{f}\ldots\wedge\mathbf{e}^{k}e_{abef\ldots k}-\kappa\int \delta A^{c}{}_{k}\left(\mathcal{R}^{k}{}_{c}-\frac{1}{2}\mathcal{R}\delta^{k}_{c}\right)\mathbf{\Phi}$$

but there is now a further variation using Eq.(18). Substituting and integrating by parts, then replacing the basis forms with the volume form gives an imposing product.

$$\begin{aligned}\delta_{e,\alpha(e)}S_{ECSK} &= \frac{\kappa}{2}\int \frac{1}{2}\left(\delta^{a}_{d}\eta^{bc}-\delta^{b}_{d}\eta^{ac}\right)\delta e_{\mu}{}^{h}\left[-\frac{1}{2}e_{g}{}^{\mu}\delta^{d}_{h}D_{c}T^{e}{}_{mn}+\frac{1}{2}e_{c}{}^{\mu}\eta_{gh}D^{d}T^{e}{}_{mn}+\frac{1}{2}e_{c}{}^{\mu}\delta^{d}_{h}\wedge D_{g}T^{e}{}_{mn}\right]\\ &\quad\times\left(\delta^{m}_{a}\left(\delta^{n}_{b}\delta^{g}_{e}-\delta^{g}_{b}\delta^{n}_{e}\right)+\delta^{n}_{a}\left(\delta^{g}_{b}\delta^{m}_{e}-\delta^{m}_{b}\delta^{g}_{e}\right)+\delta^{g}_{a}\left(\delta^{m}_{b}\delta^{n}_{e}-\delta^{n}_{b}\delta^{m}_{e}\right)\right)\mathbf{\Phi}\\ &\quad -\kappa\int \delta A^{c}{}_{k}\left(\mathcal{R}^{k}{}_{c}-\frac{1}{2}\mathcal{R}\delta^{k}_{c}\right)\mathbf{\Phi}\end{aligned}$$

Distributing and collecting terms eventually leads to

$$\delta\mathcal{S}_{ECSK} = \kappa\int\left(\delta A^{cb}\right)\left(D^a\left(\frac{1}{2}\left(T_{bac}+T_{acb}+T_{cab}\right)+\eta_{ac}T^e{}_{be}-\eta_{bc}T^e{}_{ae}\right)-\left(\mathcal{R}_{bc}-\frac{1}{2}\mathcal{R}\eta_{bc}\right)\right)\mathbf{\Phi}$$

where $\delta\mathbf{e}^a = \delta A^a{}_b\mathbf{e}^b$. Replacing $T_{abc} = \mathscr{T}_{abc} - \eta_{ac}T^e{}_{eb} + \eta_{ab}T^e{}_{ec}$ the resulting field equation takes the simpler form

$$-\kappa\left(\mathcal{R}_{bc}-\frac{1}{2}\mathcal{R}\eta_{bc}-\frac{1}{2}D^a\left(\mathscr{T}_{bac}+\mathscr{T}_{acb}+\mathscr{T}_{cab}\right)\right) = 0$$

This field equation is most revealing when written in terms of symmetric and antisymmetric parts. Together with the contorsion variation we find:

$$\begin{aligned}\mathcal{R}_{(bc)}-\frac{1}{2}\mathcal{R}\eta_{bc} &= \frac{1}{2}D^a\left(\mathscr{T}_{cba}+\mathscr{T}_{bca}\right)\\ \mathcal{R}_{bc}-\mathcal{R}_{cb} &= -D_a\mathscr{T}^a{}_{bc}\\ \mathscr{T}^a{}_{bc} &= 0\end{aligned} \tag{22}$$

Notice the tight relationship between the torsion equation and the antisymmetric part of the Ricci tensor. Combining these imposes symmetry on the Ricci tensor. This is the same conclusion as we reach from the first variation, but with the added insight that the antisymmetric part of the Ricci tensor is the divergence of the contorsion equation, hence zero.

The antisymmetric equality $\mathcal{R}_{[bc]} + \frac{1}{2}D^a\mathscr{T}_{abc} = 0$ is just what we get if we restrict the variation to an infinitesimal Lorentz transformation, $\delta A^{bc} = \varepsilon^{[bc]}$.

Without matter fields, it follows that the torsion and Einstein tensor vanish, in agreement with the purely metric variation of general relativity.

4 ECSK theory with matter

ESCK theory with sources differs from general relativity when the source action depends on the connection.

Let the action now be

$$\begin{aligned}S &= S_{ECSK}+S_{matter}\\ &= \frac{\kappa}{2(n-2)!}\int\boldsymbol{\mathcal{R}}^{ab}\wedge\mathbf{e}^c\wedge\ldots\wedge\mathbf{e}^d e_{abc\ldots d}+\int L\left(\xi^A,\mathcal{D}_\mu\xi^A,\mathbf{e}^a\right)\mathbf{\Phi}\end{aligned}$$

for fields ξ^A of any type. Returning to the Palatini approach we vary the connection $\boldsymbol{\omega}^a{}_b$ and the solder form $\mathbf{e}^a$ to find

$$\begin{aligned}0 &= -\int\delta A^{ba}\left[\kappa\left(R_{ab}-\frac{1}{2}R\eta_{ab}\right)-\frac{\delta L}{\delta e_\mu{}^b}e_\mu{}^c\eta_{ca}\right]\mathbf{\Phi}\\ &\quad+\int\delta\omega^{ab}{}_c\left[\frac{\kappa}{2}\mathscr{T}^c{}_{ab}+\frac{\delta L}{\delta\omega^{ab}{}_c}\right]\mathbf{\Phi}\end{aligned}$$

Here the solder form variation is written as $\delta\mathbf{e}^a = \delta A^a{}_b\mathbf{e}^b$ for arbitrary $\delta A^a{}_b$. With the solder form and spin connection as independent variables there is a natural association of sources with the curvature and the torsion.

$$\begin{aligned}\kappa\left(R_{ab}-\frac{1}{2}R\eta_{ab}\right) &= \frac{\delta L}{\delta e_\mu{}^b}e_\mu{}^c\eta_{ca}\\ \frac{\kappa}{2}\mathscr{T}^c{}_{ab} &= -\frac{\delta L}{\delta\omega^{ab}{}_c}\end{aligned} \tag{23}$$

The Einstein tensor is sourced by the asymmetric *canonical energy tensor* $T_{ba} = \frac{\delta L}{\delta e_\mu{}^b} e_\mu{}^c \eta_{ca}$ while the torsion is sourced by the *spin tensor*

$$\sigma^c{}_{ab} \equiv \frac{\delta L}{\delta \omega^{ab}{}_c} = \frac{\delta L}{\delta C^{ab}{}_c} \tag{24}$$

with $\sigma^c{}_{ab} = -\sigma^c{}_{ba}$.

However, this association depends on the choice of independent variables. As discussed in the previous section, these sources are mixed when we apply the fiber symmetry. For this reason, we now consider the action as a functional of the solder form and contorsion, setting $\boldsymbol{\alpha}^{ab} = \boldsymbol{\alpha}^{ab}(\mathbf{e}^c)$.

Because the contorsion variation leads to the same expression for the torsion as the $\boldsymbol{\omega}^{ab}$ variation, the $\delta\boldsymbol{\omega}^{ab}$ equation remains unchanged. The torsion now has source $\sigma^c{}_{ab}$.

$$\frac{\kappa}{2}\mathscr{T}^c{}_{ab} = -\sigma^c{}_{ab} \tag{25}$$

Before carrying out the solder form variation we show the mixing under the fiber symmetry explicitly.

4.1 Lorentz symmetry

Under local Lorentz transformation, both the solder form and spin connection change. The change in the spin connection is given by the usual gauge form $\tilde{\boldsymbol{\omega}} = g\boldsymbol{\omega}g^{-1} - \mathbf{d}gg^{-1}$. In detail, for an infinitesimal gauge transformation $g^a{}_b = \delta^a{}_b + \varepsilon^a{}_b$ where $\varepsilon_{ab} = -\varepsilon_{ba}$ the change in the spin connection is

$$\begin{aligned}
\delta_L \boldsymbol{\omega}^a{}_b &= \left[\left(\delta^a_c + \varepsilon^a{}_c\right)\boldsymbol{\omega}^c{}_d\left(\delta^d_b - \varepsilon^d{}_b\right) - \mathbf{d}\varepsilon^a{}_c\left(\delta^c_b - \varepsilon^c{}_b\right)\right] - \boldsymbol{\omega}^a{}_b \\
&= -\boldsymbol{\mathcal{D}}\varepsilon^a{}_b
\end{aligned}$$

At the same time the solder form transforms as a Lorentz tensor, $\delta_L \mathbf{e}^a = \varepsilon^a{}_b \mathbf{e}^b$. This means that under an infinitesimal gauge transformation we must include changes in both the solder form and the spin connection.

$$\begin{aligned}
\delta_L S_{matter} \equiv 0 &= \int \frac{\delta L}{\delta e_\mu{}^b}\delta_L e_\mu{}^b \boldsymbol{\Phi} + \int \frac{\delta L}{\delta\omega^{ab}{}_c}\delta_L \omega^{ab}{}_c \boldsymbol{\Phi} \\
&= \int \frac{\delta L}{\delta e_\mu{}^b}\varepsilon^b{}_c e_\mu{}^c \boldsymbol{\Phi} - \int \sigma^c{}_{ab}\mathcal{D}_c\varepsilon^{ab}\boldsymbol{\Phi} \\
&= \int \left(-e_\mu{}^c \eta_{ca}\frac{\delta L}{\delta e_\mu{}^b} + \mathcal{D}_c\sigma^c{}_{ab}\right)\varepsilon^{ab}\boldsymbol{\Phi}
\end{aligned}$$

Here we may require the variation to vanish on the boundary. Since $\varepsilon^{ab} = -\varepsilon^{ba}$ is otherwise arbitrary, the antisymmetric part of the direct solder form variation must equal the divergence of the spin tensor.

$$\mathcal{D}_c\sigma^c{}_{ab} + e_\mu{}^c\frac{\delta L}{\delta e_\mu{}^{[a}}\eta_{b]c} = 0 \tag{26}$$

More importantly, the choice of independent variables determines the form of the energy tensor. Respecting the bundle structure we include the dependence of the compatible part of the spin connection on the solder form, $\boldsymbol{\alpha}^a{}_b = \boldsymbol{\alpha}^a{}_b(e)$ when we carry out the solder form variation.

Varying the solder form and the contorsion independently, and using Eq.(24) for the contorsion variation

$$\begin{aligned}
0 &= -\kappa\int \eta^{ce}e_e{}^\mu\delta e_\mu{}^b\left(\mathcal{R}_{bc} - \frac{1}{2}\mathcal{R}\eta_{bc} - \frac{1}{2}D^a\left(\mathscr{T}_{bac} + \mathscr{T}_{acb} + \mathscr{T}_{cab}\right)\right)\boldsymbol{\Phi} \\
&\quad + \int\left(\frac{\delta L}{\delta e_\mu{}^d} + \frac{\delta L}{\delta\alpha^{ab}{}_c}\frac{\delta\alpha^{ab}{}_c}{\delta e_\mu{}^d}\right)\delta e_\mu{}^d\boldsymbol{\Phi} \\
0 &= \int_V \delta C^{ab}{}_c\left(\frac{\kappa}{2}\mathscr{T}^c{}_{ab} + \sigma^c{}_{ab}\right)\boldsymbol{\Phi}
\end{aligned}$$

Next, carry out the solder form variation $\frac{\delta L}{\delta e_{\mu}{}^{d}}+\frac{\delta L}{\delta\alpha^{ab}{}_{c}}\frac{\delta\alpha^{ab}{}_{c}}{\delta e_{\mu}{}^{d}}$ in detail.

4.2 Variation of the solder form

The source for the Einstein equation now depends on

$$\delta_{e}S_{matter} = \int\left(\frac{\delta L}{\delta e_{\mu}{}^{d}}+\frac{\delta L}{\delta\alpha^{ab}{}_{c}}\frac{\delta\alpha^{ab}{}_{c}}{\delta e_{\mu}{}^{d}}\right)e_{\mu}{}^{e}\delta A^{d}{}_{e}\mathbf{\Phi}$$

Setting $\frac{\delta L}{\delta\alpha^{ab}{}_{c}}=\frac{\delta L}{\delta C^{ab}{}_{c}}=\sigma^{c}{}_{ab}$ this becomes $\delta_{e}S_{matter}=\int\left(\frac{\delta L}{\delta e_{\mu}{}^{d}}\delta e_{\mu}{}^{d}+\sigma^{c}{}_{ab}\delta\alpha^{ab}{}_{c}\right)\mathbf{\Phi}$. Then substituting (18) and integrating by parts

$$\begin{aligned}\delta_{e}S_{matter} &= \int\left(\frac{\delta L}{\delta e_{\mu}{}^{d}}+\frac{1}{2}\left(\delta^{a}_{d}\eta^{be}-\delta^{b}_{d}\eta^{ae}\right)\left[-e_{c}{}^{\mu}D_{e}\sigma^{c}{}_{ab}+D^{d}\sigma^{c}{}_{ab}e_{e}{}^{\mu}\eta_{cd}+D_{c}\sigma^{c}{}_{ab}e_{e}{}^{\mu}\right]\right)\delta e_{\mu}{}^{d}\mathbf{\Phi}\\ &= \int\left(\frac{\delta L}{\delta e_{\mu}{}^{d}}+D^{a}\sigma^{e}{}_{ad}-D_{c}\sigma^{ce}{}_{d}-D^{a}\sigma_{d}{}^{e}{}_{a}\right)e_{e}{}^{\mu}\delta e_{\mu}{}^{d}\mathbf{\Phi}\end{aligned}$$

Combining this with the curvature contributions the field equation becomes

$$\kappa\left(\mathcal{R}_{bc}-\frac{1}{2}\mathcal{R}\eta_{bc}\right)-\frac{\kappa}{2}D^{a}\left(\mathscr{T}_{bac}+\mathscr{T}_{cab}+\mathscr{T}_{acb}\right) = T_{bc}+D^{a}\left(\sigma_{bac}+\sigma_{cab}-\sigma_{acb}\right) \tag{27}$$

The source for the gravitational part is the Belinfante-Rosenfield energy tensor

$$T^{BR}_{bc} = T_{bc}+D^{a}\left(\sigma_{cab}-\sigma_{acb}-\sigma_{bca}\right)$$

and the antisymmetric part vanishes by Lorentz invariance

$$T^{BR}_{[bc]} = e_{\mu}{}^{e}\frac{\delta L}{\delta e_{\mu}{}^{[b}}\eta_{c]e}+D^{a}\sigma_{abc}=0$$

Notice that the torsion and spin tensor terms match up exactly into $\frac{\kappa}{2}\mathscr{T}_{bac}+\sigma_{bac}$ combinations.

4.3 Collected field equations

Separating symmetric and antisymmetric parts and collecting the results,

$$\kappa\left(\mathcal{R}_{(bc)}-\frac{1}{2}\mathcal{R}\eta_{bc}\right) = e_{\mu}{}^{e}\frac{\delta L}{\delta e_{\mu}{}^{(b}}\eta_{c)e}+D^{a}\left(\left(\frac{\kappa}{2}\mathscr{T}_{bac}+\sigma_{bac}\right)+\left(\frac{\kappa}{2}\mathscr{T}_{cab}+\sigma_{cab}\right)\right) \tag{28}$$

$$\kappa\mathcal{R}_{[bc]}+\frac{\kappa}{2}D^{a}\mathscr{T}_{abc} = 0 = D^{a}\sigma_{abc}+e_{\mu}{}^{e}\frac{\delta L}{\delta e_{\mu}{}^{[b}}\eta_{c]e} \tag{29}$$

$$\frac{\kappa}{2}\mathscr{T}^{c}{}_{ab}+\sigma^{c}{}_{ab} = 0 \tag{30}$$

We see that the antisymmetric parts vanish by Lorentz invariance (29) and the source for the symmetric Einstein tensor is the *symmetrized* canonical energy, while the spin tensor provides a source for torsion. Imposing this latter condition imposed we are left with the reduced equations

$$\begin{aligned}\kappa\left(\mathcal{R}_{(bc)}-\frac{1}{2}\mathcal{R}\eta_{bc}\right) &= e_{\mu}{}^{e}\frac{\delta L}{\delta e_{\mu}{}^{(b}}\eta_{c)e}\\ \frac{\kappa}{2}\mathscr{T}^{c}{}_{ab} &= -\sigma^{c}{}_{ab}\end{aligned} \tag{31}$$

These are exactly the symmetrized version of the field equations from the $(\mathbf{e}^{a},\boldsymbol{\omega}^{a}{}_{b})$ variation, Eqs.(23).

We examine the torsion produced by scalar, Yang-Mills, Dirac, and Rarita-Schwinger fields and a certain class of bosonic fields. Then we develop notation for general half-integer spin $\frac{1}{2}(2k+1)$ valid for all k. We begin with certain exceptional cases.

5 Sources for torsion

Before considering fields with nonvanishing spin tensor, we note some classes for with $\sigma^a{}_{bc} = 0$. Fields other than these exceptional types generically drive torsion.

5.1 Exceptional cases

There are two important exceptional cases–Klein-Gordon fields and Yang-Mills fields.

5.1.1 Klein-Gordon field

For Klein-Gordon fields, the covariant derivative contains no connection, $D_\mu \phi = \partial_\mu \phi$.

$$S_{KG} = \frac{1}{2} \int \left(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2 \right) \sqrt{|g|} d^n x$$

Appropriately for a scalar field, there is no spin tensor. This holds true for internal multiplets of scalar fields ϕ^i as well.

5.1.2 Yang-Mills fields

Yang-Mills fields comprise the second important class of exceptions. Let $i, j, \ldots$ index the generators of an internal Lie symmetry $g \in \mathcal{G}$, that is, the fiber symmetry of a principal fiber bundle. Then the connection satisfies the Maurer-Cartan equation, $\mathbf{dA}^i = -\frac{1}{2} c^i{}_{jk} \mathbf{A}^j \wedge \mathbf{A}^k$ where $c^i{}_{jk}$ are the structure constants. Curving the bundle the field strength

$$\mathbf{F}^i = \mathbf{dA}^i + \frac{1}{2} c^i{}_{jk} \mathbf{A}^j \wedge \mathbf{A}^k$$

is independent of the spacetime connection and the corresponding action

$$S = \int \mathbf{F}^i \wedge {}^*\mathbf{F}_i$$

has vanishing spin density. The result also holds for p-form electromagnetism [33].

These observations mean that the Higgs and Yang-Mills fields of the standard model do not drive torsion.

5.2 Bosonic matter sources

The currents of generic bosonic sources have nonvanishing spin tensors. We consider source fields of arbitrary integer spin $\Theta^{a \ldots b}$ having quadratic kinetic energies.

When the kinetic term of the fields is symmetric in derivatives we have

$$S_{kinetic} = \frac{1}{2} \int Q_{a \ldots bc \ldots d} \boldsymbol{\mathcal{D}} \Theta^{a \ldots b}\, {}^*\boldsymbol{\mathcal{D}} \Theta^{c \ldots d}$$

where $Q_{a \ldots bc \ldots d} = Q_{c \ldots da \ldots b}$ for some invariant tensor field Q. The contracted labels play no role in the solder form variation, so we may write them collectively as $A = a \ldots b, B = c \ldots d$. The action is then

$$S_{kinetic} = \frac{1}{4} \int Q_{AB} \boldsymbol{\mathcal{D}} \Theta^A\, {}^*\boldsymbol{\mathcal{D}} \Theta^B$$

where we assume $Q_{AB} = Q_{BA}$ is independent of the connection, though it may depend on the metric.

The field equations (28)-(30) or the reduced equations (31) hold without modification. We need only find the relevant variations of the matter actions.

For these fields the solder form variation only enters through the metric variation as $\eta^{ab}\left(\delta e_a{}^{\mu} e_b{}^{\nu} + e_a{}^{\mu}\delta e_b{}^{\nu}\right) = \delta g^{\mu\nu}$ since

$$\begin{aligned}
S_{kinetic} &= \frac{1}{2}\int Q_{AB}\boldsymbol{\mathcal{D}}\Theta^{A}\,{}^{*}\boldsymbol{\mathcal{D}}\Theta^{B} \\
&= \frac{1}{2}\int Q_{AB}\left(g\right)g^{\mu\nu}\mathcal{D}_{\mu}\Theta^{A}\mathcal{D}_{\nu}\Theta^{B}\sqrt{-g}d^{n}x
\end{aligned}$$

Therefore the energy tensor takes the usual symmetric form plus any (symmetric) dependence on Q_{AB}.

$$T_{ab} = Q_{AB}\mathcal{D}_{a}\Theta^{A}\mathcal{D}_{b}\Theta^{B} - \frac{1}{4}\eta_{ab}\left(Q_{AB}g^{\mu\nu}\mathcal{D}_{\mu}\Theta^{A}\mathcal{D}_{\nu}\Theta^{B}\right) + e_a{}^{\mu}e_b{}^{\nu}\frac{\delta Q_{AB}}{\delta g^{\mu\nu}}$$

despite the asymmetric solder form variation.

However, the connection variation leads to a nonvanishing spin density. Restoring $A \to a\ldots b, B \to c\ldots d$

$$\begin{aligned}
\delta_{\omega}S_{kinetic} &= \frac{1}{4\left(n-1\right)!}\delta_{\omega}\int Q_{a\ldots bc\ldots d}\left(\mathbf{d}\Theta^{a\ldots b} + \Theta^{e\ldots b}\boldsymbol{\omega}^{a}{}_{e} + \ldots + \Theta^{a\ldots e}\boldsymbol{\omega}^{b}{}_{e}\right)\,{}^{*}\mathbf{D}\Theta^{c\ldots d} \\
&= \frac{1}{2}\int \delta\omega_{feg}Q_{am\ldots nbc\ldots d}\left(\eta^{a[f}\Theta^{e]m\ldots nb} + \ldots + \eta^{b[f}\Theta^{|am\ldots n|e]}\right)D^{g}\Theta\,\Theta^{c\ldots d}\mathbf{\Phi}
\end{aligned}$$

The spin tensor is therefore

$$\sigma_{g}{}^{fe} = \frac{1}{2}Q_{am\ldots nbc\ldots d}\left(\eta^{a[f}\Phi^{e]m\ldots nb} + \ldots + \eta^{b[f}\Phi^{|am\ldots n|e]}\right)D_{g}\Phi^{c\ldots d} \tag{32}$$

This has the form of a current density.

From Lorentz invariance Eq.(26) and the symmetry of the energy tensor $T_{[ab]} = 0$ we immediately have conservation of the spin tensor

$$D_{c}\sigma^{c}{}_{ab} = 0 \tag{33}$$

We conclude that for the types of bosonic action considered the Poincarè gauge equations take the form

$$\begin{aligned}
\kappa\left(\mathcal{R}_{(ab)} - \frac{1}{2}\mathcal{R}\eta_{ab}\right) &= Q_{AB}D_{a}\Theta^{A}D_{b}\Theta^{B} - \frac{1}{2}\eta_{ab}\left(Q_{AB}g^{\mu\nu}D_{\mu}\Theta^{A}D_{\nu}\Theta^{B}\right) \\
\frac{\kappa}{2}\mathscr{T}_{c}{}^{ab} &= -\frac{1}{2}Q_{dm\ldots nef\ldots g}\left(\eta^{d[b}\Theta^{a]m\ldots ne} + \ldots + \eta^{e[b}\Theta^{|dm\ldots n|e]}\right)D_{c}\Theta^{f\ldots g}
\end{aligned}$$

Coupling such higher spin fields to other sources may lead to failure of causality or other pathologies.

For example, for a vector field with $Q_{ab} = \eta_{ab}$ the kinetic action is simply

$$S_{kinetic} = \frac{1}{2}\int g^{\mu\nu}g^{\alpha\beta}\mathcal{D}_{\mu}\Theta_{\nu}\mathcal{D}_{\alpha}\Theta_{\beta}\sqrt{-g}d^{n}x$$

so the energy tensor has the usual form and the current density is simply $\sigma_{\mu}{}^{ab} = \frac{1}{2}\left(\Theta^{b}D_{\mu}\Theta^{a} - \Theta^{a}D_{\mu}\Theta^{b}\right)$. The field equations are

$$\begin{aligned}
T_{ab} &= \eta_{cd}D_{a}\Theta^{c}D_{b}\Theta^{d} - \frac{1}{2}\eta_{ab}\left(Q_{cd}g^{\mu\nu}D_{\mu}\Theta^{c}D_{\nu}\Theta^{d}\right) \\
\sigma_{c}{}^{ab} &= \frac{1}{4}\left(\Theta^{b}D_{c}\Theta^{a} - \Theta^{a}D_{c}\Theta^{b}\right)
\end{aligned}$$

The torsion remains nonpropagating and vanishes whenever the source field Θ^{b} vanishes.

5.3 Dirac fields with torsion

It is well-known that the Dirac field provides a source for torsion (among the earliest references see, e.g., [34, 35, 36, 37, 38, 18, 39]). The flat space Dirac action takes the same form in any dimension

$$\mathcal{S}_D = \alpha \int \left(\bar{\psi} \left(i\partial\!\!\!/ - m \right) \psi \right) e d^n x \tag{34}$$

where $\partial\!\!\!/ = \gamma^a e_a{}^{\mu} \partial_\mu$. The principal difference in dimension n is that the spinors are representations of $Spin\,(p,q)$ and therefore elements of a $2^{\left[\frac{n}{2}\right]}$-dimensional complex vector space while the γ^a satisfy the Clifford algebra relations

$$\left\{ \gamma^a, \gamma^b \right\} = -2\eta^{ab} 1 \tag{35}$$

where η_{ab} is the (p,q) metric.

However, in a curved space the spin connection introduces an additional term. The covariant derivative of a spinor is given by

$$D_\mu \psi = \partial_\mu \psi - \frac{1}{2} \omega^{bc}{}_{\mu} \sigma_{bc} \psi$$

where $\sigma_{bc} = [\gamma_b, \gamma_c]$. The action becomes

$$\begin{aligned} \tilde{\mathcal{S}}_D &= \alpha \int \left(\bar{\psi} \left(iD\!\!\!\!/ - m \right) \psi \right) e d^n x \\ &= \alpha \int \left(\psi^\dagger h \left(i e_a{}^{\mu} \gamma^a D_\mu - m \right) \psi \right) e d^n x \end{aligned}$$

where h is Hermitian $h^\dagger = h$ and reality of a vector $v^a = \psi^\dagger h \gamma^a \psi$ under $Spin\,(p,q)$ requires

$$\gamma^{a\dagger} h = h \gamma^a$$

It follows that $\sigma^{ab\dagger} h = -h \sigma^{ab}$. While h is generally taken to be γ^0 in spacetime, h transforms as a $\left(\begin{array}{c} 0 \\ 2 \end{array} \right)$ spin tensor while γ^0 transforms as a $\left(\begin{array}{c} 1 \\ 1 \end{array} \right)$ spin tensor so that $h = \gamma^0$ can hold only in a fixed basis. There exist satisfactory choices for h in any dimension or signature (see below). The solder form components $e_a{}^{\mu}$ connect the orthonormal basis of the Clifford algebra to the coordinate basis for the covariant derivative, $\gamma^a e_a{}^{\mu} D_\mu$.

The conjugate action now differs,

$$\tilde{\mathcal{S}}_D^* = \alpha \int \left(\bar{\psi} \left(-i \overleftarrow{D}_\mu \gamma^\mu - m \right) \psi \right) e d^n x$$

so we take the manifestly real combination

$$\begin{aligned} \mathcal{S}_D &= \frac{1}{2} \left(\tilde{\mathcal{S}}_D + \tilde{\mathcal{S}}_D^* \right) \\ &= \frac{\alpha}{2} \int \bar{\psi} \left(i\gamma^a \overrightarrow{\partial}_a - i \overleftarrow{\partial}_a \gamma^a - 2m - \frac{i}{2} \omega_{bca} \left\{ \gamma^a, \sigma^{bc} \right\} \right) \psi \, e d^n x \end{aligned}$$

showing that the connection now couples to a triple of Dirac matrices $-\frac{i}{2} \omega_{bca} \left\{ \gamma^a, \sigma^{bc} \right\} = -2i \omega_{bca} \gamma^{[a} \gamma^b \gamma^{c]}$. This form is valid in any dimension. In 4- or 5-dimensions the triple antisymmetrization may be shortened using γ_5. The action is now

$$\mathcal{S}_D = \alpha \int \bar{\psi} \left(\frac{i}{2} e_a{}^{\mu} \bar{\psi} \gamma^a \overleftrightarrow{\partial}_\mu \psi - m - i e_a{}^{\mu} \omega_{bc\mu} \gamma^{[a} \gamma^b \gamma^{c]} \right) \psi \, e d^n x \tag{36}$$

where $\bar{\psi}\gamma^a \overleftrightarrow{\partial}_\mu \psi = \bar{\psi}\gamma^a\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma^a\psi$.

The simple form for the anticommutator turns out to be a low-dimensional accident. In the Appendix we show that the general form for

It is convenient to define $\Gamma^{a_1 a_2 \ldots a_k} \equiv \gamma^{[a_1}\gamma^{a_2}\ldots\gamma^{a_k]}$, including the particular cases $\Gamma = 1$ and $\sigma^{ab} = \left[\gamma^a, \gamma^b\right]$ for the $Spin\,(p,q)$ generators. For $k < \frac{n}{2}$ we may write $\Gamma^{a_1 a_2 \ldots a_k}$ in terms of $\gamma_5 \equiv i^m \Gamma^{a_1 \ldots a_n}$ and $\Gamma^{a_1 a_2 \ldots a_{n-k}}$, where i^m is chosen so that $\gamma_5^\dagger = \gamma_5$.

The simple form for the anticommutator turns out to be a low-dimensional accident. In the Appendix we show that the general form for the anticommutator $\left\{\Gamma^{a_1 a_2 \ldots a_k}, \sigma^{bc}\right\}$ depends on both $\Gamma^{a_1 a_2 \ldots a_{k+1}}$ and $\Gamma^{a_1 a_2 \ldots a_{k-1}}$ with the second form absent for the Dirac $k = 1$ case.

5.3.1 Spinor metric

The Clifford relation for the gamma matrices is

$$\left\{\gamma^a, \gamma^b\right\} = -2\eta^{ab}$$

with $\eta^{ab} = diag\,(-1, \ldots, -1, 1, \ldots, 1)$. Here the γ-matrices are numbered $\gamma^1 \ldots \gamma^q \gamma^{q+1} \ldots \gamma^{q+p}$ and we take the first q matrices hermitian. Then for $a, b \leq q$ the γs satisfy the timelike Clifford relation

$$\left\{\gamma^a, \gamma^b\right\} = -\eta^{ab} = +1$$

The final p γs must be antihermitian to give hermiticities of σ^{ab} appropriate for generating both rotations and boosts.

We seek a spinor metric h such that both the spinor inner product

$$\langle\psi, \psi\rangle = \psi^{\dagger A} h_{AB} \psi^B$$

and the n-vector

$$v^a \equiv \psi^\dagger h \gamma^a \psi$$

are real. These immediately imply

$$\begin{aligned} h^\dagger &= h \\ \gamma^{a\dagger} h &= h\gamma^a \end{aligned}$$

To satisfy the second condition we take h proportional to the product of all timelike γs, $h = \lambda\gamma^1 \ldots \gamma^q$. This insures that $\gamma^{a\dagger} h = (-1)^{q-1} h\gamma^a$ with the same sign for all γ^a. Then hermiticity requires $\lambda = i^{\frac{q(q-1)}{2}}$.

This is all we need for q odd. When q is even we include an additional factor of γ_5 where $\gamma_5 = i^{p + \frac{n(n-1)}{2}} \gamma^1 \ldots \gamma^n$. In this case we must also include an additional i^q. Therefore we define

$$h = \begin{cases} i^q i^{\frac{q(q-1)}{2}} \gamma^1\gamma^2 \cdots \gamma^q \gamma_5 & q\ even \\ i^{\frac{q(q-1)}{2}} \gamma^1\gamma^2 \cdots \gamma^q & q\ odd \end{cases}$$

Adopting the usual notation, we may now let $\bar{\psi} = \psi^\dagger h$ for spinors in any dimension. We note that $\gamma_5 h = (-1)^q h\gamma_5$

5.3.2 Energy tensor and spin density from the Dirac equation

From the action (36) the energy tensor and spin current are immediate. Since the Dirac Lagrangian is proportional to the Dirac equation, there is no contribution from the volume form. Therefore the source for the Einstein tensor is

$$\frac{\delta L}{\delta e_\mu{}^b} e_\mu{}^c \eta_{ca} = -i\alpha\bar{\psi}\gamma_a e_b{}^\mu \overleftrightarrow{\partial}_\mu \psi + 2i\alpha\eta_{ac}\omega_{deb}\bar{\psi}\Gamma^{cde}\psi$$

giving the reduced curvature equation (31) the form

$$\kappa\left(R_{ab}-\frac{1}{2}R\eta_{ab}\right) = -i\alpha\bar{\psi}\gamma_{(a}e_{b)}{}^{\mu}\overleftrightarrow{D}_{\mu}\psi+2i\alpha\omega_{de(b}\eta_{a)c}\bar{\psi}\Gamma^{cde}\psi$$

with $2i\alpha\omega_{de(b}\eta_{a)c}\bar{\psi}\Gamma^{cde}\psi$ becoming the axial current $\alpha\omega^{cd}{}_{(a}\varepsilon_{b)cde}\bar{\psi}\gamma^{e}\gamma_5\psi$ in 4-dimensions.

The spin density is

$$\begin{aligned}\sigma^{cab} &\equiv \frac{\delta L}{\delta\omega_{abc}}\\ &= -i\alpha\bar{\psi}\Gamma^{abc}\psi\end{aligned}$$

so the torsion is given by

$$\frac{\kappa}{2}\mathscr{T}^{cab}=i\alpha\bar{\psi}\Gamma^{abc}\psi$$

This axial current in 4-dimensions. Many studies of torsion in ECSK and generalizations to propagating torsion are restricted to this totally antisymmetric form of $\mathscr{T}^{cab}$.

5.3.3 The general relativity limit

We wish to examine general relativity with coupled Dirac sources. This source still has a spin density, despite the absence of torsion, and it is necessary to determine whether this puts a constraint on the Dirac field.

With vanishing torsion the connection reduces the connection is compatible, $\omega^{bc}{}_{\mu}\to\alpha^{bc}{}_{\mu}$ though the action must still be made real by adding the conjugate. From the curvature field equation Eq.(27) with $\mathscr{T}_{abc}=0$,

$$\kappa\left(\mathcal{R}_{bc}-\frac{1}{2}\mathcal{R}\eta_{bc}\right) = T_{bc}+D^{a}\left(\sigma_{bac}+\sigma_{cab}-\sigma_{acb}\right)$$

Although there is nonvanishing spin density there is no second field equation. There is now an antisymmetric part to the Einstein equation.

$$0 = T_{[bc]}+D^{a}\sigma_{abc}$$

This is exactly the part that vanishes by Lorentz symmetry. The Einstein equation therefore reduces to the symmetric expression

$$\kappa\left(R_{bc}-\frac{1}{2}R\eta_{bc}\right) = T_{(bc)}+D^{a}\left(\sigma_{bac}+\sigma_{cab}\right)$$

where the spin tensor is the antisymmetric current

$$\sigma^{cab}\equiv\frac{\delta L}{\delta\omega_{abc}}=-i\alpha\bar{\psi}\Gamma^{abc}\psi$$

Because this is totally antisymmetric, $\sigma_{bac}+\sigma_{cab}=0$ and we recover the Einstein equation with the usual symmetrized energy tensor and no additional coupling.

$$\kappa\left(R_{bc}-\frac{1}{2}R\eta_{bc}\right) = T_{(bc)}$$

Therefore, despite nonvanishing spin tensor, Dirac fields make only the expected contribution to the field equation of general relativity with no additional constraint.

5.4 Rarita-Schwinger

The spin-$\frac{3}{2}$ Rarita-Schwinger field [40] is known to give rise to acausal behavior when coupled to other fields [41]. This problem is overcome when a spin-$\frac{3}{2}$ field representing the gravitino is coupled supersymmetrically. Therefore, we first examine the 11-dimensional supergravity Lagrangian.

5.4.1 11-d Supergravity

Here the basic Lagrangian

$$\mathcal{L} = \frac{1}{2\kappa^2} eR - \frac{1}{2} e\overline{\psi}_\mu \Gamma^{\mu\nu\alpha} D_\nu \psi_\alpha + \frac{1}{48} eF^2_{\mu\nu\alpha\beta}$$

includes the scalar curvature R, the spin-$\frac{3}{2}$ Majorana gravitino field ψ_α, and a complex 4-form field built from a 3-form potential as $\mathbf{F} = \mathbf{dA}$. The covariant derivative has connection $\boldsymbol{\omega}^a{}_b$ and $\gamma^\mu = e_a{}^\mu \gamma^a$.

This starting Lagrangian is augmented by ψ_α-$\mathbf{F}$ coupling terms and a Chern-Simons term required to enforce the supersymmetry ([42, 43, 45, 46]). The result is the Lagrangian for 11D supergravity, first found by Cremmer, Julia and Scherk [45].

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2\kappa^2} eR - \frac{1}{2} e\overline{\psi}_\mu \Gamma^{\mu\nu\alpha} D_\nu \left[\frac{1}{2}\left(\omega - \overline{\omega}\right)\right] \psi_\alpha \\
&\quad + \frac{1}{48} eF^2_{\mu\nu\alpha\beta} + \frac{\sqrt{2}\kappa}{384} e \left(\overline{\psi}_\mu \Gamma^{\mu\nu\alpha\beta\rho\sigma} \psi_\sigma + 12 \overline{\psi}^\nu \Gamma^{\alpha\beta} \psi^\rho\right) \left(F + \overline{F}\right)_{\nu\alpha\beta\rho} \\
&\quad + \frac{\sqrt{2}\kappa}{3456} \varepsilon^{\alpha_1 \ldots \alpha_{11}} F_{\alpha_1 \ldots \alpha_4} F_{\alpha_5 \ldots \alpha_8} A_{\alpha_9 \alpha_{10} \alpha_{11}}
\end{aligned}$$

Since we are primarily interested in sources for torsion, we will only need the kinetic term for the Rarita-Schwinger field. While is it possible that supergravity theories–which exist only in certain dimensions–are the only consistent formulation of spin-$\frac{3}{2}$ fields, there may be alternative couplings that allow them. For this reason, we will consider the original Rarita-Schwinger kinetic term in arbitrary dimension as a source for torsion, omitting additional couplings.

5.4.2 The Rarita-Schwinger equation

In flat 4-dimensional space the uncoupled Rarita-Schwinger equation may be written as

$$\varepsilon^{\mu\nu\alpha\beta} \gamma_\nu \gamma_5 \partial_\alpha \psi_\beta + \frac{1}{2} m \sigma^{\mu\beta} \psi_\beta = 0$$

with real action

$$S^0_{RS} = \int \bar{\psi}_\mu \left(\epsilon^{\mu\kappa\rho\nu} \gamma_5 \gamma_\kappa \partial_\rho - \frac{1}{2} m \sigma^{\mu\nu}\right) \psi_\nu$$

In curved spacetime, generalizing to the covariant derivative $\partial_\alpha \psi_\beta \rightarrow \mathcal{D}_\alpha \psi_\beta$ where

$$\mathcal{D}_\alpha \psi_\beta = \partial_\alpha \psi_\beta - \psi_\mu \Gamma^\mu{}_{\beta\alpha} - \frac{1}{2} \omega_{ab\alpha} \sigma^{ab} \psi_\beta$$

we must explicitly make it real. As with the Dirac field, the extra terms give an anticommutator. Noticing that

$$\varepsilon^{\mu\kappa\alpha\nu} \Gamma^\rho{}_{\nu\alpha} = \frac{1}{2} \varepsilon^{\mu\kappa\alpha\nu} T^\rho{}_{\alpha\nu}$$

we have

$$
\begin{aligned}
S_{RS} &= \frac{1}{2}\left(S+S^*\right) \\
&= S^0_{RS} - \frac{1}{2}\int\left(\epsilon^{\mu\kappa\alpha\nu}\left(\frac{1}{2}\bar{\psi}_\mu\gamma_5\gamma_\kappa\psi_\rho T^\rho{}_{\alpha\nu} + \frac{1}{2}\left[\bar{\psi}_\mu\gamma_5\gamma_\kappa\psi_\rho T^\rho{}_{\alpha\nu}\right]^\dagger\right)\right) \\
&\quad +\frac{1}{2}\int\left(-\frac{1}{2}\omega_{ab\alpha}\epsilon^{\mu\kappa\alpha\nu}\left(\bar{\psi}_\mu\gamma_5\gamma_\kappa\sigma^{ab}\psi_\nu + \left[\bar{\psi}_\mu\gamma_5\gamma_\kappa\sigma^{ab}\psi_\nu\right]^\dagger\right)\right)
\end{aligned}
$$

and therefore, taking the adjoint and rearranging

$$
\begin{aligned}
S_{RS} &= S^0_{RS} - \frac{1}{4}\int\epsilon^{\mu\kappa\alpha\nu}\left(\bar{\psi}_\mu\gamma_5\gamma_\kappa\psi_\rho T^\rho{}_{\alpha\nu} + \bar{\psi}_\rho T^\rho{}_{\alpha\nu}\gamma_5\gamma_\kappa\psi_\mu\right) \\
&\quad -\frac{1}{4}\int\omega_{ab\alpha}\epsilon^{\mu\kappa\alpha\nu}\bar{\psi}_\mu\gamma_5\left\{\gamma_\kappa,\sigma^{ab}\right\}\psi_\nu
\end{aligned}
$$

The explicit torsion coupling here is surprising, and forces us to be clear about the independent variables. We may set $\mathbf{T}^a = \mathbf{d}\mathbf{e}^a - \mathbf{e}^b\wedge\boldsymbol{\omega}^a{}_b$ and vary $(\mathbf{e}^a,\boldsymbol{\omega}^a{}_b)$ or we may write $\boldsymbol{\omega}^a{}_b = \boldsymbol{\alpha}^a{}_b(\mathbf{e}^c) + \mathbf{C}^a{}_b$ and write the torsion in terms of the contorsion $\mathbf{T}^a = \mathbf{C}^a{}_b\wedge\mathbf{e}^b$, then vary $(\mathbf{e}^a,\mathbf{C}^a{}_b)$. We choose the latter course, since this respects the Lorentz fiber symmetry and yields the Belinfante-Rosenfield tensor as source. For the spin tensor it makes no difference because

$$
\begin{aligned}
\delta_\omega\mathbf{T}^a &= -\mathbf{e}^b\wedge\delta\boldsymbol{\omega}^a{}_b \\
\delta_C\mathbf{T}^a &= -\mathbf{e}^b\wedge\delta\mathbf{C}^a{}_b
\end{aligned}
$$

Before carrying out the variation we develop Rarita-Schwinger action in higher dimensions.

5.4.3 The Rarita-Schwinger action in arbitrary dimension

To explore higher dimensions we introduce some general notation. Clearly we will need the Hodge dual, but it yields a more systematic result if we combine the dual with the gamma matrices.

Define:

$$
\begin{aligned}
\boldsymbol{\gamma} &\equiv \gamma_a\mathbf{e}^a \\
\boldsymbol{\psi} &\equiv \psi_a\mathbf{e}^a \\
(\wedge\boldsymbol{\gamma})^k &\equiv \gamma_{a_1}\ldots\gamma_{a_k}\mathbf{e}^{a_1}\wedge\ldots\wedge\mathbf{e}^{a_k} \\
\boldsymbol{\Gamma}^k &\equiv {}^*\left[\frac{1}{k!}(\wedge\boldsymbol{\gamma})^k\right]
\end{aligned}
$$

In particular, $\boldsymbol{\Gamma}^0$ is just the volume form $\boldsymbol{\Phi}$.

It is not hard to check that the Dirac case may be written as

$$
S^0_D = \int\left(\bar{\psi}\boldsymbol{\Gamma}^1\wedge i\mathbf{d}\psi - m\bar{\psi}\boldsymbol{\Gamma}^0\psi - \frac{i}{4}\left\{\boldsymbol{\Gamma}^1,\sigma^{cd}\right\}\wedge\boldsymbol{\omega}_{cd}\psi\right)
$$

by expanding the forms.

To rewrite the Rarita-Schwinger action in arbitrary dimensions we replace the volume form and set

$$
\sigma^{\mu\nu} = (-1)^q\left(\frac{1}{2}\sigma^{\rho\sigma}\right)\left(\frac{1}{2}e^{\alpha\beta\mu\nu}e_{\alpha\beta\rho\sigma}\right)
$$

Then

$$
\begin{aligned}
S^0_{RS} &= \int \bar{\psi}_\mu \left(\epsilon^{\mu\kappa\rho\nu} \gamma_5 \gamma_\kappa \partial_\rho - \frac{1}{2} m \sigma^{\mu\nu} \right) \psi_\nu \mathbf{\Phi} \\
&= \int \left(\epsilon^{\mu\kappa\rho\nu} \bar{\psi}_\mu \gamma_5 \gamma_\kappa \partial_\rho \psi_\nu \frac{(-1)^q}{4!} e_{defg} \mathbf{e}^d \wedge \mathbf{e}^e \wedge \mathbf{e}^f \wedge \mathbf{e}^g \right) \\
&\quad - \int \frac{1}{2} m \bar{\psi}_\mu (-1)^q \frac{1}{2} \sigma^{\rho\sigma} \frac{1}{2} e^{\alpha\beta\mu\nu} e_{\alpha\beta\rho\sigma} \psi_\nu \frac{1}{4!} e_{defg} \mathbf{e}^d \wedge \mathbf{e}^e \wedge \mathbf{e}^f \wedge \mathbf{e}^g
\end{aligned}
$$

This allows us to eliminate the 4-dimensional Levi-Civita tensor by reducing the Levi-Civita pairs $\frac{(-1)^q}{4!} \epsilon^{\mu\kappa\rho\nu} e_{defg}$ and $\frac{(-1)^q}{4!} e^{\alpha\beta\mu\nu} e_{defg}$, to combine a solder form with each spinor. Then

$$
\begin{aligned}
S^0_{RS} &= \int \bar{\boldsymbol{\psi}} \wedge \gamma_5 \gamma_e \wedge \mathbf{e}^e \mathbf{d} \boldsymbol{\psi} \\
&\quad - \int \frac{1}{8} m \bar{\boldsymbol{\psi}} \wedge \left(\frac{1}{8} \sigma^{\rho\sigma} e_{\rho\sigma de} \mathbf{e}^d \wedge \mathbf{e}^e \right) \wedge \boldsymbol{\psi}
\end{aligned}
$$

Now set

$$
\gamma_5 \gamma_\kappa \mathbf{e}^\kappa = \frac{i}{3!} \gamma_{[a} \gamma_b \gamma_{c]} \varepsilon^{abc}{}_{\kappa} \mathbf{e}^\kappa = i \mathbf{\Gamma}^3
$$

and

$$
\frac{1}{8} \sigma^{ab} e_{abcd} \mathbf{e}^c \wedge \mathbf{e}^d = \frac{1}{2!} {}^* \left(\gamma^a \gamma^b \mathbf{e}_a \wedge \mathbf{e}_b \right) = \mathbf{\Gamma}^2
$$

to write the action as

$$
S^0_{RS} = \int \left(\bar{\boldsymbol{\psi}} \wedge \mathbf{\Gamma}^3 \wedge i \mathbf{d} \boldsymbol{\psi} - m \bar{\boldsymbol{\psi}} \wedge \mathbf{\Gamma}^2 \wedge \boldsymbol{\psi} \right) \tag{37}
$$

By using the Hodge dual in $\mathbf{\Gamma}^2$ and $\mathbf{\Gamma}^3$ we have eliminated the specific reference to dimension. Equation (37) is the Rarita-Schwinger action in flat (p, q)-space.

5.4.4 Rarita-Schwinger in curved spaces

To generalize Eq.(37) we now replace the exterior derivative with the covariant exterior derivative

$$
\tilde{S}_{RS} = \int \left(\bar{\boldsymbol{\psi}} \wedge \mathbf{\Gamma}^3 \wedge i \boldsymbol{\mathcal{D}} \boldsymbol{\psi} - m \bar{\boldsymbol{\psi}} \wedge \mathbf{\Gamma}^2 \wedge \boldsymbol{\psi} \right)
$$

keeping the action real by taking $S_{RS} = \frac{1}{2} \left(\tilde{S}_{RS} + \tilde{S}^\dagger_{RS} \right)$. The covariant derivative 2-form $\boldsymbol{\mathcal{D}} \boldsymbol{\psi}$ is

$$
\boldsymbol{\mathcal{D}} \boldsymbol{\psi} = \mathbf{d} \boldsymbol{\psi} - \psi_\mu \mathbf{T}^\mu - \frac{1}{2} \boldsymbol{\omega}_{mn} \sigma^{mn} \wedge \boldsymbol{\psi} \tag{38}
$$

Therefore, the direct torsion-Rarita-Schwinger coupling will occur in higher dimensions as well.

Expanding the action and separating the free contribution

$$
\begin{aligned}
S_{RS} &= S^0_{RS} + \frac{1}{2} \int \left(\bar{\boldsymbol{\psi}} \wedge \mathbf{\Gamma}^3 \wedge (-i \psi_\mu \mathbf{T}^\mu) + \left(\bar{\boldsymbol{\psi}} \wedge \mathbf{\Gamma}^3 \wedge (-i \psi_\mu \mathbf{T}^\mu) \right)^\dagger \right) \\
&\quad + \frac{1}{2} \int \left(\bar{\boldsymbol{\psi}} \wedge \mathbf{\Gamma}^3 \wedge \left(-\frac{i}{2} \boldsymbol{\omega}_{mn} \wedge \sigma^{mn} \boldsymbol{\psi} \right) + \left(\bar{\boldsymbol{\psi}} \wedge \mathbf{\Gamma}^3 \wedge \left(-\frac{i}{2} \boldsymbol{\omega}_{mn} \wedge \sigma^{mn} \boldsymbol{\psi} \right) \right)^\dagger \right)
\end{aligned}
$$

The conjugate torsion piece us given by

$$\frac{1}{2}\int\left(\bar{\boldsymbol{\psi}}\wedge\boldsymbol{\Gamma}^{3}\wedge\left(-i\psi_{m}\mathbf{T}^{m}\right)\right)^{\dagger}=-\frac{i}{2}\int\left(-1\right)^{n+1}\bar{\psi}_{m}\mathbf{T}^{m}\wedge\boldsymbol{\Gamma}^{3}\wedge\boldsymbol{\psi}$$

and the conjugate spin connection piece becomes

$$\frac{1}{2}\int\left(\bar{\boldsymbol{\psi}}\wedge\boldsymbol{\Gamma}^{3}\wedge\left(-\frac{i}{2}\boldsymbol{\omega}_{mn}\wedge\sigma^{mn}\boldsymbol{\psi}\right)\right)^{\dagger} = \frac{1}{2}\int\left[\bar{\boldsymbol{\psi}}\wedge\sigma^{mn}\boldsymbol{\Gamma}^{3}\left(-\frac{i}{2}\wedge\boldsymbol{\omega}_{mn}\right)\wedge\boldsymbol{\psi}\right]$$

Therefore, the full action is

$$\begin{aligned}
S_{RS} &= \int\left(\bar{\boldsymbol{\psi}}\wedge\boldsymbol{\Gamma}^{3}\wedge i\mathbf{d}\boldsymbol{\psi}-m\bar{\boldsymbol{\psi}}\wedge\boldsymbol{\Gamma}^{2}\wedge\boldsymbol{\psi}\right)\\
&\quad-\frac{i}{2}\int\left(\bar{\boldsymbol{\psi}}\wedge\boldsymbol{\Gamma}^{3}\wedge\mathbf{T}^{a}\psi_{a}-\left(-1\right)^{n}\mathbf{T}^{a}\bar{\psi}_{a}\wedge\boldsymbol{\Gamma}^{3}\wedge\boldsymbol{\psi}\right)\\
&\quad-\frac{i}{4}\int\bar{\boldsymbol{\psi}}\wedge\left\{\boldsymbol{\Gamma}^{3},\sigma^{cd}\right\}\wedge\boldsymbol{\omega}_{cd}\wedge\boldsymbol{\psi}
\end{aligned}$$

The anticommutator is

$$\begin{aligned}
\left\{\gamma^{[a_1}\gamma^{a_2}\gamma^{a_3]},\sigma^{de}\right\} &= 4\sum_{a_1<a_2<a_3}\left(\gamma^{[a_1}\gamma^{a_2}\gamma^{a_3}\gamma^{d}\gamma^{e]}-\left(\eta^{a_1d}\eta^{a_2e}-\eta^{a_2d}\eta^{a_1e}\right)\eta^{dd}\eta^{ee}\gamma^{a_3}\right.\\
&\quad\left.+\left(\eta^{a_1d}\eta^{a_3e}-\eta^{a_3d}\eta^{a_1e}\right)\eta^{dd}\eta^{ee}\gamma^{a_2}-\left(\eta^{a_2d}\eta^{a_3e}-\eta^{a_3d}\eta^{a_2e}\right)\eta^{dd}\eta^{ee}\gamma^{a_1}\right)
\end{aligned}$$

so the Rarita-Schwinger spin tensor contains couplings involving $\boldsymbol{\Gamma}^{1},\boldsymbol{\Gamma}^{3},\boldsymbol{\Gamma}^{5}$.

5.4.5 The Rarita-Schwinger spin tensor

Varying the action with respect to the spin connection or contorsion

$$\begin{aligned}
\delta_{\omega}S_{RS} &= -\int\frac{i}{2}\bar{\boldsymbol{\psi}}\wedge\boldsymbol{\Gamma}^{3}\wedge\left(-\mathbf{e}^{b}\wedge\delta\boldsymbol{\omega}^{a}{}_{b}\right)\psi_{a}-\frac{i}{2}\int\left(-1\right)^{n+1}\left(-\mathbf{e}^{b}\wedge\delta\boldsymbol{\omega}^{a}{}_{b}\right)\bar{\psi}_{a}\wedge\boldsymbol{\Gamma}^{3}\wedge\boldsymbol{\psi}\\
&\quad+\frac{i}{4}\int\bar{\boldsymbol{\psi}}\wedge\left\{\boldsymbol{\Gamma}^{3},\sigma^{b}{}_{a}\right\}\wedge\delta\boldsymbol{\omega}^{a}{}_{b}\wedge\boldsymbol{\psi}
\end{aligned}$$

Expanding the forms, setting $\delta\boldsymbol{\omega}^{a}{}_{b}=A^{a}{}_{bc}\mathbf{e}^{c}$, and collecting the basis into volume forms this becomes

$$\delta_{\omega}S_{RS} = \frac{i}{2}\int A_{abc}\left(\bar{\psi}_{e}\gamma^{[e}\gamma^{b}\gamma^{c]}\psi^{a}-\bar{\psi}^{a}\gamma^{[b}\gamma^{e}\gamma^{c]}\psi_{e}-\frac{1}{2}\bar{\psi}_{d}\left\{\gamma^{[d}\gamma^{e}\gamma^{c]},\sigma^{ba}\right\}\psi_{e}\right)\boldsymbol{\Phi}$$

so antisymmetrizing on ab and expanding the anticommutator as

$$\begin{aligned}
\frac{i}{4}\bar{\psi}_{d}\left\{\gamma^{[d}\gamma^{e}\gamma^{c]},\sigma^{ab}\right\}\psi_{e} &= i\bar{\psi}_{d}\gamma^{[a}\gamma^{b}\gamma^{c}\gamma^{d}\gamma^{e]}\psi_{e}+i\left(\eta^{ac}\eta^{bd}-\eta^{bc}\eta^{ad}\right)\bar{\psi}_{d}\gamma^{e}\psi_{e}\\
&\quad+i\left(\eta^{ae}\eta^{bc}-\eta^{ac}\eta^{be}\right)\bar{\psi}_{d}\gamma^{d}\psi_{e}+i\left(\eta^{ad}\eta^{be}-\eta^{ae}\eta^{bd}\right)\bar{\psi}_{d}\gamma^{c}\psi_{e}
\end{aligned}$$

the spin tensor is

$$\begin{aligned}
\sigma^{cab} &= \frac{i}{4}\left(\bar{\psi}_{e}\gamma^{[e}\gamma^{b}\gamma^{c]}\psi^{a}-\bar{\psi}_{e}\gamma^{[e}\gamma^{a}\gamma^{c]}\psi^{b}+\bar{\psi}^{b}\gamma^{[a}\gamma^{e}\gamma^{c]}\psi_{e}-\bar{\psi}^{a}\gamma^{[b}\gamma^{e}\gamma^{c]}\psi_{e}\right)\\
&\quad+i\bar{\psi}_{d}\gamma^{[a}\gamma^{b}\gamma^{c}\gamma^{d}\gamma^{e]}\psi_{e}+i\left(\eta^{ac}\eta^{bd}-\eta^{bc}\eta^{ad}\right)\bar{\psi}_{d}\gamma^{e}\psi_{e}\\
&\quad+i\left(\eta^{ae}\eta^{bc}-\eta^{ac}\eta^{be}\right)\bar{\psi}_{d}\gamma^{d}\psi_{e}+i\left(\eta^{ad}\eta^{be}-\eta^{ae}\eta^{bd}\right)\bar{\psi}_{d}\gamma^{c}\psi_{e}
\end{aligned} \tag{39}$$

After using the torsion equation, the source for the Einstein tensor is always the symmetrized canonical tensor (31) but the torsion is now driven by much more than the axial current. We next use the full spin tensor , Eq.(39), to compute the source for each indepdendent part of the torsion. Since the reduced field equation shows that $\frac{\kappa}{2}\mathscr{T}^{c}{}_{ab} = -\sigma^{c}{}_{ab}$ it suffices to find the trace, totally antisymmetric, and traceless, mixed symmetry parts of σ^{cab}. The corresponding parts of the torsion are proportional to these.

First, the trace of the spin tensor reduces to a simple vector current.

$$\sigma_{c}{}^{cb} = i\left(n-2\right)\left(\bar{\psi}^{b}\gamma^{e}\psi_{e} - \bar{\psi}_{e}\gamma^{e}\psi^{b}\right)$$

For the antisymmetric part there is no change in the totally antisymmetric piece $i\bar{\psi}_{d}\gamma^{[a}\gamma^{b}\gamma^{c}\gamma^{d}\gamma^{e]}\psi_{e}$. Of the last three terms involving metrics, the first two vanish while the antisymmetrization of the third gives

$$\left(i\left(\eta^{ad}\eta^{be} - \eta^{ae}\eta^{bd}\right)\bar{\psi}_{d}\gamma^{c}\psi_{e}\right)_{[abc]} = -2i\bar{\psi}^{[a}\gamma^{b}\psi^{c]}$$

The remaining terms require the abc antisymmetrization of $\bar{\psi}_{e}\gamma^{[e}\gamma^{b}\gamma^{c]}\psi^{a}$ and $\bar{\psi}^{b}\gamma^{[a}\gamma^{e}\gamma^{c]}\psi_{e}$. This is complicated by the existing antisymmetry of ebc. Write these out in detail and collecting terms we find

$$\begin{aligned}
\left(\bar{\psi}_{e}\gamma^{[e}\gamma^{b}\gamma^{c]}\psi^{a}\right)_{[abc]} &= \frac{4}{3}\bar{\psi}_{e}\gamma^{[e}\gamma^{a}\gamma^{b}\psi^{c]} + \frac{1}{3}\bar{\psi}_{e}\gamma^{[a}\gamma^{b}\gamma^{c]}\psi^{e} \\
\left(\bar{\psi}^{a}\gamma^{[b}\gamma^{e}\gamma^{c]}\psi_{e}\right)_{[abc]} &= \frac{4}{3}\bar{\psi}^{[a}\gamma^{b}\gamma^{c}\gamma^{e]}\psi_{e} + \frac{1}{3}\bar{\psi}^{e}\left(\gamma^{[a}\gamma^{b}\gamma^{c]}\right)\psi_{e}
\end{aligned}$$

with the full contribution to $\sigma^{[abc]}$ being $\frac{i}{2}$ times these. Combining everything, the source for the totally antisymmetric part of the torsion is

$$\begin{aligned}
\sigma^{[cab]} &= i\bar{\psi}_{d}\gamma^{[a}\gamma^{b}\gamma^{c}\gamma^{d}\gamma^{e]}\psi_{e} + \frac{2i}{3}\left(\bar{\psi}_{e}\gamma^{[e}\gamma^{a}\gamma^{b}\psi^{c]} - \bar{\psi}^{[a}\gamma^{b}\gamma^{c}\gamma^{e]}\psi_{e}\right) \\
&\quad + \frac{i}{6}\left(\bar{\psi}_{e}\gamma^{[a}\gamma^{b}\gamma^{c]}\psi^{e} - \bar{\psi}^{e}\left(\gamma^{[a}\gamma^{b}\gamma^{c]}\right)\psi_{e}\right) - 2i\bar{\psi}^{[a}\gamma^{b}\psi^{c]}
\end{aligned}$$

containing 1-, 3-, and 5-gamma currents.

The traceless, mixed symmetry part $\tilde{\sigma}^{cab}$ is found by subtracting the trace and antisymmetric pieces.

$$\tilde{\sigma}^{cab} = \sigma^{cab} - \sigma^{[cab]} - \frac{1}{n-1}\left(\eta^{ac}\sigma_{c}{}^{cb} - \eta^{bc}\sigma_{c}{}^{ca}\right)$$

The result is

$$\begin{aligned}
\tilde{\sigma}^{cab} &= \frac{i}{4}\left(\bar{\psi}_{e}\gamma^{[e}\gamma^{b}\gamma^{c]}\psi^{a} - \bar{\psi}_{e}\gamma^{[e}\gamma^{a}\gamma^{c]}\psi^{b} + \bar{\psi}^{b}\gamma^{[a}\gamma^{e}\gamma^{c]}\psi_{e} - \bar{\psi}^{a}\gamma^{[b}\gamma^{e}\gamma^{c]}\psi_{e}\right) \\
&\quad - \frac{2i}{3}\left(\bar{\psi}_{e}\gamma^{[e}\gamma^{a}\gamma^{b}\psi^{c]} - \bar{\psi}^{[a}\gamma^{b}\gamma^{c}\gamma^{e]}\psi_{e}\right) + 2i\bar{\psi}^{[a}\gamma^{b}\psi^{c]} \\
&\quad + \frac{i}{n-1}\eta^{ac}\left(\bar{\psi}^{b}\gamma^{e}\psi_{e} - \bar{\psi}_{e}\gamma^{e}\psi^{b}\right) - \frac{i}{n-1}\eta^{bc}\left(\bar{\psi}^{a}\gamma^{e}\psi_{e} - \bar{\psi}_{e}\gamma^{e}\psi^{a}\right) + i\left(\bar{\psi}^{a}\gamma^{c}\psi^{b} - \bar{\psi}^{b}\gamma^{c}\psi^{a}\right)
\end{aligned}$$

The traceless, mixed symmetry piece therefore depends on 1- and 3-gamma currents.

Therefore, while the Dirac field produces only an axial vector source for torsion, the Rarita-Schwinger field provides a souce for each independent piece. Moreover, since a spin-$\frac{3}{2}$ field in n-dimensions has $n \times 2^{\left[\frac{n}{2}\right]+1}$ degrees of freedom while the torsion has $\frac{1}{2}n^{2}\left(n-1\right)$, generic solutions may be expected to produce generic torsion except in dimensions $n = 5, 7$ or 9.

5.5 Higher spin fermions

We have seen that the vacuum Dirac ($k = 0$) and Rarita-Schwinger ($k = 1$) actions for spin-$\frac{2k+1}{2}$ may be written as

$$\begin{aligned} S^0_{k=0} &= \int \left(\bar{\psi}\mathbf{\Gamma}^1 \wedge i\mathbf{d}\psi - m\bar{\psi}\mathbf{\Gamma}^0\psi\right) \\ S^0_{k=1} &= \int \left(\bar{\boldsymbol{\psi}} \wedge \mathbf{\Gamma}^3 \wedge i\mathbf{d}\boldsymbol{\psi} - m\bar{\boldsymbol{\psi}} \wedge \mathbf{\Gamma}^2 \wedge \boldsymbol{\psi}\right) \end{aligned}$$

The pattern seen here generalizes immediately to higher fermionic spins in any dimension $n \geq 2k + 1$, with the flat space kinetic term depending on $\mathbf{\Gamma}^{2k+1}$ and the mass term depending on $\mathbf{\Gamma}^{2k}$ for spin $\frac{2k+1}{2}$ fields. Including the covariant derivative then adds torsion and anticommutator couplings.

$$\begin{aligned} S_{k=0} &= \int \bar{\psi}\left(\frac{1}{2}\mathbf{\Gamma}^1 \wedge i\overleftrightarrow{\mathbf{d}} - m\mathbf{\Gamma}^0\right)\psi - \frac{i}{4}\bar{\psi}\left\{\mathbf{\Gamma}^1, \sigma^{ef}\right\}\psi \wedge \boldsymbol{\omega}_{ef} \\ S_{k=1} &= \int \bar{\boldsymbol{\psi}} \wedge \left(\mathbf{\Gamma}^3 \wedge i\mathbf{d}\boldsymbol{\psi} - m\mathbf{\Gamma}^2 \wedge \boldsymbol{\psi}\right) \\ &\quad -\frac{i}{2}\int \left(\bar{\boldsymbol{\psi}} \wedge \mathbf{\Gamma}^3 \wedge \mathbf{T}^a\psi_a + (-1)^{n+1}\mathbf{T}^a\bar{\psi}_a \wedge \mathbf{\Gamma}^3 \wedge \boldsymbol{\psi}\right) \\ &\quad -\frac{i}{4}\int \bar{\boldsymbol{\psi}} \wedge \left\{\mathbf{\Gamma}^3, \sigma^{cd}\right\} \wedge \boldsymbol{\omega}_{cd} \wedge \boldsymbol{\psi} \end{aligned}$$

5.5.1 General case definitions

The covariant derivative is similar to that for the Rarita-Schwinger field (38), but for spin $\frac{2k+1}{2}$ there are is a factor k times the torsion term. Expanding

$$\boldsymbol{\mathcal{D}\psi} = \mathbf{e}^a \wedge \mathbf{e}^{b_1} \wedge \ldots \wedge \mathbf{e}^{b_k}\mathcal{D}_a\psi_{b_1\ldots b_k}$$

the expansion is clearest in coordinates,

$$\begin{aligned} \boldsymbol{\mathcal{D}\psi} &= D_\mu\psi_{\alpha\ldots\beta}\mathbf{d}x^\mu \wedge \mathbf{d}x^\alpha \wedge \ldots \wedge \mathbf{d}x^\beta \\ &= \partial_\mu\psi_{\alpha\ldots\beta}\mathbf{d}x^\mu \wedge \mathbf{d}x^\alpha \wedge \ldots \wedge \mathbf{d}x^\beta - \psi_{\rho\ldots\beta}\Gamma^\rho{}_{\alpha\mu}\mathbf{d}x^\mu \wedge \mathbf{d}x^\alpha \wedge \ldots \wedge \mathbf{d}x^\beta \\ &\quad - \ldots - \psi_{\alpha\ldots\rho}\Gamma^\rho{}_{\beta\mu}\mathbf{d}x^\mu \wedge \mathbf{d}x^\alpha \wedge \ldots \wedge \mathbf{d}x^\beta - \frac{1}{2}\omega_{ab\mu}\sigma^{ab}\psi_{\alpha\ldots\beta}\mathbf{d}x^\mu \wedge \mathbf{d}x^\alpha \wedge \ldots \wedge \mathbf{d}x^\beta \end{aligned}$$

Antisymmetrizing each $\Gamma^\rho{}_{\alpha\mu}$ gives a torsion $\psi_{\alpha\ldots\rho}\Gamma^\rho{}_{\beta\mu}\mathbf{d}x^\mu \wedge \mathbf{d}x^\alpha \wedge \ldots \wedge \mathbf{d}x^\sigma \wedge \mathbf{d}x^\beta = \mathbf{T}^\rho \wedge \boldsymbol{\psi}_\rho$ where we define $\boldsymbol{\psi}_\rho \equiv \psi_{\rho\alpha\ldots\sigma} \wedge \mathbf{d}x^\alpha \wedge \ldots \wedge \mathbf{d}x^\sigma$. We get the same expression for each vector index so rearrangement gives

$$\boldsymbol{\mathcal{D}\psi} = \mathbf{d}\boldsymbol{\psi} - k\mathbf{T}^\rho \wedge \boldsymbol{\psi}_\rho - \frac{1}{2}\boldsymbol{\omega}_{ab} \wedge \sigma^{ab}\boldsymbol{\psi} \tag{40}$$

The same result follows in an orthogonal basis, but it is easiest to see using coordinates.

For the generalized Γs it is useful to normalize to avoid overall signs. Setting $h\mathbf{\Gamma}^k = \left(h\mathbf{\Gamma}^k\right)^\dagger$ introduces a factor of $(-1)^k$, but including the fields the adjoint of the combination $\bar{\boldsymbol{\psi}} \wedge \mathbf{\Gamma}^{2k+1} \wedge i\mathbf{d}\boldsymbol{\psi}$ introduces an additional factor of $(-1)^k$. We therefore require no phase factor and can conveniently define

$$\mathbf{\Gamma}^m \equiv \frac{1}{m!}{}^*\left[(\wedge\boldsymbol{\gamma})^m\right]$$

for all integers m.

5.5.2 Spin $\frac{2k+1}{2}$ fields

To start, we take the flat space $Spin\left(\frac{2k+1}{2}\right)$ action to be

$$S_k^0 = \int \bar{\boldsymbol{\psi}} \wedge \left(\boldsymbol{\Gamma}^{2k+1} \wedge i\mathbf{d}\boldsymbol{\psi} - m\boldsymbol{\Gamma}^{2k} \wedge \boldsymbol{\psi}\right) \tag{41}$$

after taking the conjugate and expanding the forms explicitly to check that S_k^0 is real. Notice that $\bar{\boldsymbol{\psi}} \wedge \mathbf{d}\boldsymbol{\psi}$ is a $(2k+1)$-form and therefore S_k^0 exists only for $n \geq 2k+1$. This makes Rarita-Schwinger the maximal case in 4-dimensional spacetime. Then, replacing $\mathbf{d} \Rightarrow \boldsymbol{\mathcal{D}}$ using Eq.(40) and symmetrizing, the gravitationally coupled $Spin\left(\frac{2k+1}{2}\right)$ action is

$$S_k = \frac{1}{2}\left(\tilde{S}_k + \tilde{S}_k^*\right)$$

As with the Rarita-Schwinger case, we find the real part of the torsion and σ^{ab} parts. For the torsion terms

$$\begin{aligned} S_k(T) &= \frac{1}{2}\int \bar{\boldsymbol{\psi}} \wedge \left(\boldsymbol{\Gamma}^{2k+1} \wedge (-ik\mathbf{T}^a \wedge \boldsymbol{\psi}_a)\right) + \frac{1}{2}\int \left[\bar{\boldsymbol{\psi}} \wedge \boldsymbol{\Gamma}^{2k+1} \wedge (-ik\mathbf{T}^a \wedge \boldsymbol{\psi}_a)\right]^\dagger \\ &= -\frac{ik}{2}\int \left(\bar{\boldsymbol{\psi}} \wedge \boldsymbol{\Gamma}^{2k+1} \wedge \mathbf{T}^a \wedge \boldsymbol{\psi}_a + (-1)^{n+k}\,\mathbf{T}^a \wedge \bar{\boldsymbol{\psi}}_a \wedge \boldsymbol{\Gamma}^{2k+1} \wedge \boldsymbol{\psi}\right) \end{aligned}$$

while the σ^{ab} terms still give an anticommutator

$$S_k(\sigma) = \frac{1}{2}\int \bar{\boldsymbol{\psi}} \wedge \left\{\boldsymbol{\Gamma}^{2k+1}, \sigma^{cd}\right\} \wedge \left(-\frac{i}{2}\boldsymbol{\omega}_{cd} \wedge \boldsymbol{\psi}\right)$$

Therefore, the action for gravitationally coupled $Spin\left(\frac{2k+1}{2}\right)$ fields is

$$\begin{aligned} S_k &= \int \bar{\boldsymbol{\psi}} \wedge \left(\boldsymbol{\Gamma}^{2k+1} \wedge i\mathbf{d}\boldsymbol{\psi} - m\boldsymbol{\Gamma}^{2k} \wedge \boldsymbol{\psi}\right) \\ &\quad -\frac{ik}{2}\int \left(\bar{\boldsymbol{\psi}} \wedge \boldsymbol{\Gamma}^{2k+1} \wedge \mathbf{T}^a \wedge \boldsymbol{\psi}_a + (-1)^{n+k}\,\mathbf{T}^a \wedge \bar{\boldsymbol{\psi}}_a \wedge \boldsymbol{\Gamma}^{2k+1} \wedge \boldsymbol{\psi}\right) \\ &\quad +\frac{1}{2}\int \bar{\boldsymbol{\psi}} \wedge \left\{\boldsymbol{\Gamma}^{2k+1}, \sigma^{cd}\right\} \wedge \left(-\frac{i}{2}\boldsymbol{\omega}_{cd}\boldsymbol{\psi}\right) \end{aligned} \tag{42}$$

The spin tensor always contains the anticommutator, which always brings in couplings involving Γ^{2k-1} and Γ^{2k+3} only (see the Appendix). The Dirac field has $k=0$, so only the Γ^3 term is possible, while for Rarita-Schwinger fields with $k=1$ we see both Γ^1 and Γ^5.

There are also direct torsion couplings of the form

$$k\bar{\boldsymbol{\psi}} \wedge \boldsymbol{\Gamma}^{2k+1} \wedge \mathbf{T}^a \wedge \boldsymbol{\psi}_a + c.c.$$

so the $Spin\left(\frac{2k+1}{2}\right)$ field may emit and absorb torsion. This is absent from Dirac interactions because there is no vector index on ψ, but does show up in the Rarita-Schwinger spin tensor. If the action includes a dynamical torsion term this constitutes a new interaction unless there is a consistent interpretation of torsion in terms of known interactions.

The spin tensor is given by a simple variation, followed by reducing the basis forms to a volume form. The result is

$$\begin{aligned} \sigma^{cab} &= \frac{ik}{2}(-1)^{kn-k-n+1}\left(\eta^{be}\delta^d_f - (-1)^k\,\eta^{bd}\delta^e_f\right)\bar{\psi}_{df_1\ldots f_{k-1}}\Gamma^{[acff_1\ldots f_{k-1}g_1\ldots g_{k-1}]}\psi_{eg_1\ldots g_{k-1}} \\ &\quad +\frac{i}{4}(-1)^{kn-k-n+1}\,\bar{\psi}_{a_1\ldots a_k}\left\{\Gamma^{[a_1\ldots a_k b_1\ldots b_k c]}, \sigma^{ab}\right\}\psi_{b_1\ldots b_k}\delta^{a_1\ldots a_k b_1\ldots b_k c}_{c_1\ldots c_{2k+1}} \end{aligned}$$

The anticommutator is a linear combination of $\Gamma^{2k-1}, \Gamma^{2k+3}$ (See Appendix 6) so together with the torsion contribution we have the original and both adjacent couplings $\Gamma^{2k-1}, \boldsymbol{\Gamma}^{2k+1}, \Gamma^{2k+3}$. It is extremely likely that, like the Rarita-Schwinger field, higher spin fermions drive all invariant parts of the torsion.

6 Conclusions

We implemented Poincarè gauging in arbitrary dimension n and signature (p,q) using Cartan's methods. The principal fields are the curvature and torsion 2-forms, given in terms of the solder form and local Lorentz spin connection. The inclusion of torsion produces a Riemann-Cartan geometry rather than Riemannian. We found the Bianchi identities and showed that the Riemann-Cartan identities hold if and only if the Riemannian Bianchi identities hold.

Replicating familiar results, we reproduced general relativity in Riemannian geometry by setting the torsion to zero and varying only the metric. The resulting Riemannian geometry is known to be consistent and metric variation leads to a symmetric energy tensor.

We examined sources for the ECSK theory, that is, the gravity theory in Riemann-Cartan geometry found by using the Einstein-Hilbert form of the action with the Einstein-Cartan curvature tensor. The vacuum theory agrees with general relativity even when both the solder form and connection are varied independently, but there are frequently nonvanishing matter sources for both the Einstein tensor and the torsion.

The first issue we dealt with in depth was the choice of independent variables. The spin connection was shown to be the sum of the solder-form-compatible connection and the contorsion tensor $\boldsymbol{\omega}^a{}_b = \boldsymbol{\alpha}^a{}_b + \mathbf{C}^a{}_b$. We compared and constrasted the resulting two allowed sets of independent variables: the solder form and spin connection $(\mathbf{e}^a, \boldsymbol{\omega}^a{}_b)$ on the one hand and the solder form and the contorsion tensor $(\mathbf{e}^a, \mathbf{C}^a{}_b)$ on the other. When choosing the latter pair the compatible part of the spin connection $\boldsymbol{\alpha}^a{}_b$ must be treated through its dependence on the solder form. We demonstrated explicitly how the two choices of independent variable differ in their relationship to the Lorentz fibers of the Riemann-Cartan space.

Changing independent variables changes the energy tensor. We showed that the difference between these two choices leads to the difference between the (asymmetric) canonical energy tensor and the (symmetric) Belinfante-Rosenfield energy tensor. When the field equations are combined both methods yield the same reduced system.

Our second main contribution was a more thorough analysis of sources for torsion. Many, perhaps most, of the research on ECSK theory or its generalizations to include dynamical torsion have restricted attention to Dirac fields as sources. This yields a single axial current and totally antisymmetric torsion. This amounts to only n of the $\frac{1}{2}n^2(n-1)$ degrees of freedom of the torsion.

We took the opposite approach, considering fields of *all* spin. Only scalar and Yang-Mills fields fail to determine nonvanishing torsion. In addition to these we looked at symmetric bosonic kinetic forms and found all to provide sources for torsion. We studied Dirac and Rarita-Schwinger fields in greater depth. After reproducing the well-known result for Dirac fields, we developed formalism to describe the spin-$\frac{3}{2}$ Rarita-Schwinger field in arbitrary dimension. Surprisingly, in addition dependence on the anticommutator of three gammas with the spin generator, $\left\{\gamma^{[a}\gamma^b\gamma^{c]}, \sigma^{de}\right\}$, there is a direct coupling to torsion, $\psi_a \mathbf{T}^a$. Continuing, we showed that Rarita-Schwinger fields drive all three independent parts of the torsion: the trace, the totally antisymmetric part, and the traceless, mixed-symmetry residual. Except in dimensions $5, 7$, and 9 the Rarita-Schwinger field has enough degrees of freedom to produce generic torsion.

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Appendix: Anticommutators

The anticommutator of the generators σ^{de} with any odd number of antisymmetrized γs has the form

$$\left\{\gamma^{[a_1}\gamma^{a_2}\ldots\gamma^{a_{2k+1}]},\sigma^{de}\right\}$$

We may simplify this by specifying that the a_i are all different and $d\neq e$. Then we have $\sigma^{de}=2\gamma^d\gamma^e$ and may rearrange the a_i in increasing order $a_1\leq a_i<a_j\leq a_k$ with the appropriate sign.

With these conditions there are three cases. First, if neither d nor e equals any of the a_i then the full product is antisymmetric.

$$\left\{\gamma^{[a_1}\ldots\gamma^{a_{2k+1}]},\sigma^{de}\right\}=4\gamma^{[a_1}\ldots\gamma^{a_3}\gamma^d\gamma^{e]}$$

For the second case, suppose exactly one of e,d equals one of the a_i. All other γs anticommute. Without loss of generality let $a_i=d$ with e distinct. Then with $\gamma^d\gamma^d=-\eta^{dd}$ we have

$$\begin{aligned}\left\{\gamma^{[a_1}\ldots\gamma^{a_i}\ldots\gamma^{a_{2k+1}]},\sigma^{de}\right\} &= 2\gamma^{a_1}\ldots\gamma^{a_i}\ldots\gamma^{a_{2k+1}}\gamma^d\gamma^e+2\gamma^d\gamma^e\gamma^{a_1}\ldots\gamma^{a_i}\ldots\gamma^{a_{2k+1}}\\ &= (-1)^{2k+1-i}\,2\gamma^{a_1}\ldots\gamma^{a_i}\gamma^d\ldots\gamma^{a_k}\gamma^e+(-1)^i\,2\gamma^e\gamma^{a_1}\ldots\gamma^d\gamma^{a_i}\ldots\gamma^{a_k}\\ &= (-1)^i\left(-2\gamma^{a_1}\ldots\gamma^{a_i}\gamma^d\ldots\gamma^{a_k}\gamma^e+2\gamma^e\gamma^{a_1}\ldots\gamma^d\gamma^{a_i}\ldots\gamma^{a_k}\right)\\ &= 0\end{aligned}$$

and this vanishes for all k.This holds for every instance.

The only reduced term that can occur is when both d and e match some a_i,a_j. There are two cases: $d=a_i,e=a_j$ and $d=a_j,e=a_i$ where $i<j$. For the first case

$$\begin{aligned}\left\{\gamma^{[a_1}\ldots\gamma^{a_{2k+1}]},\sigma^{de}\right\} &= 2\gamma^{a_1}\ldots\gamma^{a_i}\ldots\gamma^{a_j}\ldots\gamma^{a_{2k+1}}\gamma^d\gamma^e+2\gamma^d\gamma^e\gamma^{a_1}\ldots\gamma^{a_i}\ldots\gamma^{a_j}\ldots\gamma^{a_{2k+1}}\\ &= 2\gamma^{a_1}\ldots(-1)^{2k+1-i}\,\gamma^{a_i}\gamma^d\ldots(-1)^{2k+1-j}\,\gamma^{a_j}\gamma^e\ldots\gamma^{a_{2k+1}}\\ &\quad +2\gamma^{a_1}\ldots(-1)^{i-1}\,\gamma^d\gamma^{a_i}\ldots(-1)^{j-1}\,\gamma^e\gamma^{a_j}\ldots\gamma^{a_{2k+1}}\\ &= (-1)^{i+j}\,4\eta^{a_id}\eta^{a_je}\gamma^{a_1}\ldots\underset{\wedge}{\gamma^{a_i}}\ldots\underset{\wedge}{\gamma^{a_j}}\ldots\gamma^{a_{2k+1}}\end{aligned}$$

where $\underset{\wedge}{\gamma^{a_i}}$ indicates omission of γ^{a_i}.

For the second case, $d = a_j, e = a_i$, we simply exchange $\sigma^{de} = -\sigma^{ed}$ so we just replace $\eta^{a_i d}\eta^{a_j e} \to -\eta^{a_i e}\eta^{a_j d}$. Combining both terms

$$\left\{\gamma^{[a_1} \dots \gamma^{a_k]}, \sigma^{de}\right\} = 4\left(\eta^{a_i d}\eta^{a_j e} - \eta^{a_j d}\eta^{a_i e}\right)(-1)^{i+j}\,\gamma^{[a_1} \dots \underset{\wedge}{\gamma^{a_i}} \dots \underset{\wedge}{\gamma^{a_j}} \dots \gamma^{a_k]}$$

and we have a term like this for each $a_1 \le a_i < a_j \le a_k$.

Therefore, for any set of fixed $a_1 < a_2 < \ldots < a_k$ and $d < e$, the general result is

$$\begin{aligned}\left\{\gamma^{[a_1} \dots \gamma^{a_k]}, \sigma^{de}\right\} &= 4\gamma^{[a_1} \dots \gamma^{a_k}\gamma^{d}\gamma^{e]} \\ &\quad + 4 \sum_{a_1 \le a_i < a_j \le a_k} \left(\eta^{a_i d}\eta^{a_j e} - \eta^{a_j d}\eta^{a_i e}\right)(-1)^{i+j}\,\gamma^{[a_1} \dots \underset{\wedge}{\gamma^{a_i}} \dots \underset{\wedge}{\gamma^{a_j}} \dots \gamma^{a_k]}\end{aligned}$$

The essential feature here is that the anticommutator coupling between σ^{de} and Γ^{2k+1} always leads to a linear combination of Γ^{2k+3} and Γ^{2k-1} and only these.