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Torsion free biconformal spaces: Reducing the torsion field equations

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Reducing the torsion field equations

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Abstract

Our goal is to solve the full set of torsion and co-torsion field equations of Euclidean biconformal space, with only the assumption of vanishing torsion. Here we begin by resolving the involution constraints, symmetry conditions and torsion field equation into a single equation for further study.

In two preceding studies, we have looked at whether certain solutions flat for the connection of flat biconformal space also solve the curved space field equations. Now we take a more ambitious approach and try to solve the full set of torsion and co-torsion field equations, with only the assumption of vanishing torsion. Here we begin by resolving the involution constraints, symmetry conditions and torsion field equation into a single equation for further study.

1 Torsion-free field equations

Beginning with the field equations,

$$
\begin{aligned}
0 &= \beta\left(T^{ba}{}_{b} - T^{ab}{}_{b} + S_{b}{}^{ab}\right) \\
0 &= \beta\left(T^{b}_{ab} + S_{a}{}^{b}{}_{b} - S_{b}{}^{b}{}_{a}\right) \\
0 &= \Delta^{ap}_{qb}\left(T^{cb}{}_{a} - \delta^{c}_{a}T^{eb}{}_{e} - \delta^{c}_{a}S_{e}{}^{be}\right) \\
&\quad +\Delta^{ap}_{qb}\left(-\frac{1}{2}\partial_{a}\eta^{cb} + \frac{1}{2}\delta^{c}_{a}\partial_{f}\eta^{bf} - \eta^{cd}\mu^{b}{}_{da} + \delta^{c}_{a}\eta^{bd}\mu^{f}{}_{df} + W_{a}\eta^{cb} - \delta^{c}_{a}\eta^{bf}W_{f}\right) \\
&\quad +\Delta^{ap}_{qb}\left(-\frac{1}{2}\eta^{cd}\partial^{b}\eta_{ad} + \frac{1}{2}\delta^{c}_{a}\eta^{bd}\partial^{e}\eta_{ed} + \rho^{c}{}_{a}{}^{b} - \delta^{c}_{a}\rho^{b}{}_{e}{}^{e}\right) \\
0 &= \Delta^{ap}_{qb}\left(S_{c}{}^{b}{}_{a} - \delta^{b}_{c}S_{e}{}^{e}{}_{a} + \delta^{b}_{c}T^{e}{}_{ae}\right) \\
&\quad +\Delta^{ap}_{qb}\left(-\frac{1}{2}\partial^{b}\eta_{ac} + \delta^{b}_{c}\frac{1}{2}\partial^{e}\eta_{ea} + \eta_{ec}\rho^{e}{}_{a}{}^{b} - \delta^{b}_{c}\eta_{da}\rho^{d}{}_{e}{}^{e} - \eta_{ac}W^{b} + \delta^{b}_{c}\eta_{ea}W^{e}\right) \\
&\quad +\Delta^{ap}_{qb}\left(\frac{1}{2}\eta_{cd}\partial_{a}\eta^{bd} - \frac{1}{2}\delta^{b}_{c}\eta_{ad}\partial_{f}\eta^{df} + \eta_{cd}\eta^{de}\mu^{b}{}_{ea} - \delta^{b}_{c}\eta_{ad}\eta^{de}\mu^{f}{}_{ef}\right)
\end{aligned}
$$

and the involution conditions

$$
\begin{aligned}
T^{abc} &= \eta^{ad}\rho^{b}{}_{d}{}^{c} - \eta^{ad}\rho^{c}{}_{d}{}^{b} + \eta^{ac}u^{b} - \eta^{ab}u^{c} \\
&= \rho^{bac} - \rho^{cab} + \eta^{ac}u^{b} - \eta^{ab}u^{c} \\
S_{abc} &= k\eta_{ad}\mu^{d}{}_{bc} - k\eta_{ad}\mu^{d}{}_{cb} - k\eta_{ab}v_{c} + k\eta_{ac}v_{b} \\
&= k\left(\left(\mu_{abc} - \eta_{ab}v_{c}\right) - \left(\mu_{acb} - \eta_{ac}v_{b}\right)\right)
\end{aligned}
$$

we wish to determine the six independent parts of the torsion and co-torsion,

$$
T^{abc}, T^{ab}{}_{c}, T^{a}_{bc}, S_{a}{}^{bc}, S_{a}{}^{b}{}_{c}, S_{abc}
$$

Antisymmetrizing the involution conditions fully and noting the symmetry of ρ^{abc} and μ_{abc}, we immediately see that

$$\begin{aligned} T^{[abc]} &= 0 \\ S_{[abc]} &= 0 \end{aligned}$$

and we have the traces

$$\begin{aligned} \eta_{ab}T^{abc} &= \rho^{b\ c}_{\ b} - \rho^{c\ b}_{\ b} - (n-1)\, u^c \\ \eta^{ab}S_{abc} &= k\left(\mu^{b}_{\ bc} - \mu^{b}_{\ cb} - (n-1)\, v_c\right) \end{aligned}$$

but further progress is difficult without some assumption. In keeping with ideas from Riemannian geometry, we assume vanishing torsion,

$$\begin{aligned} T^{abc} &= 0 \\ T^{ab}_{\ \ c} &= 0 \\ T^{a}_{\ bc} &= 0 \end{aligned}$$

Choosing the metric conformally orthonormal, $\eta_{ab} = e^{2\phi}\eta^0_{ab}$, the field equations reduce to

$$\begin{aligned} 0 &= \beta S_b^{\ ab} \\ 0 &= \beta\left(S_a^{\ b}{}_{\ b} - S_b^{\ b}{}_{\ a}\right) \\ 0 &= \Delta^{ap}_{qb}\left(-\delta^c_a S_e^{\ be}\right) \\ &\quad +\Delta^{ap}_{qb}\left(\eta^{bc}\partial_a\phi - \delta^c_a\eta^{be}\partial_e\phi - \eta^{cd}\mu^{b}_{\ da} + \delta^c_a\eta^{bd}\mu^{f}_{\ df} + W_a\eta^{cb} - \delta^c_a\eta^{bf}W_f\right) \\ &\quad +\Delta^{ap}_{qb}\left(-\delta^c_a\partial^b\phi + \delta^c_a\partial^b\phi + \rho^{c\ b}_{\ a} - \delta^c_a\rho^{b\ e}_{\ e}\right) \\ &= \Delta^{ap}_{qb}\left(\rho^{c\ b}_{\ a} - \delta^c_a\rho^{b\ e}_{\ e} - \eta^{cd}\mu^{b}_{\ da} + \delta^c_a\eta^{bd}\mu^{f}_{\ df} + \eta^{bc}\left(W_a + \partial_a\phi\right) - \delta^c_a\eta^{be}\left(W_e + \partial_e\phi\right)\right) \\ 0 &= \Delta^{ap}_{qb}\left(S_{c\ a}^{\ b} - \delta^b_c S_{e\ a}^{\ e}\right) \\ &\quad +\Delta^{ap}_{qb}\left(-\eta_{ac}\partial^b\phi + \delta^b_c\eta_{ae}\partial^e\phi + \eta_{ec}\rho^{e\ b}_{\ a} - \delta^b_c\eta_{da}\rho^{d\ e}_{\ e} - \eta_{ac}W^b + \delta^b_c\eta_{ea}W^e\right) \\ &\quad +\Delta^{ap}_{qb}\left(-\delta^b_c\partial_a\phi + \delta^b_c\partial_a\phi + \eta_{cd}\eta^{de}\mu^{b}_{\ ea} - \delta^b_c\eta_{ad}\eta^{de}\mu^{f}_{\ ef}\right) \\ &= \Delta^{ap}_{qb}\left(S_{c\ a}^{\ b} - \delta^b_c S_{e\ a}^{\ e}\right) \\ &\quad +\Delta^{ap}_{qb}\left(\eta_{cd}\eta^{de}\mu^{b}_{\ ea} - \delta^b_c\eta_{ad}\eta^{de}\mu^{f}_{\ ef} + \eta_{ec}\rho^{e\ b}_{\ a} - \delta^b_c\eta_{da}\rho^{d\ e}_{\ e} - \eta_{ac}\left(W^b + \partial^b\phi\right) + \delta^b_c\eta_{ea}\left(W^e + \partial^e\phi\right)\right) \end{aligned}$$

We recognize $u^a = (W^a + \partial^a\phi)$ and $v_a = W_a + \partial_a\phi$. Collecting everything, we have two equations which are independent of the co-torsion,

$$\begin{aligned} 0 &= \eta^{ad}\rho^{b\ c}_{\ d} - \eta^{ad}\rho^{c\ b}_{\ d} + \eta^{ac}u^b - \eta^{ab}u^c \\ 0 &= \Delta^{ap}_{qb}\left(\rho^{c\ b}_{\ a} - \delta^c_a\rho^{b\ e}_{\ e} - \eta^{cd}\mu^{b}_{\ da} + \delta^c_a\eta^{bd}\mu^{f}_{\ df} + \eta^{bc}v_a - \delta^c_a\eta^{be}v_e\right) \end{aligned}$$

together with the symmetry of $\rho^{b\ c}_{\ a}$ and $\mu^{a}_{\ bc}$ and four equations involving co-torsion,

$$\begin{aligned} S_{abc} &= k\left(\eta_{ad}\mu^{d}_{\ bc} - \eta_{ad}\mu^{d}_{\ cb} - \eta_{ab}v_c + \eta_{ac}v_b\right) \\ S_b^{\ ab} &= 0 \\ S_a^{\ b}{}_{\ b} - S_b^{\ b}{}_{\ a} &= 0 \\ \Delta^{ap}_{qb}\left(S_{c\ a}^{\ b} - \delta^b_c S_{e\ a}^{\ e}\right) &= -\Delta^{ap}_{qb}\left(\eta_{cd}\eta^{de}\mu^{b}_{\ ea} - \delta^b_c\eta_{ad}\eta^{de}\mu^{f}_{\ ef} + \eta_{ec}\rho^{e\ b}_{\ a} - \delta^b_c\eta_{da}\rho^{d\ e}_{\ e} - \eta_{ac}u^b + \delta^b_c\eta_{ea}u^e\right) \end{aligned}$$

For now, we focus on the following six of these conditions, with the remaining three above taken as constraining the co-torsion. From this point on, since we are dealing only with a single form each of $Q^{c\,b}_{\ a}, \rho^{b\,c}_{\ e}, \mu^{b}_{\ ec}$ and S_{abc}, it will be far simpler to allow raising and lowering of indices using η_{ab}, η^{ab}. At the end we will return the indices to their "proper" positions. Raising all but the co-torsion indices and the Δ^{ae}_{db} equation, we have

$$
\begin{aligned}
0 &= \rho^{bac} - \rho^{cab} + \eta^{ac}u^b - \eta^{ab}u^c && (1)\\
kS_{abc} &= \mu_{abc} - \mu_{acb} - \eta_{ab}v_c + \eta_{ac}v_b && (2)\\
Q^{cab} &\equiv \mu^{bca} - \eta^{ac}\mu^{eb}_{\ \ e} - \rho^{cab} + \eta^{ac}\rho^{b\,e}_{\ e} - \eta^{bc}v^a + \eta^{ac}v^b && (3)\\
\Delta^{ae}_{db}Q^{c\,b}_{\ a} &= 0 && (4)\\
\rho^{abc} &= \rho^{bac} && (5)\\
\mu^{abc} &= \mu^{bac} && (6)
\end{aligned}
$$

These equations relate components of the connection. Notice that the only co-torsion term is the space-time piece, $S_{abc} = -S_{acb}$.

2 Solving the connection conditions

2.1 Expanding Δ^{ae}_{db}

First, notice that if $Q^{c\,b}_{\ a} = 0$, these equations cannot determine the relation between the metrics, δ_{ab} and η_{ab}. However, the arbitrariness of $Q^{c\,b}_{\ a}$ keeps us from solving the final equation for the this relationship. Since other arguments have shown us that the metric must be Lorentzian, we start by writing the inverse metric relation in the manifestly Lorentzian form

$$\delta^{ab} = A\left(\eta^{ab} - \frac{2}{w^2}w^a w^b\right)$$

for some vector w^a and amplitude A, and deriving the properties of $Q^{c\,b}_{\ a}$. It follows that the metric is

$$\delta_{ab} = \frac{1}{A}\left(\eta_{ab} - \frac{2}{w^2}\eta_{ac}\eta_{bd}w^c w^d\right)$$

and we have:

$$
\begin{aligned}
W^2 &\equiv \delta_{ab}w^a w^b\\
&= \frac{1}{A}\left(\eta_{ab}w^a w^b - \frac{2}{w^2}\eta_{ac}\eta_{bd}w^c w^d w^a w^b\right)\\
&= -\frac{1}{A}w^2
\end{aligned}
$$

where $w^2 \equiv \eta_{ab}w^a w^b$. Therefore, we have

$$
\begin{aligned}
\delta^{ab} &= -\frac{w^2}{W^2}\left(\eta^{ab} - \frac{2}{w^2}w^a w^b\right)\\
\delta_{ab} &= -\frac{W^2}{w^2}\left(\eta_{ab} - \frac{2}{w^2}\eta_{ac}\eta_{bd}w^c w^d\right)
\end{aligned}
$$

Now expand eq.(4):

$$0 = 2\Delta^{ae}_{db}Q^{c\,b}_{\ a}$$

$$
\begin{aligned}
&= \left(\delta^a_d \delta^e_b - \delta^{ae}\delta_{bd}\right) Q^{c\ b}_{\ a} \\
&= Q^{c\ e}_{\ d} - \frac{w^2}{W^2}\left(\eta^{ae} - \frac{2}{w^2} w^a w^e\right)\frac{W^2}{w^2}\left(\eta_{bd} - \frac{2}{w^2}\eta_{bf}\eta_{dg} w^f w^g\right) Q^{c\ b}_{\ a} \\
&= Q^{c\ e}_{\ d} - \left(\eta^{ae}\eta_{bd} Q^{c\ b}_{\ a} - \frac{2}{w^2} w^a w^e \eta_{bd} Q^{c\ b}_{\ a} - \frac{2}{w^2}\eta^{ae}\eta_{bf}\eta_{dg} w^f w^g Q^{c\ b}_{\ a} + \frac{2}{w^2}\frac{2}{w^2} w^a w^e w^f w^g Q^{c\ b}_{\ a}\eta_{bf}\eta_{dg}\right)
\end{aligned}
$$

Raising d,

$$
Q^{cde} - Q^{ced} = -\frac{2}{w^2} w^e w_a Q^{cad} - \frac{2}{w^2} w^d w_b Q^{ceb} + \frac{4}{\left(w^2\right)^2} w^e w^d Q^{cab} w_a w_b \tag{7}
$$

Check:

$$
\begin{aligned}
0 &= 2\Delta^{ae}_{db} Q^{c\ b}_{\ a} \\
&= \left(\delta^a_d \delta^e_b - \delta^{ae}\delta_{db}\right) Q^{c\ b}_{\ a} \\
&= Q^{c\ e}_{\ d} - \left(\eta^{ae} - \frac{2}{w^2} w^a w^e\right)\left(\eta_{db} - \frac{2}{w^2}\eta_{df}\eta_{bg} w^f w^g\right) Q^{c\ b}_{\ a} \\
&= Q^{c\ e}_{\ d} - \left(\eta^{ae}\eta_{db} Q^{c\ b}_{\ a} - \frac{2}{w^2}\eta^{ae}\eta_{df}\eta_{bg} w^f w^g Q^{c\ b}_{\ a} - \frac{2}{w^2} w^a w^e \eta_{db} Q^{c\ b}_{\ a} + \frac{2}{w^2}\eta_{df}\eta_{bg}\frac{2}{w^2} w^a w^e w^f w^g Q^{c\ b}_{\ a}\right) \\
0 &= Q^{cde} - Q^{ced} + \frac{2}{w^2} w^d w_a Q^{cea} + \frac{2}{w^2} w^e w_a Q^{cad} - \frac{4}{\left(w^2\right)^2} w^d w^e w_a w_b Q^{cab} \\
Q^{cde} - Q^{ced} &= -\frac{2}{w^2} w^d w_a Q^{cea} - \frac{2}{w^2} w^e w_a Q^{cad} + \frac{4}{\left(w^2\right)^2} w^d w^e w_a w_b Q^{cab}
\end{aligned}
$$

2.2 Symmetric and antisymmetric parts of the field equation

2.2.1 Antisymmetrize

If we antisymmetrize ed,

$$
\begin{aligned}
2\left(Q^{cde} - Q^{ced}\right) &= -\frac{2}{w^2} w^d w_a Q^{cea} - \frac{2}{w^2} w^e w_a Q^{cad} + \frac{2}{w^2} w^e w_a Q^{cda} + \frac{2}{w^2} w^d w_a Q^{cae} \\
&= \frac{2}{w^2} w^d w_a \left(Q^{cae} - Q^{cea}\right) - \frac{2}{w^2} w^e w_a \left(Q^{cad} - Q^{cda}\right) \\
&= \frac{2}{w^2} w^d w_a \delta^e_b \left(Q^{cab} - Q^{cba}\right) - \frac{2}{w^2} w^e w_a \delta^d_b \left(Q^{cab} - Q^{cba}\right) \\
Q^{cde} - Q^{ced} &= \frac{1}{w^2}\left(w^d w_a \delta^e_b - w^e w_a \delta^d_b\right)\left(Q^{cab} - Q^{cba}\right)
\end{aligned}
$$

Therefore, the antisymmetric part of Q^{abc} is a mixed projection:

$$
Q^{cde} - Q^{ced} = \frac{1}{w^2}\left(w^d w_a \delta^e_b - w^e w_a \delta^d_b\right)\left(Q^{cab} - Q^{cba}\right) \tag{8}
$$

2.2.2 Symmetrize

Symmetrizing ed,

$$
\begin{aligned}
0 &= -\frac{2}{w^2} w^d w_a Q^{cea} - \frac{2}{w^2} w^e w_a Q^{cad} + \frac{4}{\left(w^2\right)^2} w^d w^e w_a w_b Q^{cab} \\
&\quad -\frac{2}{w^2} w^e w_a Q^{cda} - \frac{2}{w^2} w^d w_a Q^{cae} + \frac{4}{\left(w^2\right)^2} w^d w^e w_a w_b Q^{cab} \\
0 &= -\frac{1}{w^2} w^d w_a \delta^e_b \left(Q^{cab} + Q^{cba}\right) - \frac{1}{w^2} w^e w_a \delta^d_b \left(Q^{cab} + Q^{cba}\right) + \frac{2}{\left(w^2\right)^2} w^d w^e w_a w_b \left(Q^{cab} + Q^{cba}\right)
\end{aligned}
$$

This gives the symmetric part as

$$0 = -\frac{1}{w^2} w^d w_a \delta^e_b \left(Q^{cab} + Q^{cba}\right) - \frac{1}{w^2} w^e w_a \delta^d_b \left(Q^{cab} + Q^{cba}\right) + \frac{2}{\left(w^2\right)^2} w^d w^e w_a w_b \left(Q^{cab} + Q^{cba}\right) \quad (9)$$

Now combine the results to check. Adding the symmetric and antisymmetric parts,

$$\begin{aligned}
\left(Q^{cde} - Q^{ced}\right) &= \frac{1}{w^2}\left(w^d w_a \delta^e_b - w^e w_a \delta^d_b\right)\left(Q^{cab} - Q^{cba}\right) \\
0 &= -\frac{1}{w^2}\left(w^d w_a \delta^e_b + w^e w_a \delta^d_b\right)\left(Q^{cab} + Q^{cba}\right) + \frac{2}{\left(w^2\right)^2} w^d w^e w_a w_b \left(Q^{cab} + Q^{cba}\right)
\end{aligned}$$

we recover the field equation:

$$\begin{aligned}
\left(Q^{cde} - Q^{ced}\right) &= \frac{1}{w^2}\left(w^d w_a \delta^e_b - w^e w_a \delta^d_b\right)\left(Q^{cab} - Q^{cba}\right) \\
&\quad -\frac{1}{w^2}\left(w^d w_a \delta^e_b + w^e w_a \delta^d_b\right)\left(Q^{cab} + Q^{cba}\right) + \frac{2}{\left(w^2\right)^2} w^d w^e w_a w_b \left(Q^{cab} + Q^{cba}\right) \\
&= -\frac{2}{w^2}\left(w^d w_a Q^{cea} + w^e w_a Q^{cad}\right) + \frac{4}{\left(w^2\right)^2} w^d w^e w_a w_b Q^{cab}
\end{aligned}$$

3 Solving for ρ^{abc} and μ^{abc}

We have

$$\begin{aligned}
0 &= \rho^{bac} - \rho^{cab} + \eta^{ac} u^b - \eta^{ab} u^c \\
kS_{abc} &= \mu_{abc} - \mu_{acb} - \eta_{ab} v_c + \eta_{ac} v_b \\
Q^{cab} &\equiv \mu^{bca} - \eta^{ac} \mu^{eb}_{\ \ e} - \rho^{cab} + \eta^{ac} \rho^{b\ e}_{\ e} - \eta^{bc} v^a + \eta^{ac} v^b \\
\Delta^{ae}_{db} Q^{c\ b}_{\ a} &= 0 \\
\rho^{abc} &= \rho^{bac} \\
\mu^{abc} &= \mu^{bac}
\end{aligned}$$

Consider the first and fifth:

$$\begin{aligned}
0 &= \rho^{bac} - \rho^{cab} + \eta^{ac} u^b - \eta^{ab} u^c \\
0 &= \rho^{abc} - \rho^{bac} \\
\rho^{abc} &= \rho^{bac} \\
&= \rho^{cab} - \eta^{ac} u^b + \eta^{ab} u^c \\
&= \rho^{acb} - \eta^{ac} u^b + \eta^{ab} u^c \\
\rho^{abc} - \rho^{acb} &= \eta^{ab} u^c - \eta^{ac} u^b
\end{aligned}$$

Therefore, the symmetric part is

$$\begin{aligned}
3p^{abc} &\equiv \rho^{abc} + \rho^{cab} + \rho^{bca} \\
&= \rho^{abc} + \left(\rho^{bac} + \eta^{ac} u^b - \eta^{ab} u^c\right) + \left(\rho^{bac} + \eta^{bc} u^a - \eta^{ba} u^c\right) \\
&= 3\rho^{abc} + \eta^{ac} u^b - \eta^{ab} u^c + \eta^{bc} u^a - \eta^{ba} u^c
\end{aligned}$$

and therefore,

$$\rho^{abc} = p^{abc} + \frac{1}{3}\left(2\eta^{ab} u^c - \eta^{ca} u^b - \eta^{cb} u^a\right)$$

Similarly,

$$\begin{aligned}
kS_{abc} &= \mu_{abc} - \mu_{acb} - \eta_{ab}v_c + \eta_{ac}v_b \\
\mu_{abc} - \mu_{acb} &= kS_{abc} + \eta_{ab}v_c - \eta_{ac}v_b \\
\mu_{bac} - \mu_{cab} &= kS_{abc} + \eta_{ab}v_c - \eta_{ac}v_b \\
\mu_{abc} - \mu_{bac} &= 0
\end{aligned}$$

Then with

$$\begin{aligned}
3u_{abc} &= \mu_{abc} + \mu_{cab} + \mu_{bca} \\
&= \mu_{abc} + \mu_{bac} - (kS_{abc} + \eta_{ab}v_c - \eta_{ac}v_b) + (\mu_{bac} + kS_{bca} + \eta_{bc}v_a - \eta_{ba}v_c) \\
&= 3\mu_{abc} + kS_{bca} - kS_{abc} - 2\eta_{ab}v_c + \eta_{ac}v_b + \eta_{bc}v_a \\
\mu_{abc} &= u_{abc} + \frac{1}{3}\left(kS_{abc} + kS_{bac} + 2\eta_{ab}v_c - \eta_{ac}v_b - \eta_{bc}v_a\right)
\end{aligned}$$

Therefore, we have:

$$\begin{aligned}
\rho^{abc} &= p^{abc} + \frac{1}{3}\left(2\eta^{ab}u^c - \eta^{ca}u^b - \eta^{cb}u^a\right) \\
\mu_{abc} &= u_{abc} + \frac{1}{3}\left(2\eta_{ab}v_c - \eta_{ac}v_b - \eta_{bc}v_a\right) + \frac{1}{3}\left(kS_{abc} + kS_{bac}\right)
\end{aligned}$$

with traces

$$\begin{aligned}
\rho^{b\,e}_{\ \ e} &= p^{b\,e}_{\ \ e} - \frac{1}{3}(n-1)\,u^b \\
\mu^{eb}_{\ \ e} &= u^{eb}_{\ \ e} - \frac{1}{3}(n-1)\,v^b + \frac{1}{3}kS^{eb}_{\ \ e}
\end{aligned}$$

and so

$$\begin{aligned}
Q^{cab} &\equiv \mu^{bca} - \eta^{ac}\mu^{eb}_{\ \ e} - \rho^{cab} + \eta^{ac}\rho^{b\,e}_{\ \ e} - \eta^{bc}v^a + \eta^{ac}v^b \\
&= u^{bca} + \frac{1}{3}\left(2\eta^{bc}v^a - \eta^{ab}v^c - \eta^{ac}v^b\right) + \frac{1}{3}\left(kS^{bca} + kS^{cba}\right) - \eta^{ac}\left(u^{eb}_{\ \ e} - \frac{1}{3}(n-1)\,v^b + \frac{1}{3}kS^{eb}_{\ \ e}\right) \\
&\quad - \left(p^{abc} + \frac{1}{3}\left(2\eta^{ca}u^b - \eta^{bc}u^a - \eta^{ba}u^c\right)\right) + \eta^{ac}\left(p^{b\,e}_{\ \ e} - \frac{1}{3}(n-1)\,u^b\right) - \eta^{bc}v^a + \eta^{ac}v^b \\
&= u^{bca} - p^{abc} - \eta^{ac}u^{eb}_{\ \ e} + \eta^{ac}p^{b\,e}_{\ \ e} + \frac{1}{3}\left(kS^{bca} + kS^{cba}\right) - \frac{1}{3}k\eta^{ac}S^{eb}_{\ \ e} \\
&\quad + \frac{1}{3}\left((n+1)\,\eta^{ac}v^b - \eta^{bc}v^a - \eta^{ab}v^c - (n+1)\,\eta^{ac}u^b + \eta^{bc}u^a + \eta^{ba}u^c\right) \\
&= U^{bca} - \eta^{ac}U^{eb}_{\ \ e} + \frac{k}{3}\left(S^{bca} + S^{cba} - \eta^{ac}S^{eb}_{\ \ e}\right) \\
&\quad + \frac{1}{3}\left((n+1)\,\eta^{ac}\left(v^b - u^b\right) - \eta^{bc}\left(v^a - u^a\right) - \eta^{ab}\left(v^c - u^c\right)\right)
\end{aligned}$$

Define $k^a \equiv v^a - u^a$. Then Q^{cab} is

$$Q^{cab} = U^{bca} - \eta^{ac}U^{eb}_{\ \ e} + \frac{k}{3}\left(S^{bca} + S^{cba} - \eta^{ac}S^{eb}_{\ \ e}\right) + \frac{1}{3}\left((n+1)\,\eta^{ac}k^b - \eta^{bc}k^a - \eta^{ab}k^c\right)$$

where the totally symmetric part of Q^{cab} is

$$U^{abc} = u^{bca} - p^{abc}$$

This Q^{cab} is the only combination of μ^{abc}, ρ^{abc} and S^{abc} that is constrained by the field equation, $\Delta^{ae}_{db}Q^{c\,b}_{\ \ a} = 0$.

4 Conclusion: the field equation

Expanding the field equation, we require

$$
\begin{aligned}
0 &= Q^{cab} - Q^{cba} + \frac{2}{w^2} w^b w_e Q^{cea} + \frac{2}{w^2} w^a w_e Q^{cbe} - \frac{4}{(w^2)^2} w^a w^b Q^{cde} w_d w_e \\
&= U^{bca} - \eta^{ac} U^{eb}_{\ \ e} + \frac{k}{3}\left(S^{bca} + S^{cba} - \eta^{ac} S^{eb}_{\ \ e}\right) + \frac{1}{3}\left((n+1)\,\eta^{ac} k^b - \eta^{bc} k^a - \eta^{ab} k^c\right) \\
&\quad - U^{acb} + \eta^{bc} U^{ea}_{\ \ e} - \frac{k}{3}\left(S^{acb} + S^{cab} - \eta^{bc} S^{ea}_{\ \ e}\right) - \frac{1}{3}\left((n+1)\,\eta^{bc} k^a - \eta^{ac} k^b - \eta^{ab} k^c\right) \\
&\quad + \frac{2}{w^2} w^b \left(w_e U^{ace} - w^c U^{fa}_{\ \ f} + \frac{k}{3}\left(w_e S^{ace} + w_e S^{cae} - w^c S^{fa}_{\ \ f}\right) + \frac{1}{3}\left((n+1)\,w^c k^a - \eta^{ac} k^e w_e - w^a k^c\right)\right) \\
&\quad + \frac{2}{w^2} w^a \left(w_e U^{ecb} - \eta^{bc} w_e U^{fe}_{\ \ f} + \frac{k}{3}\left(w_e S^{ecb} + w_e S^{ceb} - \eta^{bc} w_e S^{fe}_{\ \ f}\right) + \frac{1}{3}\left((n+1)\,\eta^{bc} k^e w_e - w^c k^b - w^b k^c\right)\right) \\
&\quad - \frac{4}{(w^2)^2} w^a w^b \left(w_d w_e U^{cde} - w^c w_e U^{fe}_{\ \ f} + \frac{k}{3}\left(w_d w_e S^{ecd} - w^c w_e S^{fe}_{\ \ f}\right) + \frac{1}{3}\left((n+1)\,w^c k^e w_e - w^c k^d w_d - w^2 k^c\right)\right) \\
&= \eta^{bc} U^{ea}_{\ \ e} - \eta^{ac} U^{eb}_{\ \ e} + \frac{2}{w^2} w^b \left(w_e U^{ace} - w^c U^{fa}_{\ \ f}\right) + \frac{2}{w^2} w^a \left(w_e U^{ecb} - \eta^{bc}\left(w_e U^{fe}_{\ \ f}\right)\right) \\
&\quad - \frac{4}{(w^2)^2} w^a w^b w_d w_e U^{cde} + \frac{4}{(w^2)^2} w^a w^b w^c w_e U^{fe}_{\ \ f} \\
&\quad + \frac{k}{3}\left(S^{bca} + S^{cba} - \eta^{ac} S^{eb}_{\ \ e}\right) - \frac{k}{3}\left(S^{acb} + S^{cab} - \eta^{bc} S^{ea}_{\ \ e}\right) + \frac{2}{3w^2}\left(w^b w_e S^{ace} + w^b w_e S^{cae} - w^b w^c S^{fa}_{\ \ f}\right) \\
&\quad + \frac{2}{3w^2}\left(w^a w_e S^{ecb} + w^a w_e S^{ceb} - w^a \eta^{bc}\left(w_e S^{fe}_{\ \ f}\right)\right) - \frac{4}{3\,(w^2)^2} w^a w^b w_d w_e k S^{ecd} + \frac{4}{3\,(w^2)^2} w^a w^b w^c \left(w_e k S^{fe}_{\ \ f}\right) \\
&\quad + \frac{1}{3}\left((n+1)\,\eta^{ac} k^b - \eta^{bc} k^a - \eta^{ab} k^c\right) - \frac{1}{3}\left((n+1)\,\eta^{bc} k^a - \eta^{ac} k^b - \eta^{ab} k^c\right) + \frac{2}{3w^2}\left((n+1)\,w^b w^c k^a - w^b \eta^{ac} k^e w_e - w^a w^b \right. \\
&\quad + \frac{2}{3w^2}\left((n+1)\,w^a \eta^{bc}\left(k^e w_e\right) - w^a w^c k^b - w^a w^b k^c\right) - \frac{4}{3\,(w^2)^2} n w^a w^b w^c \left(k^e w_e\right) + \frac{4}{3w^2} w^a w^b k^c \\
&= \eta^{bc} U^{ea}_{\ \ e} - \eta^{ac} U^{eb}_{\ \ e} + \frac{2}{w^2} w^b w_e U^{ace} + \frac{2}{w^2} w^a w_e U^{ecb} - \frac{2}{w^2} w^b w^c U^{fa}_{\ \ f} - \frac{2}{w^2} w^a \eta^{bc}\left(w_e U^{fe}_{\ \ f}\right) \\
&\quad - \frac{4}{(w^2)^2} w^a w^b w_d w_e U^{cde} + \frac{4}{(w^2)^2} w^a w^b w^c \left(w_e U^{fe}_{\ \ f}\right) \\
&\quad - k S^{cab} + \frac{2}{3w^2}\left(w^b w_e\left(k S^{ace} + k S^{cae}\right) + w^a w_e\left(k S^{ecb} + k S^{ceb}\right)\right) - \frac{2}{3w^2} w^b w^c k S^{fa}_{\ \ f} + \frac{1}{3}\left(\eta^{bc} k S^{ea}_{\ \ e} - \eta^{ac} k S^{eb}_{\ \ e}\right) \\
&\quad - \frac{4}{3\,(w^2)^2} w^a w^b w_d w_e k S^{ecd} + \frac{4}{3\,(w^2)^2} w^a w^b w^c \left(w_e k S^{fe}_{\ \ f}\right) - \frac{2}{3w^2} w^a \eta^{bc}\left(k w_e S^{fe}_{\ \ f}\right) \\
&\quad + \frac{1}{3}\left((n+2)\,\eta^{ac} k^b - (n+2)\,\eta^{bc} k^a\right) + \frac{2}{3w^2}\left((n+1)\,w^b w^c k^a - w^a w^c k^b + (n+1)\,w^a \eta^{bc}\left(k^e w_e\right) - w^b \eta^{ac}\left(k^e w_e\right)\right) \\
&\quad - \frac{4}{3\,(w^2)^2} n w^a w^b w^c \left(k^e w_e\right)
\end{aligned}
$$

In the next report, we therefore consider the equation

$$
\begin{aligned}
0 &= \eta^{bc} U^{ea}_{\ \ e} - \eta^{ac} U^{eb}_{\ \ e} + \frac{2}{w^2} w^b w_e U^{ace} + \frac{2}{w^2} w^a w_e U^{ecb} - \frac{2}{w^2} w^b w^c U^{fa}_{\ \ f} - \frac{2}{w^2} w^a \eta^{bc}\left(w_e U^{fe}_{\ \ f}\right) \\
&\quad - \frac{4}{(w^2)^2} w^a w^b w_d w_e U^{cde} + \frac{4}{(w^2)^2} w^a w^b w^c \left(w_e U^{fe}_{\ \ f}\right) \\
&\quad - k S^{cab} + \frac{2}{3w^2}\left(w^b w_e\left(k S^{ace} + k S^{cae}\right) + w^a w_e\left(k S^{ecb} + k S^{ceb}\right)\right) - \frac{2}{3w^2} w^b w^c k S^{fa}_{\ \ f} + \frac{1}{3}\left(\eta^{bc} k S^{ea}_{\ \ e} - \eta^{ac} k S^{eb}_{\ \ e}\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{4}{3\left(w^2\right)^2} w^a w^b w_d w_e k S^{ecd} + \frac{4}{3\left(w^2\right)^2} w^a w^b w^c \left(w_e k S^{fe}{}_f\right) - \frac{2}{3w^2} w^a \eta^{bc} \left(k w_e S^{fe}{}_f\right) \\
&+\frac{1}{3}\left((n+2)\,\eta^{ac}k^b - (n+2)\,\eta^{bc}k^a\right) + \frac{2}{3w^2}\left((n+1)\,w^b w^c k^a - w^a w^c k^b + (n+1)\,w^a \eta^{bc}\left(k^e w_e\right) - w^b \eta^{ac}\left(k^e w_e\right)\right) \\
&-\frac{4}{3\left(w^2\right)^2} n w^a w^b w^c \left(k^e w_e\right)
\end{aligned}
$$

This, among other variables, relates the time vector w^a to the connection vectors u^a and v^a. Once the consequences of this equation are quantified, we will consider the remaining field equations for the cross co-torsion and the momentum co-torsion.

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