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Gauge theories of general relativity

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Abstract

General relativity can be seen as a gauge theory of the Lorentz, Poincaré, Weyl, de Sitter, or conformal groups. In most of these, there is little or no difference from the standard formulation in Riemannian geometry, but the higher symmetries — de Sitter and conformal — introduce new features and explain old ones. The potential presence of a cosmological constant, the spacetime metric, cosmological dust, symplectic structure, Kähler structure and even the existence of a timelike direction can all be seen to arise from the underlying group structure.

Overview:

1. Vacuum relativity as vanishing Weyl-Schouten tensor in a trivial Weyl geometry
2. Seven gauge theories of general relativity
3. Properties of the conformal cases

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- JTW, Weyl gravity as general relativity, Phys Rev D 2014 90 025027; ArXiv 1310.0526
1 Vanishing Weyl-Schouten tensor as general relativity

We argue that the vanishing of the Schouten tensor of a pure-gauge Weyl geometry should be regarded as the correct field equation for general relativity.

1.1 Why bother?

General relativity is already globally scale invariant, and accounts well for most gravitational phenomena. Though we can easily write the field equation of scale-invariant general relativity, we are justified in ignoring it because:

1. We rarely use spacetime dependent measures (e.g., the temperature of the universe as a time coordinate)

2. The simplest ways to include scale invariance lead only to more complicated equations without obvious consequence.

Here are some reasons to bother:

I. A lesson from quantum physics

The essential difference between quantum and classical systems is that the classical description assumes a unique preferred path even though that path is not measured, while a quantum mechanical path integral takes into account what might happen along any possible, but unobserved, part of the motion.

Bearing this in mind, Bell’s Theorem teaches us an important lesson. Not only does quantum mechanics give us more precise measurements: a classical picture of Einstein-Podolsky-Rosen type experiments can give us false predictions.

If there is something that we can not know about the world, we should not build it into our physical models

This suggests that we include all the symmetries of spacetime before settling on the form of our gravity theory.

It turns out that there are several ways to write gravitational gauge theories which are precisely equivalent to general relativity once we choose something like the “Cesium gauge”. Which one we use may have deep consequences when we make the correction to quantum gravity.

II. Differing ground states

When a gauge theory of gravity is based on the de Sitter or the conformal group, we find that the ground states differ. The de Sitter and anti de Sitter cases are well known, giving hyperbolic or hyperspherical ground states in place of Minkowski space. Currently, the evidence for a cosmological constant (perhaps with additional dark energy) is favored. For the conformal group, we find both a cosmological constant and cosmological dust. These models can radically change predictions about the need for dark matter and dark energy.

III. New insights, variables, consequences

With a gauge-theoretic approach to gravity, properties of the underlying group can survive into the gravity theory. For example, unlike the Poincaré group, the de Sitter and conformal groups have nondegenerate Killing forms. When the restriction of this Killing metric to the spacetime submanifold is also nondegenerate, we gain a “prediction” of the metric. Other new structures occur as well, including complex and Kähler structures in certain conformal cases.
1.2 Why Weyl-Schouten?

The symmetry of spacetime: what we can’t know.

There is no preferred Lorentz frame of reference in empty space. Furthermore:

I. Following matter and light

In 1972, Ehlers, Pirani, and Schild argued that physical measurements lead us to two connections on spacetime:

1. The paths of massive particles give a projective connection

2. The paths of photons give a conformal connection

Assuming that these connections agree in the limit of light-speed particles requires the connection of a Weyl geometry.

Last year, Matveev and Trautman found the necessary and sufficient conditions for compatibility of conformal and projective connections. Specifically, the projective equivalence class of the difference of the two connections, $T_{jk}^i$ must satisfy

$$T_{jk}^i = g_{jk}g^{im}T_m - \frac{1}{n+1} \left( \delta_j^i T_k + \delta_k^i T_j \right)$$

$$\partial_i T_j - \partial_j T_i = 0$$

where $T_i \equiv \frac{n+1}{(n+2)(n-1)}g^{jk}T_{jk}^i$. A short calculation shows that these conditions hold if and only if the connection is projectively equivalent to the connection of a trivial Weyl geometry,

$$\tilde{\Gamma}_{jk}^i = \Gamma_{jk}^i + \left( \delta_j^i W_k + \delta_k^i W_j - g_{jk}g^{im}W_m \right)$$

$$W_{k,m} - W_{m,k} = 0$$

This triviality condition differs from the original Ehlers, Pirani, Schild claim. Therefore, reasonable conditions on what can be known about the world geometry suggest use of a trivial Weyl geometry.

II. Relative measurement

The same conclusion follows from a consideration of scale invariance. The inclusion of dilatational symmetry is consistent with our ability to choose local units of measurement — ultimately, we compare all measurements to a defined scale. Once we define the unit of length (e.g., via the frequency of a certain electron spin flip in Cesium), the local dilatational symmetry reduces to the familiar global unit symmetry of general relativity. Note that nontrivial Weyl geometries allow measurable size change, so once again we are motivated to consider Weyl geometries in which the dilatational gauge vector is pure-gauge, $W_{\mu,\nu} - W_{\nu,\mu} = 0$. 
III. From gauge theory

Surprisingly, standard treatments of general relativity lead to similar conclusions. For example, writing the Palatini variation as a Poincaré gauge theory the Cartan structure equations are

\[
\begin{align*}
\text{d} \omega^a_{\ b} &= \omega^c_{\ b} \wedge \omega^a_{\ c} + R^a_{\ b} \\
\text{d} e^a &= e^b \wedge \omega^a_{\ b} + T^a
\end{align*}
\]

Specifying the Palatini form of the action and varying,

\[
S = \int R^{ab} \wedge e^c \wedge e^d \varepsilon_{abcd}
\]

\[
0 = \delta \int R^{ab} \wedge e^c \wedge e^d \varepsilon_{abcd}
\]

\[
= \int \delta \omega^{ab} \wedge D (e^c \wedge e^d \varepsilon_{abcd}) + 2 \int R^{ab} \wedge e^c \wedge \delta e^d \varepsilon_{abcd}
\]

The metric variation gives the usual vacuum Einstein equation, while the consequence of the spin connection variation depends on our assumptions. All we know of the spin connection is that it is antisymmetric, but (unlike the explicit assumptions in excellent standard references by MTW, Wald or Weinberg), we do not know that the connection is compatible with the volume form. If, maintaining generality, we set

\[
D \varepsilon_{abcd} = \omega \varepsilon_{abcd}
\]

for some 1-form \( \omega \), the torsion turns out to be

\[
T^a = \omega \wedge e^a
\]

Substituting into the structure equations we have

\[
\begin{align*}
\text{d} \omega^a_{\ b} &= \omega^c_{\ b} \wedge \omega^a_{\ c} + R^a_{\ b} \\
\text{d} e^a &= e^b \wedge \omega^a_{\ b} + \omega \wedge e^a
\end{align*}
\]

If we define

\[
\Omega \equiv \text{d} \omega
\]

then the three equations together describe a Weyl geometry rather than Riemannian.

Solving these equations for the curvature, the (Wey) conformal curvature remains unchanged but the vanishing Ricci tensor generalizes to

\[
\Omega_{cd} = R^a_{\ cad}
\]

\[
= R_{cd} - D_d W_c - W_c W_d + \frac{1}{2} \eta_{cd} W^2
\]

\[
= 0
\]

where the \( R_{cd} \) is the Schouten tensor. The Ricci tensor is given by \( R_{ab} = - (n - 2) \left( R_{cd} + \frac{1}{n-2} \right) \eta_{ab} \). For a trivial Weyl geometry, we have (locally) \( W_\mu = \partial_\mu \varphi \). The trivial form was first derived by Levi-Civita.

This expression is invariant under local conformal transformations of the metric, and in the trivial case we can find a gauge in which the Ricci tensor vanishes. The gauge vector \( W_\mu \) is then unchanged by global changes of gauge, reducing us to the usual form of general relativity.
2 Gauge theories of general relativity

2.1 Clarification

By a gauge theory of general relativity, we mean a gravitational action based on a local spacetime symmetry which reduces to a trivial Weyl geometry relativity with vanishing Weyl-Schouten tensor.

Just to be clear:

• A symmetry is a transformation leaving the action functional invariant. Local symmetries may be different at different points.

• We do not consider “alternative theories of gravity” or “internal” symmetries. We seek gauge theories which reduce to general relativity via:
  – use of the field equations
  – defining the second

• One exception: we may take geometries to be torsion-free, but this must be a gauge-invariant condition and is done only after a full variation of the connection.

• Geometries may differ in their background vacuum solutions (e.g., by naturally including a cosmological constant)
2.2 Gauge theories of general relativity

There are gauge theories of gravity which are equivalent to general relativity, in the sense described above, based on the following groups:

- Lorentz (Utiyama, 1956)
- Poincaré (Kibble, 1961)
- Weyl geometry (Dirac, 1973) is a specific case of Brans-Dicke (1961) but with a different geodesic equation. Both contain an extra scalar field
- de Sitter or anti-de Sitter (MacDowell and Mansouri, 1977). Includes a cosmological constant
- Conformal (Weyl) gravity as a gauge theory (JTW, 2014). May differ from GR for Petrov type O or N through an additional scalar field.
- Lorentzian biconformal gravity (A. Wehner and JTW, 1999) with orthogonal Lagrangian submanifolds
  - Lagrangian submanifolds
  - Null metric
  - Spacetime metric by hand
- Euclidean biconformal gravity (Spencer and JTW, Hazboun and JTW) with orthogonal Lagrangian submanifolds
  - Timelike direction is induced, spacetime metric derived
  - Cosmological constant and cosmological dust
  - No assumptions; general (generic) solution

We look in more detail at the conformal cases. These are based on quotients of the conformal group by either the homogeneous Weyl group or the inhomogeneous Weyl group. The resulting homogeneous manifolds serve as a local model for more general curved spaces.
3 Properties of the conformal cases

3.1 Weyl gravity

The quotient of the conformal group of compactified Minkowski space by the inhomogeneous Weyl group a 4-dim spacetime. Writing the action

\[ S = \int \alpha \Omega_b^a \wedge^* \Omega^b_a + \beta \Omega \wedge^* \Omega \]

gives the action for Weyl gravity considered extensively by Bach (1924). The metric variation leads to fourth-order field equations. All solutions to the vacuum Einstein equations are also solutions to Weyl gravity but the converse is false: there are known solutions to Weyl gravity which do not solve the Einstein equation.

The situation changes when we consider \( S \) as a gauge theory and vary all fifteen of the conformal gauge fields independently. The Palatini-style variation leads to an extra field equation which turns out to be the integrability condition for the fourth order equations to be exactly the scale-invariant Einstein equation. In generic spacetimes, there exists a gauge in which both the Ricci tensor and the dilatational gauge vector vanish.

In Petrov type O and type N spaces, there is a gauge with vanishing Ricci tensor, but it may differ from the gauge in which the Weyl vector vanishes. The difference between these gauges is then a conformally invariant scalar, which might give a way to distinguish between this theory and the Poincaré gauge theory formulation of general relativity.
3.2 Biconformal gravity

We construct a principal fiber bundle as the quotient of the conformal group of compactified $SO(p, q)$ invariant, $n = (p + q)$-dimensional space $S$ by its homogeneous Weyl subgroup. This gives a $2n$-dimensional manifold with local Weyl symmetry, $B_{p,q}$. The $2n$-dim base manifolds are called biconformal spaces.

Torsion-free biconformal spaces

The solution for torsion-free biconformal spaces, when the basis is null, are entirely described by the solder form and a trivial Weyl vector of a conformally Ricci flat configuration space.

The signature theorem

Spencer and JTW ask for a biconformal space with the following properties:

1. There exist canonically conjugate Lagrangian subspaces
2. The subspaces are submanifolds
3. The conformal Killing metric restricted to each of these submanifolds is nondegenerate
4. The submanifolds are orthogonal to one another in the restricted Killing metric

We prove the following for biconformal spaces $B_{p,q}$ satisfying 1, 2, 3:

1. The original space $S$ must be Euclidean, $p = n, q = n$, or signature zero, $p = q$.
2. The signature of the Lagrangian submanifolds is Lorentzian if $S$ is Euclidean, or $(\frac{n}{2} + 1, \frac{n}{2} - 1)$ if $S$ if $p = q$. 
Homogeneous Euclidean biconformal spaces

We have considerably strengthened these results by restricting our attention to Euclidean $S$.

In Hazboun and JTW we give the detailed construction of a class of gravity theories having the curvature-linear action

$$S = \int \left( \alpha \Omega^a e^a e^b e^c e^d + \beta \delta^a e^a \Omega^b f^e f^d + \gamma e^a f^c e^b f^d \right) e^e \cdots e^f f^g \cdots f^h \cdots \epsilon_{a b e \cdots f}^{c d g \cdots h}$$

For a certain choice of $\gamma$, these theories allow homogeneous spaces (i.e., with vanishing curvature) as solutions. These homogeneous have the following properties with no further assumptions:

1. The biconformal space is symplectic
2. The Killing metric is nondegenerate
3. The biconformal space has a complex structure
4. The biconformal space is Kähler (but not in the Killing metric)
5. There are always conjugate Lagrangian submanifolds with nondegenerate restricted Killing metric
6. The submanifolds are always Lorentzian
7. A Lorentzian connection is induced on the submanifolds
8. Two new tensor fields emerge
   a. A vector field given by the difference between the gauge in which the Weyl vector vanishes and the gauge in which the basis is orthonormal. This vector field gives the emergent timelike directions.
   b. A rank-3 symmetric tensor
9. The Weyl vector is pure gauge on the configuration and momentum submanifolds
10. Despite the vanishing conformal curvature, the Riemann curvature of the configuration and momentum spaces has
   a. Cosmological constant
   b. Cosmological dust

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