2002

Analytically Continued Hypergeometric Expression of the Incomplete Beta Function

Jack C. Straton, *Portland State University*

Analytically Continued Hypergeometric Expression of the Incomplete Beta Function

Jack C. Straton

[This is the accepted version of an article appearing in Results in Mathematics May 2002, Volume 41, Issue 3, pp 394-395, DOI 10.1007/BF03322781.]

Abstract The Incomplete Beta Function is rewritten as a Hypergeometric Function that is the analytic continuation of the conventional form, a generalization of the finite series, which simpifies the Stieltjes transform of powers of a monomial divided by powers of a binomial.

1991 Mathematics Subject Classification: 33B20, 33C05, 44A15

Key Words: Incomplete beta function, hypergeometric function, Stieltjes transforms, definite integrals

The finite hypergeometric series expression for the Incomplete Beta Function, [1]

$$ {}_2F_1(-n,1;c;z) = (1-c)z^{1-c}(z-1)^{n+c-1}B_{1-1/z}(1-c-n,n+1) \ , \tag{1} $$

may be generalized to

Theorem

$$ \begin{aligned} {}_2F_1(-\nu,1;\gamma;z) = (1-\gamma)z^{1-\gamma}(z-1)^{\nu+\gamma-1} \quad & \big[B_{1-1/z}(1-\gamma-\nu,\nu+1) \\ & - B(1-\gamma-\nu,\nu+1)\left(1-\frac{(-1)^{-\nu}\sin[\pi(\gamma+\nu)]}{\sin(\pi\gamma)}\right)\bigg] \ . \end{aligned} \tag{2} $$

The Incomplete Beta Function [2] is conventionally defined [3] with real parameters for statistical problems,

$$ B_x(p,q) = \int_0^x t^{p-1}(1-t)^{q-1}\,dt \qquad (0 \le x \le 1, \quad p,q > 0) \ , \tag{3} $$

but is a smooth function of p, q or x when any or all are taken off the real axis (though it diverges as x takes on large, real values). Its hypergeometric expression [4] is likewise well-behaved for complex parameters, so we rewrite this expression in its more general form

$$ \begin{aligned} {}_2F_1(\alpha,\beta;\beta+1;w) &= \beta w^{-\beta}B_w(\beta,1-\alpha) = \beta w^{-\beta}B(\beta,1-\alpha)\left(1-I_{1-w}(1-\alpha,\beta)\right) \\ &= \beta w^{-\beta}\left[B(1-\alpha,\beta) - B_{1-w}(1-\alpha,\beta)\right] \ . \end{aligned} \tag{4} $$

One may analytically continue the left-hand side to [5]

$$ \begin{aligned} {}_2F_1(\alpha,\beta;\beta+1;w) &= (-1)^{-\alpha}(w)^{-\alpha}\frac{\Gamma(\beta+1)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\beta+1-\alpha)}\,{}_2F_1(\alpha,\alpha-\beta;\alpha+1-\beta;1/w) \\ &+ (-1)^{-\beta}(w)^{-\beta}\frac{\Gamma(\beta+1)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(1)}\,{}_2F_1(\beta,0;\beta+1-\alpha;1/w) \ , \end{aligned} \tag{5} $$

Then equating right-hand sides of (4) and (5) and transforming the nontrivial hypergeometric function again [6] gives

$$
\begin{aligned}
(B(1-\alpha,\beta)-B_{1-w}(1-\alpha,\beta)) &= (-1)^{-\alpha}w^{-\alpha+\beta}\frac{1}{(\beta-\alpha)}\left(1-\frac{1}{w}\right)^{1-\alpha}{}_2F_1(1-\beta,1;\alpha+1-\beta;1/w) \\
&+ (-1)^{-\beta}B(1-\alpha,\beta)\frac{\Gamma[1-(\alpha-\beta)]\Gamma(\alpha-\beta)}{\Gamma(1-\alpha)\Gamma(\alpha)}\ , \qquad (6)
\end{aligned}
$$

Letting $z = 1/w$ this simplifies [7] to

$$
\begin{aligned}
B_{1-1/z}(1-\alpha,\beta) &= z^{\alpha-\beta}\frac{1}{(\beta-\alpha)}(z-1)^{1-\alpha}{}_2F_1(1-\beta,1;\alpha+1-\beta;z) \\
&+ B(1-\alpha,\beta)\left(1+(-1)^{1-\beta}\frac{\sin[\pi\alpha)]}{\sin[\pi(\alpha-\beta)]}\right)\ , \qquad (7)
\end{aligned}
$$

Finally one substitutes $\beta = \nu + 1$ and $\alpha = \gamma + \beta - 1$ and rearranges sides to obtain Eq. (2).

In addition, if one substitutes $\beta = 1 - \nu$, $\alpha = 2 - \mu$, and $z = \frac{\beta}{\gamma}$ and analytically continues the Gauss function, [8] one may obtain a more useful form for the known [9] Stieltjes transform [10] of powers of a monomial divided by powers of a binomial,

Corollary

$$
\begin{aligned}
\int_0^\infty \frac{x^{\nu-1}(\beta+x)^{1-\mu}}{\gamma+x}dx = 2\int_0^\infty \frac{x^{\nu-1/2}(\beta+x^2)^{1-\mu}}{\gamma+x^2}dx &= \pi\gamma^{\nu-1}(\beta-\gamma)^{1-\mu}\csc(\nu\pi)I_{1-\frac{\gamma}{\beta}}(\mu-1,1-\nu) \\
&= \pi\gamma^{\nu-1}(\beta-\gamma)^{1-\mu}\csc(\nu\pi)\left(1+(-1)^{\nu}\frac{\sin[\pi(2-\mu)]}{\sin[\pi(1+\nu-\mu)]}\right) \\
&- \frac{\pi\csc(\nu\pi)\beta^{\nu+1-\mu}}{(\mu-1-\nu)(\beta-\gamma)B(\mu-1,1-\nu)}\ {}_2F_1(2-\mu,1;2-\mu+\nu;\frac{\beta}{\beta-\gamma})\ , \qquad (8)
\end{aligned}
$$

$(|arg\gamma| < \pi|, |arg\beta| < \pi|, 0 < Re\ \nu < Re\ \mu)$ which is a finite series for integer $\mu > 1$.

---

[1] A.P. Prudnikov, Yu. A. Brychkov and O.I. Marichev, *Integrals and Series* (Gorden and Breach, New York, 1990) v3, p. 466, No. 7.3.1.177.

[2] *Handbook of mathematical functions with formulas, graphs, and mathematical tables,* ed M. Abramowitz and I. A. Stegun(National Bureau of Standards, Applied mathematics series, 55, Washington, 1970), p. 263, Nos. 6.6.8, 6.6.2, and 6.6.3; see also K. Pearson, *Biometrika* **16**, 202 (1924); M. J. Tretter and G. W. Walster, *SIAM J. Sci. Stat. Comput.* **1**, 321 (1980); N. M. Temme, *SIAM J. Math Anal.* **18**, 1638 (1987); B. W. Brown and L. B. Levy, *ACM Trans. Math. Software* **20**, 393 (1994); and B. G. S. Doman, *Math. Comput.* **65**, 1283 (1996).

[3] Larry C. Andrews, *Special Functions for Engineers and Applied Mathematics,* (MacMillan, New York, 1985) p. 70.

[4] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 1980), p. 950, No. 8.391.

[5] *ibid.*, p. 1043, No. 9.132.2.

[6] *ibid.*, p. 1043, No. 9.131c.

[7] *ibid.*, p. 937, No. 8334.3.

[8] *ibid.*, p. 1043, No. 9.131b.

[9] *ibid.*, p. 290, No. 3.227.2; A. Erdélyi *et al., Tables of Integral Transforms,* Vol. II (McGraw Hill, New York, 1954), p. 217, No. 10.

[10] See, for instance, D. B. Sumner, *Bull. Am. Math. Soc.* **55**, 174 (1949); A. Byrne and E. R. Love *J. Austral. Math. Soc.* **18**, 328 (1974) and *J. Math. Anal. & Appl.* **190**, 428 (1995).

Jack C. Straton
University Studies
Portland State University
Portland, OR, 97207-0751
USA

Eingegangen am 10. November 2000