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On Measurable Semigroups in R

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In the discussion in [[**1**] p. 315], of possible versions of the famous Spitzer identity for random walks on subsets D of $\mathbb{R}$, the following alternative is listed, among others: "both D and D^c have void interiors but neither is empty." This alternative is called "pathological" in [**1**]. Here D is assumed to be Borel-measurable and $D^c = \mathbb{R} \setminus D$.

It is shown in [**1**] that, in order for a version of Spitzer's identity to hold, both D and D^c must be additive semigroups. However, then the mentioned alternative is, not just "pathological," but plainly nonexistent—even if the condition of the Borel measurability of D is weakened to that of the Lebesgue measurability. Indeed, if D is Lebesgue-measurable and λ denotes the Lebesgue measure, then $\lambda(D) \vee \lambda(D^c) > 0$, which contradicts

Proposition 2. *Let D be any Lebesgue-measurable subset of $\mathbb{R}$ with $\lambda(D) > 0$ which is an additive semigroup. Then D has a nonvoid interior.*

Proof. By the regularity of the measure λ and the condition $\lambda(D) > 0$, one can find a nonempty interval $(a_0, a_1) \subset \mathbb{R}$ such that $\lambda\big(D \cap (a_0, a_1)\big) > \frac{3}{4}(a_1 - a_0)$. For simplicity, consider the normalized measure $\mu := \frac{1}{a_1 - a_0}\lambda$, so that $\mu(D_1) > \frac{3}{4}$, where $D_t := D \cap (a_0, a_t)$ and $a_t := a_0 + (a_1 - a_0)t$ for $t \in [0, 1]$.

Take now any $t \in (\frac{1}{2}, 1)$. Then

$$\mu(D_t) = \mu(D_1) - \mu\big(D \cap [a_t, a_1)\big) \geq \mu(D_1) - (1 - t) > \frac{t}{2} = \frac{1}{2}\mu\big((a_0, a_t)\big).$$

Let $C_t := a_0 + a_t - D_t = \{a_0 + a_t - x : x \in D_t\}$. Then $C_t \subseteq (a_0, a_t)$, $\mu(C_t) = \mu(D_t) > \frac{1}{2}\mu\big((a_0, a_t)\big)$, and $D_t \subseteq (a_0, a_t)$. Hence, $\mu(C_t \cap D_t) \geq \mu(C_t) + \mu(D_t) - \mu\big((a_0, a_t)\big) > 0$. So, there is some $y \in C_t \cap D_t$. For any such y, one has $a_0 + a_t - y \in D_t$ and therefore

$$a_0 + a_t = y + (a_0 + a_t - y) \in D_t + D_t \subseteq D + D \subseteq D.$$

We conclude that

$$\emptyset \neq (a_0 + a_{1/2}, a_0 + a_1) = \{a_0 + a_t : t \in (\tfrac{1}{2}, 1)\} \subseteq D,$$

so that D has a nonvoid interior. ■

REFERENCE

1. J. F. C. Kingman, Spitzer's identity and its use in probability theory. *J. London Math. Soc.* **37** (1962) 309–316.

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