Winter 2023

An exact bound for the inner product of vectors in C^n

Iosif Pinelis, *Michigan Technological University*

An exact bound for the inner product of vectors in $\mathbb{C}^n$

Iosif Pinelis

Abstract. It is shown that $1/2$ is the exact upper bound on the difference $\sum_{k=1}^n |x_k|^2|y_k|^2 - \big|\sum_{k=1}^n x_k y_k\big|^2$ for any unit vectors $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in $\mathbb{C}^n$.

Proposition 1. *Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be any two unit vectors in $\mathbb{C}^n$. Then*

$$\langle |x|^2, |y|^2\rangle - |\langle x, y\rangle|^2 \le 1/2, \tag{1}$$

where $\langle x, y\rangle := \sum_{k=1}^n x_k\overline{y_k}$ is the canonical inner product of x and y, with $\overline{z}$ denoting the complex conjugate of z, and $|z|^2 := (|z_1|^2, \dots, |z_n|^2)$ for $z = (z_1, \dots, z_n) \in \mathbb{C}^n$. The upper bound $1/2$ on $\langle |x|^2, |y|^2\rangle - |\langle x, y\rangle|^2$ in (1) is exact, as it is attained if, e.g., $n \ge 2$, $x = \frac{1}{\sqrt{2}}(1, 1, 0, \dots, 0)$, and $y = \frac{1}{\sqrt{2}}(1, -1, 0, \dots, 0)$.

Inequality (1) may be viewed as providing a lower bound on $|\langle x, y\rangle|$. As such, it may remind one of a reverse to the Cauchy–Schwarz inequality $|\langle x, y\rangle|^2 \le \langle x, x\rangle\langle y, y\rangle$ for any x and y in $\mathbb{C}^n$; see, e.g., [**1, 2**] and many references therein. One may also note here that the lower bound on the modulus $|\langle x, y\rangle|$ of the inner product $\langle x, y\rangle$ of vectors x and y in $\mathbb{C}^n$ is provided by (1) in terms of the inner product $\langle |x|^2, |y|^2\rangle$ of vectors $|x|^2$ and $|y|^2$ in $\mathbb{R}^n$.

Proof of Proposition 1. The last assertion in Proposition 1 is obvious.

As for inequality (1), it can be obtained using Lagrange multipliers [**3**], at least for x and y in $\mathbb{R}^n$. However, analyzing a "thoughtless" Lagrange multiplier solution, one may find a more elegant, "clever" solution. At least, this happened here.

First, suppose that x and y are in $\mathbb{R}^n$. Then the problem is to show that

$$\sum x_k^2 y_k^2 \le 1/2 + \Big(\sum x_k y_k\Big)^2 \tag{2}$$

given that $\sum x_k^2 = 1$ and $\sum y_k^2 = 1$.

Let

$$s_+ := \sum_{k:\ x_k y_k > 0} x_k y_k, \quad s_- := -\sum_{k:\ x_k y_k < 0} x_k y_k.$$

Then $s_+ - s_- = \sum x_k y_k$ and $s_+ + s_- = \sum |x_k y_k| \le \sqrt{\sum x_k^2}\sqrt{\sum y_k^2} = 1$, by the Cauchy–Schwarz inequality. Also, $\sum x_k^2 y_k^2 \le s_+^2 + s_-^2$. So, (2) reduces to $s_+^2 + s_-^2 \le 1/2 + (s_+ - s_-)^2$, which simplifies to $s_+ s_- \le 1/4$, and the latter inequality holds because $s_\pm \ge 0$ and $s_+ + s_- \le 1$. Thus, (2) is proved.

Consider now the general case, with any unit vectors x and y in $\mathbb{C}^n$. Replacing $\overline{y_k}$ by y_k and noting that $|\overline{y_k}| = |y_k|$, we see that the remaining problem is to show that

$$|s|^2 - \sum_{k=1}^n |x_k|^2|y_k|^2 \ge -1/2,$$

where

$$s := \sum_{k=1}^{n} v_k \quad \text{and} \quad v_k := x_k y_k.$$

For any fixed values of the $|x_j|$'s and $|y_k|$'s (so that values of the $|v_k|$'s are also fixed), minimize $|s|^2$. Let $((x_j),(y_k))$ be a corresponding minimizer, which exists by compactness and continuity. Suppose first that the minimum value of $|s|^2$ is nonzero, so that $s \neq 0$. By the triangle inequality for complex numbers s and v_1, we have $|s| \geq \big||v_1| - |s - v_1|\big|$, with the equality iff v_1 is a real multiple of s. So,

$$|s|^2 \geq (|v_1| - |s - v_1|)^2 = (|v_1| - |v_2 + \cdots + v_n|)^2,$$

with the equality (in place of the just displayed inequality) iff v_1 is a real multiple of s. Fixing now the vectors $v_2, \ldots, v_n$, we see that, for any minimizing $((x_j),(y_k))$, the vector v_1 must be a real multiple of s. Similarly, all the v_j's must be real multiples of s.

Therefore, if $s \neq 0$, then all the vectors $v_k = x_k y_k$ are real multiples of s, and the problem reduces to the already considered case when the x_j's and y_k's are real numbers.

The yet remaining problem, corresponding to the case $s = 0$, is this: show that

$$\sum_{k=1}^{n} |x_k|^2 |y_k|^2 \leq 1/2$$

for unit vectors x and y in $\mathbb{C}^n$ with $\sum_{k=1}^{n} x_k y_k = 0$.

The condition $\sum_{k=1}^{n} x_k y_k = 0$ obviously implies $|x_j y_j| \leq \sum_{k\in[n]\setminus\{j\}} |x_k y_k|$ – or, equivalently, $2|x_j y_j| \leq \sum_{k\in[n]} |x_k y_k|$ – for all $j \in [n] := \{1, \ldots, n\}$.

Letting now $a_j := |x_j|$ and $b_j := |y_j|$, we see that the problem reduces to proving the following.

Claim. *If $a_j \geq 0$, $b_j \geq 0$, and $2a_j b_j \leq \sum_{i\in[n]} a_i b_i$ for all $j \in [n]$, and if $\sum_{j\in[n]} a_j^2 = 1 = \sum_{j\in[n]} b_j^2$, then $\sum_{j\in[n]} a_j^2 b_j^2 \leq 1/2$.*

The proof of this claim is easy. Indeed, by the Cauchy–Schwarz inequality, $\sum_{i\in[n]} a_i b_i \leq 1$. So, for all $j \in [n]$ we have $2a_j b_j \leq \sum_{i\in[n]} a_i b_i \leq 1$ and hence $a_j b_j \leq 1/2$ and $\sum_{j\in[n]} a_j^2 b_j^2 \leq \sum_{j\in[n]} a_j b_j / 2 \leq 1/2$.

This completes the proof of the claim and thus the entire proof of Proposition 1. ■

REFERENCES

1. Dragomir, S. S. (2015). Reverses of Schwarz inequality in inner product spaces with applications. *Math. Nachr.* 288(7): 730–742. URL `https://doi.org/10.1002/mana.201300100`.
2. Otachel, Z. (2018). Reverse Schwarz inequalities and some consequences in inner product spaces. *J. Math. Anal. Appl.* 458(2): 1409–1426. URL `https://doi.org/10.1016/j.jmaa.2017.10.038`.
3. Pourciau, B. H. (1980). Modern multiplier rules. *Amer. Math. Monthly* 87(6): 433–452. URL `http://dx.doi.org/10.2307/2320250`.

Department of Mathematical Sciences, Michigan Technological University, Houghton, Michigan 49931
ipinelis@mtu.edu