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AN OPTIMAL UPPER BOUND ON THE TAIL PROBABILITY FOR SUMS OF RANDOM VARIABLES

IOSIF PINELIS

Abstract. Let s be any given real number. An explicit construction is provided of random variables (r.v.'s) X and Y for which $\sup \mathsf{P}(X+Y\geqslant s)$ is attained, where the sup is taken over all r.v.'s X and Y with given distributions.

Let X and Y be random variables (r.v.'s) with given distributions. Let s be a real number. Then the tail probability $\mathsf{P}(X+Y\geqslant s)$ for the sum $X+Y$ of r.v.'s X and Y can be obviously bounded from above by the sum $\mathsf{P}(X\geqslant x)+\mathsf{P}(Y>s-x)$ of the "marginal" tail probabilities for X and Y, where x is any real number; the bound $\mathsf{P}(X\geqslant x)+\mathsf{P}(Y>s-x)$ can replaced here by $\mathsf{P}(X>x)+\mathsf{P}(Y\geqslant s-x)$. It seems plausible that one cannot get a better upper bound on $\mathsf{P}(X+Y\geqslant s)$ without additional information on the joint distribution of X and Y. Indeed, using duality arguments – see e.g. [2, 1], it is not hard to show the following.

Proposition 1.

$$\sup \mathsf{P}(X+Y\geqslant s)=q(s):=\inf_{x\in\mathbb{R}}[Q_s(x)\wedge Q_s(x+)], \tag{1}$$

where the sup *is taken over all r.v.'s X and Y with given distributions, and*

$$Q_s(x):=Q_{s;X,Y}(x):=\mathsf{P}(X\geqslant x)+\mathsf{P}(Y>s-x), \quad \textit{so that} \quad Q_s(x+):=\mathsf{P}(X>x)+\mathsf{P}(Y\geqslant s-x). \tag{2}$$

The main result of this paper is an explicit construction of r.v.'s X and Y with given distributions for which the sup in (1) is attained. It will also follow that the sup equals $q(s)$, which will furnish a more direct and explicit proof of Proposition 1. Perhaps surprisingly, the mentioned construction and even the corresponding proof are rather nontrivial.

One may note here that $Q_s(x)$ is left-continuous and $Q_s(x+)$ is right-continuous in $x\in\mathbb{R}$. Replacing X by $X-s$, note also that without loss of generality (wlog)

$$s=0, \tag{3}$$

which will be assumed in the rest of this paper, unless otherwise noted.

Concerning the best bound

$$q:=q(0) \tag{4}$$

on $\mathsf{P}(X+Y\geqslant 0)$, two cases should be distinguished, depending on whether $q=1$.

Case 1: $q=1$. This is the easy case. Indeed, here for all $x\in\mathbb{R}$ one has $\mathsf{P}(X>x)+\mathsf{P}(Y\geqslant -x)=Q_0(x+)\geqslant 1$ or, equivalently, $\mathsf{P}(X>x)\geqslant \mathsf{P}(-Y>x)$.

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That is, $F_X \leqslant F_{-Y}$, where F_Z denotes the cumulative distribution function (cdf) of a r.v. Z. Now we can use the standard construction $\tilde{X} := F_X^{-1}(U)$ and $\tilde{Y} := -F_{-Y}^{-1}(U)$, where U is a r.v. uniformly distributed on the interval $(0,1)$ and

$$F^{-1}(u) := \inf\{x \in \mathbb{R}\colon F(x) \geqslant u\} = \min\{x \in \mathbb{R}\colon F(x) \geqslant u\}$$

for any cdf F and $u \in (0,1)$. Note that for all $u \in (0,1)$ and all $x \in \mathbb{R}$ one has $F^{-1}(u) \leqslant x \iff F(x) \geqslant u$. Therefore, $\mathsf{P}(\tilde{X} \leqslant x) = \mathsf{P}(F_X^{-1}(U) \leqslant x) = \mathsf{P}(F_X(x) \geqslant U) = F_X(x)$ for all $x \in \mathbb{R}$, so that $\tilde{X} \overset{\text{D}}{=} X$, where $\overset{\text{D}}{=}$ denotes the equality in distribution. Similarly, $\tilde{Y} \overset{\text{D}}{=} Y$. Moreover, for all $u \in (0,1)$ and all $x \in \mathbb{R}$ one has the implications $F_X^{-1}(u) \leqslant x \iff F_X(x) \geqslant u \implies F_{-Y}(x) \geqslant u \iff F_{-Y}^{-1}(u) \leqslant x$. Choosing here $x = F_X^{-1}(u)$, one see that $F_{-Y}^{-1} \leqslant F_X^{-1}$, whence $-\tilde{Y} = F_{-Y}^{-1}(U) \leqslant F_X^{-1}(U) = \tilde{X}$, so that $\mathsf{P}(\tilde{X} + \tilde{Y} \geqslant 0) = 1 = q$. Thus, in the case $q = 1$, we have constructed r.v.'s $\tilde{X}$ and $\tilde{Y}$ with the same distributions as X and Y, respectively, such that the tail probability $\mathsf{P}(\tilde{X} + \tilde{Y} \geqslant 0)$ equals the best upper bound q.

Case 2: $q < 1$. Note that $Q_s(x) \to 1$ as $|x| \to \infty$. So, the condition $q < 1$ implies that there exist some $a \in \mathbb{R}$ and a sequence (x_n) in $\mathbb{R}$ converging to a such that $Q_0(x_n) \wedge Q_0(x_n+) \to q$ (as $n \to \infty$). By passing to a subsequence, we see that wlog one of the following three cases takes place:

(i) $x_n = a$ for all n, which implies $Q_0(a) \wedge Q_0(a+) = q$;
(ii) $x_n < a$ for all n, which implies $Q_0(a)[= Q_0(a-)] = \lim_n \big(Q_0(x_n) \wedge Q_0(x_n+)\big) = q$; or
(iii) $x_n > a$ for all n, which implies $Q_0(a+) = \lim_n \big(Q_0(x_n) \wedge Q_0(x_n+)\big) = q$.

So, one always has $Q_0(a) = q$ or $Q_0(a+) = q$. Each of the latter two cases is obtained from the other, since $Q_{0;X,Y}(x+) = Q_{0;Y,X}(-x)$ for all $x \in \mathbb{R}$.

So, in Case 2 wlog

$$Q_0(a) = q < 1. \tag{5}$$

Moreover, by replacing X and Y respectively by $X - a$ and $Y + a$, wlog

$$a = 0. \tag{6}$$

Consider now the following construction:

$$(\tilde{X}, \tilde{Y}) := \begin{cases} \big(H_X(U), -H_{-Y}(U)\big) & \text{if } U < u_0, \\ \big(-G_{-X}(U - u_0), G_Y(U - u_0)\big) & \text{if } u_0 < U < u_1, \\ \big(-G_{-X}(U - u_0), -H_{-Y}(U - u_1 + u_0)\big) & \text{if } U > u_1, \end{cases} \tag{7}$$

where, as before, U is a r.v. uniformly distributed on the interval $(0,1)$,

$$G_Y(u) := \inf\{x \geqslant 0\colon \mathsf{P}(0 < Y \leqslant x) \geqslant u\}, \tag{8}$$

$$H_X(u) := \sup\{x \geqslant 0\colon \mathsf{P}(0 \leqslant X < x) \leqslant u\}, \tag{9}$$

$$u_0 := \mathsf{P}(X \geqslant 0), \quad u_1 := \mathsf{P}(X \geqslant 0) + \mathsf{P}(Y > 0). \tag{10}$$

In the above definitions of $G_Y(u)$ and $H_X(u)$, we follow standard conventions concerning inf and sup, which imply that $\inf\emptyset = \infty$ and $\sup[0,\infty) = \infty$. However, it will be clear from the proof of Theorem 2 below that, given conditions (5) and (6), the r.v.'s $\tilde{X}$ and $\tilde{Y}$ defined by formula (7) take only (finite) real values. On the event $\{U = u_0 \text{ or } U = u_1\}$, the random pair $(\tilde{X}, \tilde{Y})$ is not defined. Yet, formula (7)

is enough to define a pair of r.v.'s $\tilde{X}$ and $\tilde{Y}$, because the event $\{U = u_0 \text{ or } U = u_1\}$ is of zero probability.

Theorem 2. *Assuming* (5) *and* (6), *for the r.v.'s* $\tilde{X}$ *and* $\tilde{Y}$ *as in* (7) *one has the following:*

$$\tilde{X} \overset{\mathrm{D}}{=} X, \quad \tilde{Y} \overset{\mathrm{D}}{=} Y, \tag{11}$$

and

$$\mathsf{P}(\tilde{X} + \tilde{Y} \geqslant 0) = \sup \mathsf{P}(X + Y \geqslant 0) = q = \inf_{x \in \mathbb{R}} [Q_0(x) \wedge Q_0(x+)], \tag{12}$$

with the sup *as in Proposition 1.*

Proof.

Step 1. The main point at this step is to derive key comparisons between the functions G_Y and G_{-X} and between the functions H_X and H_{-Y}, defined according to (8) and (9). Conditions (5) and (6) imply that for all $x \in \mathbb{R}$ one has $Q_0(0) \leqslant Q_0(x)$, which in turn implies

$$\mathsf{P}(0 < -X \leqslant x) \geqslant \mathsf{P}(0 < Y \leqslant x) \quad \text{and} \quad \mathsf{P}(0 \leqslant -Y < x) \geqslant \mathsf{P}(0 \leqslant X < x). \tag{13}$$

In particular, by letting $x \to \infty$, it follows that

$$\mathsf{P}(X < 0) \geqslant \mathsf{P}(Y > 0) \quad \text{and, equivalently,} \quad \mathsf{P}(Y \leqslant 0) \geqslant \mathsf{P}(X \geqslant 0).$$

By (8) and the right continuity of $\mathsf{P}(0 < Y \leqslant x)$ in x, for any real $x \geqslant 0$ and any $u \in (0, 1)$ one has

$$x \geqslant G_Y(u) \iff \mathsf{P}(0 < Y \leqslant x) \geqslant u, \tag{14}$$

whence (by letting $x = G_Y(u)$ in (14))

$$G_Y(u) > 0 \tag{15}$$

and, by (13),

$$G_Y(u) \geqslant G_{-X}(u). \tag{16}$$

Similarly, again for any real $x \geqslant 0$ and any $u \in (0, 1)$ one has

$$x \leqslant H_X(u) \iff \mathsf{P}(0 \leqslant X < x) \leqslant u, \tag{17}$$

$$H_X(u) \geqslant 0, \tag{18}$$

and

$$H_X(u) \geqslant H_{-Y}(u). \tag{19}$$

Step 2. At this step, we are going to use tools prepared at Step 1 to verify (11). First here, by (7) and (18),

$$\mathsf{P}(\tilde{X} < 0, U < u_0) = 0 = \mathsf{P}(\tilde{Y} > 0, U < u_0). \tag{20}$$

In the rest of the proof, let x stand for an arbitrary positive real number. Then one has the following. By (7), (18), (17), and (10),

(21)

$$\begin{aligned} \mathsf{P}(0 \leqslant \tilde{X} < x, U < u_0) &= \mathsf{P}\big(H_X(U) < x, U < u_0\big) = \mathsf{P}\big(\mathsf{P}(0 \leqslant X < x) > U, U < u_0\big) \\ &= \mathsf{P}(0 \leqslant X < x) \end{aligned}$$

and

$$\begin{aligned}\mathsf{P}(-x<\tilde Y\leqslant 0,U<u_0)&=\mathsf{P}\big(H_{-Y}(U)<x,U<u_0\big)\\&=\mathsf{P}\big(\mathsf{P}(0\leqslant -Y<x)>U,U<u_0\big)=u_0\wedge\mathsf{P}(0\leqslant -Y<x)\\&=\mathsf{P}(X\geqslant 0)\wedge\mathsf{P}(-x<Y\leqslant 0).\end{aligned}\tag{22}$$

By (7) and (15),

$$\mathsf{P}(\tilde X\geqslant 0,u_0<U<u_1)=0=\mathsf{P}(\tilde Y\leqslant 0,u_0<U<u_1).\tag{23}$$

By (7), (15), (14), and (10),

$$\begin{aligned}\mathsf{P}(-x\leqslant\tilde X<0,u_0<U<u_1)&=\mathsf{P}\big(G_{-X}(U-u_0)\leqslant x,u_0<U<u_1\big)\\&=\mathsf{P}\big(\mathsf{P}(0<-X\leqslant x)\geqslant U-u_0,u_0<U<u_1\big)\\&=(u_1-u_0)\wedge\mathsf{P}(-x\leqslant X<0)\\&=\mathsf{P}(Y>0)\wedge\mathsf{P}(-x\leqslant X<0)\end{aligned}\tag{24}$$

and

$$\begin{aligned}\mathsf{P}(0<\tilde Y\leqslant x,u_0<U<u_1)&=\mathsf{P}\big(G_{Y}(U-u_0)\leqslant x,u_0<U<u_1\big)\\&=\mathsf{P}\big(\mathsf{P}(0<Y\leqslant x)\geqslant U-u_0,u_0<U<u_1\big)\\&=(u_1-u_0)\wedge\mathsf{P}(0<Y\leqslant x)\\&=\mathsf{P}(Y>0)\wedge\mathsf{P}(0<Y\leqslant x)\\&=\mathsf{P}(0<Y\leqslant x).\end{aligned}\tag{25}$$

By (7), (15), and (18),

$$\mathsf{P}(\tilde X\geqslant 0,U>u_1)=0=\mathsf{P}(\tilde Y>0,U>u_1).\tag{26}$$

By (7), (15), (14), and (10),

$$\begin{aligned}\mathsf{P}(-x\leqslant\tilde X<0,U>u_1)&=\mathsf{P}\big(G_{-X}(U-u_0)\leqslant x,U>u_1\big)\\&=\mathsf{P}\big(\mathsf{P}(0<-X\leqslant x)\geqslant U-u_0,U>u_1\big)\\&=0\vee\{1\wedge[u_0+\mathsf{P}(-x\leqslant X<0)]-u_1\}\\&=0\vee\{\mathsf{P}(-x\leqslant X<0)-\mathsf{P}(Y>0)\}\\&=\mathsf{P}(Y>0)\vee\mathsf{P}(-x\leqslant X<0)-\mathsf{P}(Y>0)\end{aligned}\tag{27}$$

and, by (7), (18), (17), and (10),

$$\begin{aligned}\mathsf{P}(-x<\tilde Y\leqslant 0,U>u_1)&=\mathsf{P}\big(H_{-Y}(U-u_1+u_0)<x,U>u_1\big)\\&=\mathsf{P}\big(\mathsf{P}(0\leqslant -Y<x)>U-u_1+u_0,U>u_1\big)\\&=0\vee\{1\wedge[u_1-u_0+\mathsf{P}(-x<Y\leqslant 0)]-u_1\}\\&=0\vee\{\mathsf{P}(-x<Y\leqslant 0)-\mathsf{P}(X\geqslant 0)\}\\&=\mathsf{P}(X\geqslant 0)\vee\mathsf{P}(-x<Y\leqslant 0)-\mathsf{P}(X\geqslant 0).\end{aligned}\tag{28}$$

By (21), (23), and (26),

$$
\begin{aligned}
&\mathsf{P}(0 \leqslant \tilde{X} < x) \\
&= \mathsf{P}(0 \leqslant \tilde{X} < x, U < u_0) + \mathsf{P}(0 \leqslant \tilde{X} < x, u_0 < U < u_1) + \mathsf{P}(0 \leqslant \tilde{X} < x, U > u_1) \\
&= \mathsf{P}(0 \leqslant X < x) + 0 + 0 = \mathsf{P}(0 \leqslant X < x).
\end{aligned} \tag{29}
$$

Similarly, by (20), (24), and (27),

$$
\begin{aligned}
\mathsf{P}(-x \leqslant \tilde{X} < 0) = 0 &+ \mathsf{P}(Y > 0) \wedge \mathsf{P}(-x \leqslant X < 0) \\
&+ \mathsf{P}(Y > 0) \vee \mathsf{P}(-x \leqslant X < 0) - \mathsf{P}(Y > 0) \\
&= \mathsf{P}(-x \leqslant X < 0).
\end{aligned} \tag{30}
$$

By (20), (25), and (26),

$$
\mathsf{P}(0 < \tilde{Y} \leqslant x) = 0 + \mathsf{P}(0 < Y \leqslant x) + 0 = \mathsf{P}(0 < Y \leqslant x). \tag{31}
$$

By (22), (23), and (28),

$$
\begin{aligned}
\mathsf{P}(-x < \tilde{Y} \leqslant 0) &= \mathsf{P}(X \geqslant 0) \wedge \mathsf{P}(-x < Y \leqslant 0) + 0 + \\
&+ \mathsf{P}(X \geqslant 0) \vee \mathsf{P}(-x < Y \leqslant 0) - \mathsf{P}(X \geqslant 0) \\
&= \mathsf{P}(-x < Y \leqslant 0).
\end{aligned} \tag{32}
$$

Since x was assumed to be an arbitrary positive real number, (11) follows immediately from (29), (30), (31), and (32).

Step 3. At this step, we shall verify (12). The last equality there follows immediately from (4) and the definition of $q(s)$ in (1). The inequality $\mathsf{P}(\tilde{X}+\tilde{Y} \geqslant 0) \leqslant \sup \mathsf{P}(X+Y \geqslant 0)$ (cf. the first equality in (12)) follows immediately from (11). The inequality $\sup \mathsf{P}(X+Y \geqslant 0) \leqslant \inf_{x\in\mathbb{R}}[Q_0(x) \wedge Q_0(x+)]$ (cf. the last two equalities in (12)) is obvious; cf. the discussion preceding Proposition 1. By (5), (6), and (10),

$$
q = Q_0(0) = \mathsf{P}(X \geqslant 0) + \mathsf{P}(Y > 0) = u_1. \tag{33}
$$

So, to complete Step 3 of the proof and thus the entire proof of Theorem 2, it suffices to check that $\mathsf{P}(\tilde{X}+\tilde{Y} \geqslant 0) = u_1$, which follows because, by (7), (19), (16), (15), and (18),

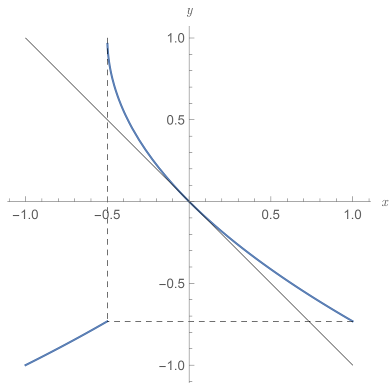
$$
\begin{aligned}
\mathsf{P}(\tilde{X}+\tilde{Y} \geqslant 0) &= \mathsf{P}(\tilde{X}+\tilde{Y} \geqslant 0, U < u_0) + \mathsf{P}(\tilde{X}+\tilde{Y} \geqslant 0, u_0 < U < u_1) \\
&+ \mathsf{P}(\tilde{X}+\tilde{Y} \geqslant 0, U > u_1) = \mathsf{P}(U < u_0) + \mathsf{P}(u_0 < U < u_1) + 0 = u_1.
\end{aligned}
$$

Theorem 2 is now completely proved. □

Example 1. Let X be any r.v. uniformly distributed on the interval $(-1,1)$, and let Y be any r.v. with the "triangular" density function $x \mapsto \frac{1-x}{2}\,\mathrm{I}\{-1 < x < 1\}$, where $\mathrm{I}\{\cdot\}$ denotes the indicator function. Straightforward but tedious calculations show that here $q(s) = \mathrm{I}\big\{s \leqslant -\frac{1}{2}\big\} + \frac{3-2s}{4}\,\mathrm{I}\big\{-\frac{1}{2} < s \leqslant 1\big\} + \frac{(2-s)^2}{4}\,\mathrm{I}\{1 < s \leqslant 2\}$ for all $s \in \mathbb{R}$, in accordance with the definition of $q(s)$ in (1). Moreover (recall (3) and (4)), in this case $q = q(0) = Q_0(0) = \frac{3}{4}$, so that conditions (5) and (6) hold. Also, here $u_0 = \frac{1}{2}$ and $u_1 = \frac{3}{4}$. Furthermore, in accordance with (7), here

$$(\tilde{X},\tilde{Y}) = (2U, 1-\sqrt{1+4U})\,\mathrm{I}\{U<\tfrac{1}{2}\} + (1-2U, 1-\sqrt{3-4U})\,\mathrm{I}\{\tfrac{1}{2}\leqslant U<\tfrac{3}{4}\} \\ + (1-2U, 1-2\sqrt{U})\,\mathrm{I}\{U>\tfrac{3}{4}\}.$$

The support (set) of the joint distribution of the random pair $(\tilde{X},\tilde{Y})$ is shown in the picture here, as well as the line $\{(x,-x)\colon\ -1<x<1\}$; in this case, the joint distribution is completely determined by its support and the condition that the r.v. $\tilde{X}$ is uniformly distributed on $(-1,1)$.



Example 2. Suppose that $\mathsf{P}(X=-1)=\mathsf{P}(X=1)=1/2$, $\mathsf{P}(Y=-1)=2/3=1-\mathsf{P}(Y=0)$. Then $q=q(0)=Q_0(0)=\frac{1}{2}$, so that conditions (5) and (6) again hold. Also, here $u_0=u_1=\frac{1}{2}$. Furthermore, in accordance with (7), here

$$(\tilde{X},\tilde{Y}) = (1,0)\,\mathrm{I}\{U<\tfrac{1}{3}\} + (1,-1)\,\mathrm{I}\{\tfrac{1}{3}\leqslant U<\tfrac{1}{2}\} + (-1,-1)\,\mathrm{I}\{U>\tfrac{1}{2}\}.$$

So, the support (set) of the joint distribution of the random pair $(\tilde{X},\tilde{Y})$ is the three-point set $\{(1,0),(1,-1),(-1,-1)\}$; more specifically, $(\tilde{X},\tilde{Y})$ takes values $(1,0),(1,-1),(-1,-1)$ with probabilities $\frac{1}{3},\frac{1}{6},\frac{1}{2}$, respectively.

Remark 3. In contrast with $\sup\mathsf{P}(X+Y\geqslant s)$, considered in this note, $\sup\mathsf{P}(X+Y>s)$ is not attained in general – even when the given cdf's F_X and F_Y are continuous. For instance, suppose that $s=0$ and $F_X=F_{-Y}=:F$, and the cdf F is continuous (and hence uniformly continuous) on $\mathbb{R}$. Then, for real $\varepsilon>0$,

$$Q_\varepsilon(x)=\mathsf{P}(X\geqslant x)+\mathsf{P}(Y>\varepsilon-x)=1-\mathsf{P}(x-\varepsilon\leqslant X<x)\underset{\varepsilon\downarrow 0}{\longrightarrow}1$$

uniformly in $x\in\mathbb{R}$. So, again for real $\varepsilon>0$,

$$\sup\mathsf{P}(X+Y>0)\geqslant\sup\mathsf{P}(X+Y\geqslant\varepsilon)=\inf_{x\in\mathbb{R}}[Q_\varepsilon(x)\wedge Q_\varepsilon(x+)]\underset{\varepsilon\downarrow 0}{\longrightarrow}1,$$

whence

$$\sup\mathsf{P}(X+Y>0)=1.$$

Suppose now that the latter sup is attained, so that $\mathsf{P}(X+Y>0)=1$ for some r.v.'s X and Y with $F_X=F_{-Y}=F$. Since the event $\{X+Y>0\}$ is the union of events $\{X>r\geqslant -Y\}$ over all rational r, we will have $\mathsf{P}(X>r\geqslant -Y)>0$ for some such r. Therefore and because $X>-Y$ almost surely,

$$0<\mathsf{P}(X>r\geqslant -Y)=\mathsf{P}(-Y\leqslant r)-\mathsf{P}(X\leqslant r)=F(r)-F(r)=0,$$

which is a contradiction, confirming the non-attainment.

In conclusion, let us present

Proposition 4. *The common value, $q(s)$, of the* sup *and the* inf *in* (1) *is left-continuous in $s \in \mathbb{R}$.*

Proof. One may observe that $\mathsf{P}(X+Y \geqslant s)$ is obviously left-continuous and nonincreasing in $s \in \mathbb{R}$. However, this observation does not seem helpful in this situation, because the supremum of a family of left-continuous nonincreasing functions does have to be left-continuous. E.g., consider the family of continuous nonincreasing functions $\mathbb{R} \ni s \mapsto 0 \vee (1 \wedge (-s/\varepsilon))$ for real $\varepsilon > 0$; the supremum of this family is the function $\mathbb{R} \ni s \mapsto \mathrm{I}\{s < 0\}$, which is not left-continuous at 0.

So, a more subtle approach is needed in this proof. First here, it suffices to prove that $q(s)$ is left-continuous in s at $s = 0$; recall the sentence containing formula (3). Recall also (4).

Case 1: $q = 1$, introduced on page 1, is easy here as well. Indeed, then $1 \geqslant \lim_{t\uparrow 0} \sup \mathsf{P}(X+Y \geqslant t) = q(0-) \geqslant q(0) = q = 1$, whence $q(0-) = q(0)$.

So, it remains to consider

Case 2: $q < 1$. Then, as before, wlog we have (5) and (6), that is,

$$q(0) = g(0-) + h(0), \tag{34}$$

where

$$g(x) := \mathsf{P}(X > x), \quad h(x) := \mathsf{P}(Y > x).$$

The key observation is that, in view of (1) and (2), for all $t \in \mathbb{R}$ we have

$$q(t) = \inf_{x,y\in\mathbb{R}} m_t(x,y), \quad \text{where} \quad m_t(x,y) := [g(x-)+h(t-x)] \wedge [g(y)+h((t-y)-)],$$

with y possibly different from x. Therefore and because $q(s)$ is nonincreasing in s and the function h is right-continuous, for real $t < 0$ we have

$$\begin{aligned} q(0) \leqslant q(t) &= \inf_{x,y\in\mathbb{R}} m_t(x,y) \leqslant m_t(t,2t) \\ &= [g(t-)+h(0)] \wedge [g(2t)+h((-t)-)] \underset{t\uparrow 0}{\longrightarrow} [g(0-)+h(0)] \wedge [g(0-)+h(0)] \\ &\qquad\qquad\qquad\qquad\qquad = g(0-)+h(0) = q(0), \end{aligned}$$

by (34). So, in Case 2 as well, we have $q(0-) = q(0)$. $\square$

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