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cos,sin-ineqs.pdf

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IDENTITIES AND INEQUALITIES FOR THE COSINE AND SINE FUNCTIONS

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ABSTRACT. Identities and inequalities for the cosine and sine functions are obtained.

1. STATEMENTS AND DISCUSSION

The basic result of this note is

Theorem 1.1. *For any real x*

$$\cos \pi x = \sum_{j=1}^{\infty} t_j \pi^{2j} (1/4 - x^2)^j, \tag{1.1}$$

where

$$t_j := \sum_{k=0}^{\infty} a_{j,k}, \quad a_{j,k} := \frac{(-\pi^2/4)^k}{(2j+2k)!} \binom{j+k}{j}. \tag{1.2}$$

Moreover, one has the recurrence

$$t_j = \frac{2(2j-3)}{\pi^2 j} t_{j-1} - \frac{1}{\pi^2 j(j-1)} t_{j-2} \quad \text{for } j = 2, 3, \dots, \tag{1.3}$$

with $t_0 = 0$ and $t_1 = 1/\pi$.

Furthermore, for all natural j

$$0 < t_j < \frac{1}{(2j)!}, \tag{1.4}$$

and

$$t_j \sim \frac{1}{(2j)!} \quad \text{as } j \to \infty.$$

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The necessary proofs will be given in Section 2.

Recurrence (1.3) allows one to compute the coefficients t_j in (1.1) quickly and efficiently. In particular, we see that

$$\begin{aligned}(t_1,\dots,t_5) &= \left(\frac{1}{\pi},\frac{1}{\pi^3},\frac{12-\pi^2}{6\pi^5},\frac{10-\pi^2}{2\pi^7},\frac{1680-180\pi^2+\pi^4}{120\pi^9}\right)\\ &\approx \left(0.318, 0.0323, 1.16\times10^{-3}, 2.16\times10^{-5}, 2.46\times10^{-7}\right).\end{aligned}$$

On the other hand, inequalities (1.4) together with identity (1.1) will serve as the source of other inequalities, which begin with the following:

Corollary 1.2. *For each natural m, consider the polynomial*

$$P_m(x) := \sum_{j=1}^{m} t_j \pi^{2j}(1/4-x^2)^j, \tag{1.5}$$

which is the mth partial sum of the series in (1.1)*. Then for all* $x\in(-1/2,1/2)$

$$P_m(x) < P_{m+1}(x) < \cos\pi x \tag{1.6}$$

and

$$\begin{aligned}0<\delta_m(x) := \cos\pi x - P_m(x) &< \frac{\pi^{2m+2}(1/4-x^2)^{m+1}}{(2m+2)!}\,\frac{1}{1-q_m}\\ &\underset{m\to\infty}{\sim} \delta_m^*(x) := \frac{\pi^{2m+2}(1/4-x^2)^{m+1}}{(2m+2)!},\end{aligned} \tag{1.7}$$

where

$$q_m := \frac{\pi^2/4}{(2m+4)(2m+3)}$$

and the asymptotic relation holds uniformly in $x\in(-\frac{1}{2},\frac{1}{2})$.

Remark 1.3. Note that the function δ_m is analytic. So, in view of [3, Proposition I] (proved e.g. in [1, page 29]), it follows from (1.7) that, for each natural m, $P_m(x)$ is the Hermite interpolating polynomial (HIP) (of degree $2m$) determined by the $2m+2$ conditions $\delta_m^{(j)}(\pm\frac{1}{2})=0$ for $j=0,\dots,m$; in fact, by Pólya's theorem [3, Theorem I], the polynomial $P_m(x)$ is already determined by any $2m+1$ of the just mentioned $2m+2$ conditions.

Explicit expressions of the general HIP were given, in particular, in [2, 4]. It is unclear, though, how to use those results to show that the polynomial $P_m(x)$, as defined in (1.5), is the HIP; nor is it seen how to derive monotonicity property (1.6) or the bound in (1.7) from the mentioned expressions. □

One also has

Proposition 1.4. *For all natural j*

$$t_j = \frac{\pi^{1-j}}{2j!} J_{j-1/2}(\pi/2), \tag{1.8}$$

where

$$J_\nu(z) := \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)} \tag{1.9}$$

is an expression defining the Bessel function (of the first kind) – as e.g. is done in [5, page 359].

In view of the identity $\sin\pi x = \cos\pi(x-1/2)$, one immediately obtains the corresponding results for $\sin\pi x$ instead of $\cos\pi x$. More specifically, we have

Corollary 1.5. *Take any real x. Then*

$$\sin\pi x = \sum_{j=1}^{\infty} t_j \pi^{2j} (x(1-x))^j. \tag{1.10}$$

Also, for all natural m and all $x \in (0,1)$

$$Q_m(x) < Q_{m+1}(x) < \sin\pi x \tag{1.11}$$

and

$$\begin{aligned} 0 < \sin\pi x - Q_m(x) &< \frac{\pi^{2m+2}(x(1-x))^{m+1}}{(2m+2)!} \frac{1}{1-q_m} \\ &\underset{m\to\infty}{\sim} \frac{\pi^{2m+2}(x(1-x))^{m+1}}{(2m+2)!}, \end{aligned} \tag{1.12}$$

where

$$Q_m(x) := P_m(x-1/2) = \sum_{j=1}^{m} t_j \pi^{2j} (x(1-x))^j. \tag{1.13}$$

Remark 1.6. One may compare expansion (1.10) with the Maclaurin expansion

$$\sin\pi x = -\sum_{j=1}^{\infty} \frac{(-\pi x)^{2j-1}}{(2j-1)!}. \tag{1.14}$$

For any natural m, the approximation of $\sin\pi x$ by the corresponding Maclaurin polynomial

$$S_m(x) := -\sum_{j=1}^{m} \frac{(-\pi x)^{2j-1}}{(2j-1)!}$$

is exact to order $2m$ at $x=0$, but it is not exact to any order at $x=1$. In contrast, in view of (1.11), the approximation of $\sin\pi x$ by $Q_m(x)$ is exact to order m at both $x=0$ and

$x = 1$. Also, in view of (1.11), the approximation of $\sin \pi x$ by $Q_m(x)$ is monotonic in m, whereas the approximation of $\sin \pi x$ by $S_m(x)$ is alternating:

$$S_{2j}(x) < S_{2j+2}(x) < \sin \pi x < S_{2j-1}(x) < S_{2j+1}(x)$$

for all natural j and all real $x > 0$. These observations are illustrated in Fig. 1.

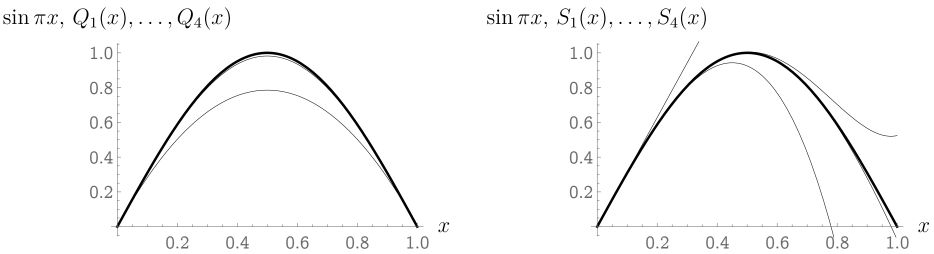

FIGURE 1. Left panel: Graphs $\{(x, \sin \pi x) \colon x \in [0,1]\}$ (thick) and $\{(x, Q_m(x)) \colon x \in [0,1]\}$ (thin) for $m = 1,2,3,4$. Right panel: Graphs $\{(x, \sin \pi x) \colon x \in [0,1]\}$ (thick) and $\{(x, S_m(x)) \colon x \in [0,1]\}$ (thin) for $m = 1,2,3,4$.

We see that $Q_3(x)$ and $Q_4(x)$ are visually indistinguishable from $\sin \pi x$ for $x \in [0,1]$; in contrast, $S_1(x), S_2(x), S_3(x), S_4(x)$ are all visually distinguishable from $\sin \pi x$ for $x \in [0,1]$. □

Remark 1.7. Inequalities (1.6) and (1.11) can of course be used to prove other inequalities, which may have exactness or near-exactness properties. For example, we can quickly prove that

$$f(x) := \frac{4}{9} + 15x^2 - 8x + \frac{4\left(2\sin^2(\pi x) + \sin^2(2\pi x)\right)}{\pi^2} > 0$$

for $x \in [0, 1/2]$. Indeed, by (1.11), we have $f \geqslant f_4$ on $[0, 1/2]$, where f_4 is the polynomial function obtained from f by replacing the function $u \mapsto \sin \pi u$ in the above expression for f by the polynomial function Q_4. The positivity of any polynomial on any interval can be verified purely algorithmically, which in this case gives $f_4 > 0$ on $(0, 1/2]$, and hence $f > 0$ on $[0, 1/2]$. The graphs of the functions f and $f - f_4$ are shown in Fig. 2. □

2. PROOFS

Proof of Theorem 1.1. Take any real x and let

$$y := 1/4 - x^2, \tag{2.1}$$

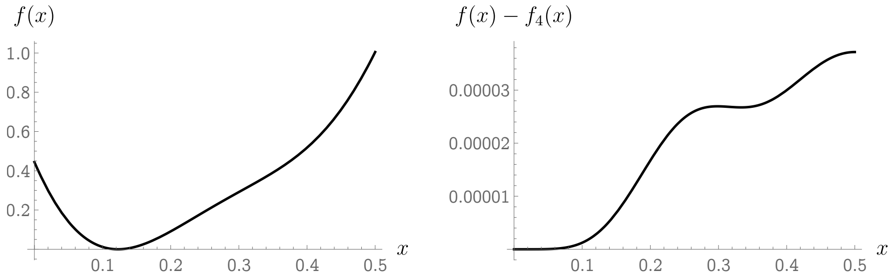

FIGURE 2. Graphs of of the functions f (left panel) and $f - f_4$ (right panel).

so that $y \leqslant 1/4$ and

$$\begin{aligned}
\cos \pi x = f(y) := \cos\Big(\frac{\pi}{2}\sqrt{1-4y}\Big) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\Big(\frac{\pi}{2}\sqrt{1-4y}\Big)^{2n} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\Big(\frac{\pi^2}{4}\Big)^n (1-4y)^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\Big(\frac{\pi^2}{4}\Big)^n \sum_{j=0}^{n}\binom{n}{j}(-4y)^j \\
&= \sum_{j=0}^{\infty}(-4y)^j \sum_{n=j}^{\infty} \frac{(-1)^n}{(2n)!}\Big(\frac{\pi^2}{4}\Big)^n \binom{n}{j} \qquad (2.2) \\
&= \sum_{j=0}^{\infty}(\pi^2 y)^j t_j = \sum_{j=1}^{\infty}(\pi^2 y)^j t_j; \qquad (2.3)
\end{aligned}$$

the equality in (2.2) follows by the Fubini theorem, the first equality in (2.3) follows by the definition of t_j in (1.2), and the second equality in (2.3) follows because $t_0 = f(0) = 0$. Thus, identity (1.1) is proved.

We have already noticed that $t_0 = f(0) = 0$. Similarly, $t_1 = f'(0)/\pi^2 = 1/\pi$. As for (1.3), it is the special case, with $z = \pi^2/4$, of the recurrence

$$T_j(z) = \frac{2j-3}{2jz}\,T_{j-1}(z) - \frac{1}{4j(j-1)z}\,T_{j-2}(z) \quad \text{for } j = 2, 3, \dots, \tag{2.4}$$

where

$$T_j(z) := \sum_{k=0}^{\infty} \frac{(-z)^k}{(2j+2k)!}\binom{j+k}{j}, \tag{2.5}$$

so that $t_j = T_j(\pi^2/4)$. In turn, identity (2.4) can be verified by a straightforward comparison of the coefficients of the powers of z on both sides of the identity.

Next, by (1.2), for $j = 1, 2, \ldots$ and $k = 0, 1, \ldots$ the ratio

$$\frac{a_{j,k+1}}{-a_{j,k}} = \frac{\pi^2}{8(k+1)(2j+2k+1)}$$

is positive and less than 1, and this ratio tends to 0 uniformly in $k = 0, 1, \ldots$ as $j \to \infty$. Therefore, $0 < t_j < a_{j,0} = \frac{1}{(2j)!}$ for all $j = 1, 2, \ldots$, and $t_j \sim a_{j,0} = \frac{1}{(2j)!}$ as $j \to \infty$. which verifies the last sentence of Theorem 1.1. □

Proof of Corollary 1.2. The inequalities in (1.6) follow immediately from (1.5), (1.1), and the first inequality in (1.4).

Recalling the definition of $\delta_m(x)$ in (1.7), identity (1.1), the definition (2.1) of y, and the second inequality in (1.4), we see that

$$\delta_m(x) = \sum_{j=m+1}^{\infty} t_j \pi^{2j} y^j < \sum_{j=m+1}^{\infty} b_j(y)$$

for all $x \in (-1/2, 1/2)$, where

$$b_j(y) := \frac{(\pi^2 y)^j}{(2j)!}.$$

Moreover, for any natural m, any natural $j \geqslant m+1$, and any $y \in (0, 1/4]$,

$$\frac{b_{j+1}(y)}{b_j(y)} = \frac{\pi^2 y}{(2j+2)(2j+1)} \leqslant \frac{\pi^2/4}{(2m+4)(2m+3)} = q_m < 1,$$

and $q_m \to 0$ as $m \to \infty$. Thus, we have verified (1.7), which completes the proof of Corollary 1.2. □

Proof of Proposition 1.4. Identity (1.8) is a special case, with $z = \pi^2/4$, of the identity

$$T_j(z) = \frac{\sqrt{\pi}}{j!\,2^{j+1/2}} z^{1/4-j/2} J_{j-1/2}(\sqrt{z}) \tag{2.6}$$

for real $z > 0$, with $T_j(z)$ as defined in (2.5). In turn, to verify identity (2.6), it is enough to compare the coefficients of the corresponding powers of z in both sides of (2.6), which is done with the help of the identity

$$\Gamma(n+1/2) = \frac{\sqrt{\pi}\,(2n)!}{4^n n!}$$

for $n = 0, 1, \ldots$, which in turn is easy to check by induction. □

REFERENCES

[1] P. J. Davis. *Interpolation and approximation*. Blaisdell Publishing Co. Ginn and Co. New York-Toronto-London, 1963.

[2] T. N. E. Greville. A generalization of Waring's formula. *Ann. Math. Statistics*, 15:218–219, 1944.

[3] S. Karlin and J. M. Karon. On Hermite-Birkhoff interpolation. *J. Approximation Theory*, 6:90–115, 1972. Collection of articles dedicated to J. L. Walsh on his 75th birthday, V (Proc. Internat. Conf. Approximation Theory, Related Topics and their Applications, Univ. Maryland, College Park, Md., 1970).

[4] A. Spitzbart. A generalization of Hermite's interpolation formula. *Amer. Math. Monthly*, 67:42–46, 1960.

[5] E. T. Whittaker and G. N. Watson. *A course of modern analysis*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1996. An introduction to the general theory of infinite processes and of analytic functions; with an account of the principal transcendental functions, Reprint of the fourth (1927) edition.

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