Fall October 19, 2016

# power-series.pdf

Iosif Pinelis, *Michigan Technological University*

https://terrytao.wordpress.com/2016/10/18/a-problem-involving-power-series/

Let us prove the following:

**Theorem 1.** *Let $(a_0, a_1, \dots)$ be a bounded sequence in $\mathbb{C}$, and suppose that for the power series*

$$f(z) := \sum_{n=0}^{\infty} a_n \frac{z^n}{n!}$$

*one has $f(x) = O(e^{-x})$ as $\mathbb{R} \ni x \to \infty$. Then $a_n = C(-1)^n$ for some $C \in \mathbb{C}$ and all $n = 0, 1, \dots$.*

*Proof.* Consider the sets

$$S_1 := \{z \in \mathbb{C} \colon \Re z > -1\} \quad \text{and} \quad S_2 := \{z \in \mathbb{C} \colon |z| > 1\}.$$

Define functions $g_1 \colon S_1 \to \mathbb{C}$ and $g_2 \colon S_2 \to \mathbb{C}$ by the formulas

$$g_1(z) := (z+1) \int_0^\infty e^{-zt} f(t) dt$$

for $z \in S_1$ and

$$g_2(z) := (z+1) \sum_{n=0}^{\infty} a_n / z^{n+1}$$

for $z \in S_2$. These functions are well defined and analytic, since $f(x) = O(e^{-x})$ as $\mathbb{R} \ni x \to \infty$ and the $a_n$'s are bounded. Moreover, because $\int_0^\infty e^{-zx} x^n dx = n!/z^{n+1}$ for all $z \in S_0 := \{w \in \mathbb{C} \colon \Re w > 1\} \subset S_1 \cap S_2$, one has $g_1 = g_2$ on $S_0$. So, $g_1$ and $g_2$ are the restrictions to $S_1$ and $S_2$ of an analytic function $g \colon S \to \mathbb{C}$, where $S := S_1 \cup S_2 = \mathbb{C} \setminus \{-1\}$. Moreover, $g(u) = g_1(u) = (u+1) \int_0^\infty e^{-ut} f(t) dt = O((u+1) \int_0^\infty e^{-ut-t} dt) = O(1)$ for real $u > -1$ and $|g(z)| = |g_2(z)| = O(|z| \sum_{n=0}^\infty 1/|z|^{n+1}) = O(1)$ as $|z| \to \infty$.

We shall show that $-1$ is a pole of $g$. Hence, by considering (say) the Laurent series for the function $g$ at the point $-1$, we conclude that $g$ is a complex constant, say $C$. Thus,

$$\sum_{n=0}^{\infty} a_n / z^{n+1} = \frac{g_2(z)}{z+1} = \frac{C}{z+1} = C \frac{1}{z(1+1/z)} = \sum_{n=0}^{\infty} C(-1)^n / z^{n+1}$$

for $z \in S_2$, and so, indeed $a_n = C(-1)^n$ for all $n = 0, 1, \dots$.

It remains to prove

**Lemma 2.** *Let $K := \sup_n |a_n| \vee \sup_{t \ge 0} |f(t)| e^t < \infty$. Take any $z \in \mathbb{C}$ such that $0 < |z+1| < 1/2$. Then*

$$|g(z)| \le 6K/|z+1|^2.$$

*So, $-1$ is a pole of $g$.*

*Proof.* Let $z$ be as in the statement of the lemma. Then $1/2 \le |z| \le 2$ and $x \le -1/2 < 0$, where $x := \Re z$ and $y := \Im z$. Consider the following three possible cases.

**Case 1:** $x = \Re z \le -1$. Then $z \in S_2$, whence

$$|g(z)| = |g_2(z)| \le \frac{1}{2} K \sum_{n=0}^{\infty} 1/|z|^{n+1} \le \frac{K}{|z|-1}.$$

Moreover, here

$|z+1|^2 = (|x|-1)^2+y^2 \le |x|-1+y^2 \le (|x|-1)(|x|+1)+y^2 = |z|^2-1 \le 3(|z|-1).$

So, in Case 1

$$|g(z)| \le \frac{3K}{|z+1|^2}.$$

**Case 2:** $|z| \le 1$. Then $z \in S_1$, whence

$$|g(z)| = |g_1(z)| \le \frac{1}{2}\,K \int_0^\infty e^{-xt-t}dt \le \frac{K}{1+x}.$$

Moreover, here $y^2 \le 1-x^2 \le 2(1+x)$, whence

$$|z+1|^2 = (1+x)^2+y^2 \le (1+x)^2+2(1+x) \le 4(1+x).$$

So, in Case 2

$$|g(z)| \le \frac{4K}{|z+1|^2}.$$

**Case 3:** $x = \Re z > -1$ **and** $|z| > 1$. Then $z \in S_1 \cap S_2$. Moreover, here

$$\begin{aligned} |z+1|^2 &= (1+x)^2+y^2 = 2(1+x)+x^2+y^2-1 = 2(1+x)+|z|^2-1 \\ &\le 2(1+x)+3(|z|-1) \le [4(1+x)] \vee [6(|z|-1)]. \end{aligned}$$

So, either (i) $|z+1|^2 \le 6(|z|-1)$ and then we bound $|g(z)|$ as in Case 1, getting here $|g(z)| \le \frac{6K}{|z+1|^2}$ or (ii) $|z+1|^2 \le 4(1+x)$ and then we bound $|g(z)|$ as in Case 2, getting here $|g(z)| \le \frac{4K}{|z+1|^2}$.

Thus, the proof of the lemma is complete. □

Thus, the proof of the theorem is complete. □

*Remark.* As the example of

$$f(x) = \exp\{-(a+i\sqrt{1-a^2})x\} \quad \text{or} \quad f(x) = \Re\exp\{-(a+i\sqrt{1-a^2})x\}$$

for $a \in (0,1)$ shows, the condition $f(x) = O(e^{-x})$ in Theorem 1 cannot be relaxed to $f(x) = O(e^{-ax})$, for any real $a < 1$.