Andreoni-McGuire Algorithm and the Limits of Warm-Glow Giving

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Abstract

We provide a full equilibrium characterization of warm-glow giving à la Andreoni (1989, 1990) by extending the Andreoni-McGuire (1993) algorithm. We then generalize and offer an intuitive meaning to the large-economy crowding-out results by Ribar and Wilhelm (2002). The algorithm indexes individuals according to their free-riding levels of the public good. This level is finite for an individual whose donation is always dictated by some altruism or concern for charity. We show that if all individuals have finite free-riding levels, then the crowding-out is complete in a large economy. If, on the other hand, a non-negligible fraction of the population never free rides, then the crowding-out is zero in a large economy. We discuss implications of these extreme crowding-out predictions for charitable behavior and fund-raising strategies.

Keywords: altruism, warm-glow, crowding-out
JEL Classification: H00, H30, H50

1 Introduction

Despite its intuitive appeal, modeling charity as a pure public good has proved unrealistic. Empirical evidence refutes two chief predictions from this model of pure altruism: complete (dollar-for-dollar) crowding-out of private donations by government grants (Warr, 1983; Bergstrom et al. 1986), and giving only by the very rich in a large economy (An-

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To reconcile the evidence, Andreoni (1989, 1990) has introduced to preferences a direct benefit or “warm-glow” from the act of giving itself – calling the extended model warm-glow giving. Empirical research has strongly corroborated the presence of both altruistic and warm-glow motives for giving. Despite its prevalent use, a full equilibrium characterization of warm-glow giving is, however, absent in the literature. This is the gap we fill in this paper. In light of our characterization, we extend the important large-economy results of Ribar and Wilhelm (2002) to heterogenous donors, and establish a simple necessary and sufficient condition for complete and zero crowding-out. As a by-product of our analysis, we also demonstrate that the equilibrium theory of warm-glow giving can be as tractable as its polar case of pure altruism.

The key to our equilibrium characterization is to first identify the set of contributors. We achieve this by modifying Andreoni and McGuire’s (1993; henceforth A-M) elegant algorithm tailored to pure altruism. The A-M algorithm, and our modification of it, ranks each individual according to her free-riding threshold of the public good, or her “dropout” amount. We show that an individual becomes a contributor (in the whole economy) if and only if the subeconomy consisting of those ranked higher than her fail to supply her dropout amount of the public good. Therefore, one’s decision to contribute depends solely on those less likely to free ride than her while her gift size depends on the entire set of contributors. Indeed, the equilibrium public good occurs at the level that neutralizes the marginal contributor’s incentive to give.

A noteworthy feature of a person’s dropout amount is that it is pinned down by her own preferences and income. We prove that this amount is finite if her donation is always dictated by some altruism or concern for charity. We then prove that if everyone

\footnote{Most empirical studies find little crowding-out – often 0 to 35% – (e.g., Kingma, 1989; Okten and Weisbrod 2000; Ribar and Wilhelm, 2002; Manzoor and Straub, 2005; and Gruber and Hungerman, 2007). Moreover, most Americans give. For instance, in 2000, 9 out of 10 U.S. households donated to at least one charitable organization. See Andreoni (2006) and Vesterlund (2006) for an overview of the charitable sector.}

\footnote{See also Cornes and Sandler (1984) and Steinberg (1987).}

\footnote{The incomplete crowding-out observed in empirical studies (see footnote 1) is often taken to be the evidence of warm-glow. For experimental evidence of the incomplete crowding-out, see Andreoni, 1993; Bolton and Katok, 1998. Direct experimental tests of warm-glow are, however, also available; e.g., Palfrey and Prisbrey 1997; Andreoni and Miller, 2002; Crumpler and Grossman, 2008; and Korenok et al., 2013.}

\footnote{Andreoni (1989) proves the equilibrium existence and uniqueness by restricting attention to positive contributions. Kotchen (2007) extends his result to the free-riding possibility. Neither, however, offers a full characterization.}

\footnote{Assuming identical donors, Ribar and Wilhelm show the sufficiency of the marginal warm-glow incentive for zero crowding-out. They do not show its necessity. Moreover, unlike ours, Ribar and Wilhelm’s conditions for limit crowding-out rely on equilibrium gifts and thus they are more difficult to verify.}
in the economy has a finite dropout amount, in a large economy, the crowding-out must be complete. The reason is that the equilibrium public good, including any government grants, cannot, by definition, exceed a contributor’s dropout amount. And as the population size increases, the public good converges to the highest dropout amount, which is determined independently of government grants. This means that in a large economy, incomplete crowding-out can result only because a non-negligible fraction of the population always gives. But in the limit, such individuals must behave as though they were pure warm-glow givers and in turn the crowding-out must be zero. These extreme crowding-out predictions generalize Ribar and Wilhelm’s in an intuitive way. Nevertheless, since, on its face, the warm-glow model does not explain partial crowding-out observed in (large-sample) empirical studies, we argue in Section 4 that the model needs to incorporate other important features of the charitable sector such as fundraising and informational asymmetry.

The rest of the paper is organized as follows. We set up the basic model in the next section, and provide an equilibrium characterization in Section 3. In Section 4, we extend the analysis to include government grants and perform a limit economy analysis. In Section 5, we briefly discuss our findings and possible modifications of the model. We conclude in Section 6. Proofs that do not appear in the text are relegated to the appendix.

2 Warm-Glow Giving

Consider the model of warm-glow giving studied by Andreoni (1989, 1990). An economy consists of \( n \) individuals who each allocates her monetary income, \( m_i > 0 \), between a private good consumption, \( x_i \geq 0 \), and a gift to the public good or charity, \( g_i \geq 0 \), without observing others’ decisions. Both the public and private goods are normal. Units are measured in dollars so that \( x_i + g_i = m_i \). Let \( G = \sum_j g_j \) be the supply of the public good, and \( u^i(x_i, G, g_i) \) be a twice continuously differentiable, strictly increasing and strictly quasi-concave utility function for individual \( i \). This specification admits as polar cases pure altruism, \( u^i(x_i, G) \), where individual \( i \) cares only about the charity, and pure warm-glow, \( u^i(x_i, g_i) \), where she cares only about the act of giving. In general, individual \( i \) enjoys her gift, \( g_i \), through the public good and also as a private good.

To determine gifts at a Nash equilibrium, we first determine person \( i \)'s optimal gift or reaction function, \( g_i \), given others' \( G_{-i} = \sum_{j \neq i} g_j \). Formally, substituting for \( x_i = m_i - g_i \),
\(\hat{g}_i\) solves
\[
\max_{g_i \geq 0} u^i(m_i - g_i, G_{-i} + g_i, g_i).
\]
(P)

The first-order condition for \(\hat{g}_i\) is
\[
\frac{d}{dg_i} u^i(m_i - g_i, G_{-i} + g_i, g_i) \leq 0 \quad (= 0 \text{ if } g_i > 0).
\]
(1)

The strict quasi-concavity of \(u^i\) ensures the second-order condition: \(\frac{d^2}{dg_i^2} u^i(.) < 0\).

As with Andreoni, an alternative (and often more intuitive) approach to (P) is to substitute for \(g_i = G - G_{-i}\):
\[
\max_{G \geq G_{-i}} u^i(m_i + G_{-i}, G, G - G_{-i}).
\]
(P')

Let \(\hat{G} = \max\{f^i(m_i + G_{-i}, G_{-i}), G_{-i}\}\) be the unique solution to (P'), where we call \(f^i\) Nash supply of the public good.\(^6\) Then,
\[
\hat{g}_i = \max\{f^i(m_i + G_{-i}, G_{-i}) - G_{-i}, 0\}.
\]
(2)

Let \(f_a^i\) and \(f_w^i\) denote \(f^i\)'s partial derivatives with respect to its first and second arguments, signifying person \(i\)'s marginal propensity to give due to altruism and due to warm-glow, respectively. Normality of public and private goods imply that \(0 < f_a^i < 1\) and \(f_w^i \geq 0\). For \(f_w^i = 0\), the warm-glow model reduces to pure altruism while for \(f_a^i + f_w^i = 1\), it reduces to pure warm-glow giving. Since we are interested in settings with both motives present, we assume \(0 < f_a^i + f_w^i < 1\). From (2), this implies a downward-sloping reaction function because \(\frac{d}{dG_{-i}} \hat{g}_i = f_a^i + f_w^i - 1 < 0\) and an upward-sloping total contribution because \(\frac{d}{dG_{-i}} (\hat{G} + G_{-i}) = f_a^i + f_w^i > 0\); that is, others’ contribution crowds out one’s own but only partially. To see what this implies about preferences, note from (1) that
\[
\frac{d}{dG_{-i}} \hat{g}_i = -\frac{\frac{d}{dg_i} u_G^i(.)}{\frac{d^2}{dg_i^2} u^i(.)},
\]
(3)

where subscripts of \(u^i\) refer to respective partial derivatives.\(^7\) Since \(\frac{d}{dG_{-i}} \hat{g}_i < 0\) and \(\frac{d^2}{dg_i^2} u^i(.) < 0\), we must have
\[
\frac{d}{dg_i} u_G^i(.) = -u_{xG}^i(.) + u_{GG}^i(.) + u_{Gg_i}^i(.) < 0.
\]
(4)

---

\(^6\)This term is meant to suggest Nash behavior by person \(i\).

\(^7\)More explicitly, for \(\hat{g}_i > 0\), (1) implies \(\frac{d}{dg_i} u^i(m_i - \hat{g}_i, G_{-i} + \hat{g}_i, \hat{g}_i) = 0\). Differentiating both sides with respect to \(G_{-i}\), we obtain \(\frac{d}{dg_i} u^i(.) \times \frac{d}{dG_{-i}} \hat{g}_i + \frac{d}{dg_i} u_G^i(.) = 0\), which yields (3).
The condition in (4) has implications for substitution properties between \(x\), \(G\), and \(g_i\). If \(u^i\) is concave and separable in private consumption, i.e., \(u^i_{xG}(.) = 0\), then (4) requires \(u^i_{GG}(.) < -u^i_{Gg}(.)\), which allows substitution and some complementarity between warm-glow and altruistic preferences. For instance, “impact philanthropy” where an individual’s warm-glow comes from “making a difference” may mean substitutes, \(u^i_{GG}(.) < 0\), because others’ contribution reduces the marginal impact of one’s own whereas social institutions coaxing donations may mean complements, \(u^i_{Gg}(.) > 0\), because individuals may be concerned about their social images and audience effects. If, on the other hand, \(u^i\) is concave and separable in warm-glow, i.e., \(u^i_{Gg}(.) = 0\), then (4) requires \(u^i_{GG}(.) < u^i_{xG}(.)\), which allows complementarity and some substitution between public and private goods. Complementarity may arise, for instance, between watching TV and supporting public TV while substitution may arise between investing in home security and supporting police.

It is also worth noting that for the special case of pure warm-glow preferences, the three terms in (4) sum to zero, implying no substitution: \(u^i_{GG}(.) = u^i_{xG}(.) = 0\), except for the knife-edge case in which the three terms are not zero but exactly cancel out.\(^9\)

To avoid trivial noncontributing, we assume that each person has a positive standalone demand for the public good, i.e., \(f^i(m_i, 0) > 0\). A necessary and sufficient condition for this is that utility be strictly increasing at the margin: \(\frac{d}{dg} u^i(m_i, 0, 0) > 0\).

3 Equilibrium Characterization: Andreoni-McGuire Algorithm Revisited

By definition, equilibrium gifts lie at the intersection of individuals’ reaction functions and can be found by simultaneously solving \(n\) equations described by (2). This direct approach, however, yields limited insight into giving behavior. In particular, it does not systematically identify the set of contributors. Within the purely altruistic model, Andreoni and McGuire (1993) provide an elegant algorithm that does exactly that. We show that their analysis extends to warm-glow giving without losing tractability.

As in Andreoni and McGuire, let \(G^0_i\) denote others’ threshold contribution, or the pub-
lic good level, that induces person $i$ to free ride or “drop out” of the contributor set; that is, $g_i = 0$ if and only if $G_{-i} \geq G_i^0$. Eq.(2) then implies,

$$f_i^0(m_i + G_i^0, G_i^0) - G_i^0 = 0.$$  (5)

Since $f_i^0 + f_i^w - 1 < 0$ and $f_i(m_i, 0) > 0$, if a solution to (5) exists, it must be unique (and strictly positive). If a solution does not exist, it simply means that person $i$ never free rides or, by convention, has $G_i^0 = \infty$. Alternatively, from (1), $G_i^0$ must satisfy: $\frac{d}{d G_i} u^i(m_i, G_i^0, 0) = 0$. Since $\frac{d}{d G_i} u^i(m_i, 0, 0) > 0$, $G_i^0$ exists if and only if $\frac{d}{d G_i} u^i(m_i, G, 0) \leq 0$ for some $G$. Thus, person $i$ never free rides if and only if the warm-glow motive remains active at all levels of the public good. An issue that is especially pertinent to large economies is whether warm-glow becomes the sole charitable motive at very high levels of the public good. The following result shows this is the case.

**Proposition 1** Let $\lim_{G_i \to \infty} f_i^a + f_i^w = \theta_i$.

(a) If $\theta_i < 1$, then $G_i^0$ is finite.

(b) If $G_i^0$ is infinite, then $\theta_i = 1$ and thus $\lim_{G_i \to \infty} \frac{d}{d G_i} \hat{G}_i = 0$.

(c) Suppose $u^i$ is homothetic. Then, $\theta_i = f_i^0(1,1)$. For $\theta_i = 1$, $\lim_{G_i \to \infty} \hat{G}_i = f_i^a(1,1) \times m_i$.

**Proof.** See the appendix. ■

Recall that pure warm-glow giving refers to $f_i^a + f_i^w = 1$ for all $G_{-i}$, or equivalently a flat reaction function $\frac{d}{d G_i} \hat{G}_i = 0$. Part (a) indicates that even after very generous contribution by others, if a person’s last dollar to charity is partially driven by a concern for its output, or altruism, then she will turn a free rider at some threshold donation by others.\textsuperscript{10} Conversely, part (b) says that a person who always gives some will grow unresponsive to others’ contribution and behave as though she were a pure warm-glow giver. With homothetic preferences, part (c) shows that such a person’s contribution will asymptote to a fixed percentage of her income, which is the gift that would be chosen by a pure warm-glow donor. To see this, note that under homothetic preferences, the Nash supply $f_i$ is linear homogenous and thus person $i$’s gift in (2) can be decomposed as:

$$\hat{G}_i = f_i^a \times m_i - (1 - f_i^a - f_i^w) \times G_{-i}.$$  

\textsuperscript{10}This is, of course, a counter-factual for person $i$ because she does not observe others’ contributions while choosing her own.
In words, person $i$’s gift is a fixed percentage of her income, $f_i^a \times m_i$, minus her free-riding amount as a fraction $(1 - f_i^a - f_i^w)$ of others’ contribution. If $f_i^a + f_i^w \to 1$, the person does not free ride in the limit and the second term vanishes. Moreover, the resulting gift, $f_i^a \times m_i$, would also be optimal for a pure warm-glow giver who is, by definition, unresponsive to any contribution by others – not just in the limit.

To illustrate Proposition 1, consider these homothetic utilities similar to those used by Ribar and Wilhelm (2002).

$$u_i = (1 - \alpha) \ln x_i + \alpha \ln \left( (1 - \omega)G + \omega g_i \right). \quad (6)$$

$$u_i = (1 - \alpha - \omega) \ln x_i + \alpha \ln G + \omega \ln g_i. \quad (7)$$

Under specification (6), $f_i^a = \alpha (m_i + G_{-i}) + \omega (1 - \alpha) G_{-i}$, and thus $f_i^a + f_i^w = \alpha + \omega (1 - \alpha) < 1$ and $G_1^0 = \frac{\alpha}{1 - \alpha} m_i < \infty$. Under specification (7), $f_i^a + f_i^w \to \frac{\omega}{1 - \alpha} + \frac{1 - \alpha - \omega}{1 - \alpha} = 1$ and thus $G_1^0 = \infty$ and $G_{-i} \to \infty$. Consistent with part (c), $G_i = \frac{\omega}{1 - \alpha} m_i$ is the optimal gift by a pure warm-glow giver whose utility is $\Pi_i = (1 - \alpha - \omega) \ln x_i + \omega \ln g_i$.

An important feature of $G_1^0$ is that it depends on one’s own preferences and income – not on the equilibrium. Without loss of generality, index individuals in a descending order of their dropout levels:

$$G_1^0 \geq G_2^0 \geq \ldots \geq G_n^0.$$ 

A pivotal step in generalizing the A-M algorithm to warm-glow giving is to define the inverse function $\phi^i(G, m_i)$ from (2):

$$f_i^i(m_i + \phi^i, \phi^i) = G.$$ 

Such an inverse exists because $f_i^a + f_i^w > 0$. It is readily verified that partial derivatives satisfy $\phi_{G}^i = \frac{1}{f_i^a + f_i^w} > 1$ and $-1 < \phi_{m}^i = -\frac{f_i^w}{f_i^a + f_i^w} < 0$. By definition, $\phi^i(G, m_i)$ is the minimum contribution person $i$ needs from others to supply the rest of the public good $G$ herself. As a result, person $i$’s optimal gift in eq.(2) can be re-written:

$$g_i = \max\{G - \phi^i(G, m_i), 0\}, \quad (8)$$

\[\text{Evidently, } f_i^i(1, 1) = 1.\]
and eq. (5) can be re-stated: 
\[ G_0^i - \phi^i(G_0^i, m_i) = 0. \]
Clearly, \( i \)'s gift is decreasing in \( G \) and increasing in \( m_i \) but both at a lower rate for a warm-glow giver than a pure altruist. The reason is as explained in Andreoni (2006, p. 1221): warm-glow creates a “stickiness” to giving – people are no longer indifferent to the source of the gift. It is worth noting that with pure altruism, \( \phi^i(G, m_i) = \overline{\phi}^i(G) - m_i \) where \( \overline{\phi}^i \) denotes the inverse demand for public good.

Andreoni and McGuire prove and explain five facts to formulate their algorithm. We show that these facts extend to warm-glow giving, and record them here for exposition. Adopting the A-M notation, let \( G^* \) denote the equilibrium public good and \( C \) denote the set of contributors.

**Fact 1** \( i \in C \) if and only if \( G^* < G_0^i \).

This fact directly follows from the definition of \( G_0^i \) and (8). It indicates that the equilibrium public good – not just others’ contribution – must stay strictly below a contributor’s dropout point. Therefore if \( G_0^i \) is bounded for all \( i \), so is the equilibrium public good. This implication will prove instrumental in studying crowding-out. The next two facts are immediate from Fact 1. In particular, Fact 3 establishes that a contributor cannot turn a free-rider when a new person who is more likely to drop out joins the group.

**Fact 2** If \( i \in C \) and \( G_0^j \geq G_0^i \), then \( j \in C \).

**Fact 3** Let \( G_{0,\text{min}} = \min\{G_0^i \text{ s.t. } i \in C\} \). If \( G_0^k \leq G_{0,\text{min}} \) and person \( k \) is added to the economy, none of the original members of \( C \) will be replaced, although person \( k \) may be added to \( C \).

Fact 4 makes the obvious point that as new individuals are added to the group, public good provision cannot decrease despite the intensified free riding.

**Fact 4** Let \( S' \) and \( S'' \) be two sets of individuals such that \( S' \subset S'' \), with the respective equilibrium levels of public goods, \( G' \) and \( G'' \). Then, \( G' \leq G'' \).

**Proof.** Suppose, to the contrary, that \( G' > G'' \). Then, by Fact 1, \( C' \subseteq C'' \). Next, for \( i \in C', g'_i = G' - \phi^i(G', m_i) < G'' - \phi^i(G'', m_i) = s''_i \) because \( \phi_G^i > 1 \). But, given \( C' \subseteq C'' \), this implies \( G' \leq G'' \), a contradiction. Thus, \( G' \leq G'' \). 

The final fact determines the equilibrium public good that can be produced among the contributors, which directly obtains from aggregating (8).
**Fact 5** Let $C_i = \{1, \ldots, i\}$ be a set of individuals and $G < G_i^0$. $G$ is an equilibrium in $C_i$ if and only if

$$\Delta_i(G, m) \equiv G - \sum_{j=1}^{i} (G - \phi^j(G, m_j)) = 0. \quad (9)$$

**Proof.** Suppose $G$ is an equilibrium in $C_i$. Then $g_j = G - \phi^j(G, m_j)$ for $j \in C_i$. Summing over all $j \in C_i$ yields $\Delta_i(G, m) = 0$. Conversely, suppose $\Delta_i(G, m) = 0$, and let $g_j = G - \phi^j(G, m_j)$. Since $G < G_i^0$, we have $g_j > 0$ and this gift profile constitutes an equilibrium in $C_i$. \(\blacksquare\)

To understand (9), recall from (8) that the amount $G - \phi^j(G, m_j)$ is person $j$’s gift toward the public good $G$. Clearly, these gifts form a Nash equilibrium among contributors $j = 1, \ldots, i$ if indeed they add up to $G$, as required by (9). In general, the function $\Delta_i(G, m)$ accounts for the (potential) shortfall of the subeconomy $j = 1, \ldots, i$ to voluntarily supply $G$.

As with the A-M algorithm, a systematic approach to characterizing equilibrium is to first pinpoint the set of contributors. Intuitively, person $i$ should become a contributor if those ranked higher than her cannot reach her dropout amount, $G_i^0$, or equivalently if $\Delta_i(G_i^0, m) > 0$. Proposition 2 formalizes this intuition and generalizes the A-M algorithm for warm-glow giving.

**Proposition 2** Let $\Delta_i^0 \equiv \Delta_i(G_i^0, m)$. Then,

(a) $\Delta_1^0 \geq \Delta_2^0 \geq \ldots \geq \Delta_n^0$ (with $\Delta_i^0 = \infty$ if $G_i^0 = \infty$).

(b) $i \in C$ if and only if $\Delta_i^0 > 0$.

(c) If $C = \{1, \ldots, k\}$, then the equilibrium public good uniquely solves: $\Delta_k^0(G^*, m) = 0$. In equilibrium, $i \in C$ contributes $g_i^* = G^* - \phi^i(G^*, m_i)$.

**Proof.** Note that $\Delta_i(G, m)$ is strictly increasing in $G$. Thus

$$\Delta_i^0 \equiv \Delta_i(G_i^0, m) \geq \Delta_i(G_{i+1}^0, m) \geq \Delta_{i+1}(G_{i+1}^0, m) \equiv \Delta_{i+1}^0,$$

where the last inequality follows because $G_i^0 - \phi^i(G_i^0, m_i) = 0 \leq G_{i+1}^0 - \phi^i(G_{i+1}^0, m_i)$. To complete part (a), note that if $G_i^0 = \infty$, then for $j \leq i$, $\lim_{G \to G_i^0} \frac{G - \phi^j(G, m_j)}{G} = \lim_{G \to G_i^0} 1 - \frac{\phi^j(G, m_j)}{G} = \lim_{G \to G_i^0} (1 - \frac{1}{f_i + f_j}) = 0$, where the last equality obtains since $f_i + f_j \to 1$ for $G_j^0 = \infty$ by Proposition 1(a). Thus, eq.(9) implies that if $G_i^0 = \infty$, it must be that $\Delta_i^0 = G_i^0 = \infty$, as stated.
To prove part (b), suppose $\Delta_i^0 > 0$. Then, since $\Delta_i(G, m)$ is strictly increasing in $G$, there is a unique $G_i^* < G_i^0$ such that $\Delta_i(G_i^*, m) = 0$. By Fact 5, $G_i^*$ is an equilibrium among $k = 1, ..., i$. Since $G_i^* < G_i^0$, person $i$ is a contributor in this equilibrium. Fact 3 implies that she must stay a contributor in the whole economy. Conversely, suppose $i \in C$. Then, $G_i^* < G_i^0$ by Fact 1, and thus $0 \leq \Delta_i(G_i^*, m) < \Delta_i(G_i^0, m) = \Delta_i^0$. The last part directly follows from Fact 5 and eq.(8).

Part (a) says that the search for contributors follows the same descending order as their dropout levels of the public good. As alluded to above, person $i$’s incentive to become a contributor is, however, measured by $\Delta_i^0$ in order to account for the free-riding among the contributors. It is this free-riding that creates a wedge between $G_i^0$ and $\Delta_i^0$, which grows as one moves down in the order. Consider individual 1. If she were alone in the economy, she would contribute because $\Delta_1^0 = G_1^0 > 0$. In the presence of individual 1, individual 2 is less eager to contribute since at the public good level $G_2^0$, she expects $G_2^0 - \phi^1(G_2^0, m_1) > 0$, leaving her $G_2^0 - [G_2^0 - \phi^1(G_2^0, m_1)]$ or equivalently $\Delta_2^0$ since $G_2^0 - \phi^2(G_2^0, m_2) = 0$. Iteratively applied, this logic reveals that after the gifts from $j = 1, ..., i - 1$, $\Delta_i^0$ is the residual amount individual $i$ needs to bear in order to reach the public good level $G_i^0$. Part (b) indicates that $i$ becomes a contributor whenever there is a shortfall, i.e., $\Delta_i^0 > 0$. In the special case of pure altruism, recall that $\phi^i(G, m_i) = \phi^i(G) - m_i$, and thus $\Delta_i^0 > 0$ if and only if $\sum_{j=1}^i \phi^j(G_i^0) - (i - 1)G_i^0 > \sum_{j=1}^i m_j$, which is exactly the A-M condition for pure altruism.

Once the set of contributors is determined from the model primitives, part (c) reveals that the equilibrium supply of the public good $G^*$ occurs at the level that drives the marginal contributor’s incentive to give down to zero. $G^*$ must be strictly less than the marginal contributor’s dropout level in order for her to stay as a contributor (see Fact 1). Using $G^*$, a contributor $i$’s gift is then $g_i^* = G^* - \phi^i(G^*, m_i)$ from (8).

Proposition 2 reveals an interesting implication of the theory that could be of empirical relevance: one’s decision to become a contributor depends solely on those less likely to free ride than her while her actual contribution depends on the entire set of contributors. We believe that identifying this dichotomy also adds to Bergstrom et al.’s (1986) main message that “... adjustments on the ‘extensive margin’– the decision of whether or not to become a contributor – are at least as important as adjustments on the ‘intensive margin’ – the decision of how much to contribute. (p. 27)”

Proposition 2 is also useful to reproduce and extend previous results such as Proposi-
tion 1 in Andreoni (1990) on income redistribution and nonneutrality of public good provision: assume everyone is a contributor and totally differentiate \( \Delta_n(G^*, m) = 0 \), and then note that \( -\phi_i m = \frac{f_i}{f_i + f_w} \) is Andreoni’s “altruism coefficient”. Using Proposition 2, one can, however, extend Andreoni’s finding even when free-riding is possible. Our equilibrium characterization of warm-glow giving is also amenable to extensions. In the next section, we include government grants and investigate the issue of crowding-out.

4 Crowding-out in Limit Economies

To what degree do government grants crowd out private donation? This is a fundamental question in public economics because it shapes policy and helps understand charitable motives. The evidence is, however, mixed. While empirical studies find small crowding-out – often 0 to 35% – experimental data shows otherwise – up to 70% – (see footnotes 1 and 3). This discrepancy has mostly been attributed to donor preferences for giving. In particular, empirical studies point to a stronger warm-glow motive. In an interesting paper, Ribar and Wilhelm (2002) have offered a positive reconciliation of the conflicting evidence based on the sample size differences between field and lab data. By performing a limit analysis of warm-glow giving, they show that under reasonable conditions, the crowding-out can be either complete or zero. Though enlightening, the generality of Ribar and Wilhelm’s conditions is difficult to ascertain as they depend on equilibrium gifts.¹²

By augmenting Proposition 2, we re-address the issue of the crowding-out and derive a simple criterion for the limit crowding-out: if \( G_0^i \) is finite for all \( i \), then crowding-out must be complete in the limit; if, on the other hand, \( G_0^i \) is infinite for a nonnegligible percentage of the population, then crowding-out is zero.

Suppose the charity receives a direct government grant \( R \geq 0 \), which raises the public good supply to \( G + R \). Person \( i \)’s maximization then becomes

\[
\max_{g_i \geq 0} u^i(m_i - g_i, G + R, g_i).
\]

Letting \( G = G + R, \bar{G}_{-i} = G_{-i} + R, \) and \( g_i = \bar{G} - \bar{G}_{-i}, \)

\[
\max_{\bar{G} \geq \bar{G}_{-i}} u^i(m_i + \bar{G}_{-i} - \bar{G}, \bar{G}, \bar{G} - \bar{G}_{-i}).
\]

¹²With identical donors, Ribar and Wilhelm determine the rate at which the equilibrium value of \((f_a + f_w - 1)\) converges. If the rate is faster than \( n \), then the crowding-out is zero; if it is slower than \( n \), the crowding-out is complete. They also provide a sufficient condition for the zero crowding-out, which is slightly more restrictive than that: \( \frac{d}{dG} u^i(m_i, G; 0) > 0 \) for all \( G \).
From here, i’s optimal gift is found to be $g_i = \max\{f(m_i + \overline{G}_{-i}, \overline{G}_{-i}) - \overline{G}_{-i}, 0\}$, or equivalently

$$g_i = \max\{\overline{G} - \phi^i(\overline{G}, m_i), 0\}. \quad (10)$$

Eq.(10) closely tracks (8), which enables us to reproduce the five facts above by replacing $G^*$ with $\overline{G}^*$, and incorporate the grant into Proposition 2.

Proposition 3 Let $R \geq 0$ be the government grant. Then,

(a) $i \in C$ if and only if $\Delta_i^0 - R > 0$.

(b) If $C = \{1, \ldots, k\}$, then the equilibrium public good uniquely solves: $\Delta_k(\overline{G}^*, m) - R = 0$. In equilibrium, $i \in C$ contributes $g_i^* = \overline{G}^* - \phi^i(\overline{G}^*, m_i)$.

(c) If $G_i^0 > R$, then $G^*$ is strictly decreasing in $R$, but $\overline{G}^*$ is strictly increasing in $R$.

Proof. See the appendix. ■

To understand Proposition 3, simply treat the government as another contributor whose gift $R$ lowers person $i$’s shortfall for the public good $G_i^0$ from $\Delta_i^0 - R$. Part (a) thus says that with the grant, person $i$ is less likely to contribute. Part (b) says that as in the case without the grant, the equilibrium public good drives the marginal giver’s incentive to zero. More importantly, part (c) records that in a finite economy, private giving is partially crowded out: the grant displaces some private giving but the total provision increases.\(^\text{13}\)

The obvious question, however, remains: what is the degree of the crowding-out in a large economy? The following result answers this question. In its exposition, we call donor $i$ type $t$ if her dropout amount is $G_{t,0}$. Recall from (5) that two donors with different preferences and different incomes can possess the same dropout amount.

Proposition 4 Without loss of generality, let $G_{1,0} > G_{2,0} > \ldots > G_{T,0}$ represent donor types, and $\lambda_t \neq 0$ be the fraction of type $t$. Also let $G_{1,0} > R$. Then, as $n \to \infty$,

(a) only type 1 donors contribute;

(b) (complete crowding-out) if $G_{1,0}$ is finite, then $G^*(n, R) + R \to G_{1,0}$ and $g^*_{1,i}(n, R) \to 0$;

\(^{13}\)Note that we ignore the financing issue of the government grant here. As such, Proposition 3 slightly underestimates the crowding-out. If person $i$ is taxed $\tau_i$ to finance $R$, it can be shown that our analysis will still hold except that $G_i^0$ will use the reduced income $m_i - \tau_i$. 

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(c) (zero crowding-out) if $G^{1,0}$ is infinite, each type 1 donor’s gift becomes independent of $R$.
If, in addition, $u^{1,1}$ is homothetic, then $g^{1*}_i(n, R) \to f^{1,1}_a(1, 1) \times m_i$, which would be the contribution of a pure warm-glow giver.

**Proof.** Since $\Delta^{0}_i = G^{1,0} > R$, all type 1 individuals are contributors for any $n$. If $G^{2,0} \leq R$, clearly no other type contributes. Suppose $G^{2,0} > R$ and, to the contrary, some type $t \neq 1$ remains a contributor as $n \to \infty$. Then, $G^*(\infty, R) + R \leq G^{2,0}$ by Fact 1. But since $G^{2,0} < G^{1,0}$, type 1 individuals would give a strictly positive amount in the limit, implying that $G^*(\infty, R) + R = \infty > G^{2,0}$, a contradiction. Thus, no type other than 1 should contribute in a limit economy. Next, if $G^{1,0} < \infty$, a similar limit argument shows that $G^*(n, R) + R \to G^{1,0}$ as $n \to \infty$. Since $G^{1,0}$ is independent of $R$, this means complete crowding-out. Moreover, from Proposition 3b, $g^{1*}_i(n, R) \to G^{1,0} - \phi^i(G^{1,0}, m_i) = 0$, since $G^{1,0} < \infty$.

Finally, suppose $G^{1,0} = \infty$. Then, as $n \to \infty$, $G^*(n, R) \to \infty$; otherwise, $G^*(\infty, R) = G_1 < \infty$ would imply $g^{1*}_i(\infty, R) = 0$ for some person $i$ of type 1, which would imply $(G_1 + R) - \phi^i(G_1 + R, m_i) = 0$ by (10). But then, there must be some $G > G_1$ such that $(G + R) - \phi^i(G + R, m_i) < 0$, contradicting $G^{1,0} = \infty$. Since $G^*(n, R) \to \infty$, so is $G^*(n, R) + R \to \infty$. From (10), this means that $g^{1*}_i(\infty, R)$ is independent of $R$. If, in addition, $u^{1}_i$ is homothetic, Proposition 1(c) reveals that $g^{1*}_i(n, R) \to f^{1,1}_a(1, 1) \times m_i$, which would be the amount by a pure warm-glow giver.

Proposition 4 makes two points. First, as the economy grows large, only the “most willing” individuals contribute. The reason is as explained in Proposition 1: donors with a finite dropout must feel some altruism and ultimately free ride at a sufficiently high level of the public good that is reached in a large economy. As a result, the degree of the crowding-out in a large economy is determined by the most willing type’s motives for giving. If these donors’ altruism is also persistent so that their dropout amount is finite, then the total provision of the public good including the grant asymptotically approaches this amount, $G^{1,0}$. Since $G^{1,0}$ is independent of $R$, this implies that partial crowding-out identified for a finite economy becomes complete in a limit economy. The intuition is that due to the residual altruism, a type 1 donor’s contribution is crowded out some by each additional donor and driven to zero in the limit. This means that for the crowding-out

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14 This observation also generalizes Andreoni’s (1988) limit result for pure altruism (Theorem 1.1) to warm-glow giving. Specifically, with identical tastes, Proposition 4 implies that only the very rich contribute, though with warm-glow, the average donation need not converge to zero.
to be incomplete, no altruism should be left at high levels of the public good. Indeed, from Proposition 1b, we know that individuals who continue to give at high public good levels must behave as though they were pure warm-glow givers. This in turn implies that in a large economy, the incomplete crowding-out can only be no crowding-out at all. With homothetic preferences, the last part of Proposition 4 shows the exact asymptotic contribution of each individual. As is clear, limit contributions can vary across donors, depending on their warm-glow preferences and incomes.

Aside from generalizing Ribar and Wilhelm’s large-economy results to heterogeneous donors and deriving a simple necessary and sufficient condition for complete and zero crowding-out based on dropout amounts, Proposition 4 points to the necessity of warm-glow as the sole charitable motive for zero crowding-out. This follows because if \( G^1 \) is infinite so that type 1 donors always contribute, then not only is there zero crowding-out but also these donors must grow unresponsive to others’ contributions, including government grants, in a large economy (Proposition 1b).

To illustrate Proposition 4, suppose donors are identical so that \( G_0^i = G_0 \) for all \( i \) (there is only one type). Consider first the utility in (6). Routine computation shows

\[
G^*(n, R) = \theta_n(G_0^0 - R) \quad \text{and} \quad G^*(n, R) + R = \theta_n G_0^0 + (1 - \theta_n) R,
\]

where \( G_0 = \frac{a}{(1 - \alpha)(1 - \omega)m} \) and \( \theta_n = \frac{1}{1 + \frac{\omega + \alpha}{m(1 - \omega)(1 - \alpha)}} \in (0, 1) \). Clearly, \( \frac{\partial}{\partial R} G^*(n, R) = -\theta_n \) and \( \frac{\partial}{\partial R} (G^*(n, R) + R) = 1 - \theta_n \) (partial crowding-out for a fixed \( n \)), but \( G^*(n, R) \to G_0^0 - R \) as \( n \to \infty \) (complete crowding-out in the limit).

Consider next the utility in (7). Directly calculating the equilibrium gifts, we find:

\[
g^* = \frac{(n\omega + \alpha)m - (1 - \alpha)R + \sqrt{((n\omega + \alpha)m - (1 - \alpha)R)^2 + 4\omega(\alpha + (1 - \alpha)n)RM}}{2(\alpha + (1 - \alpha)n)}.
\]

As \( n \to \infty \), \( g^* \to \frac{\omega}{1 - \alpha}m \), which, as predicted by Proposition 1, would be the gift by a pure warm-glow giver whose utility is \( u_i = (1 - \alpha - \omega) \ln x_i + \omega \ln g_i \). Thus, there is zero crowding-out in this case.

Within these examples, we can also illustrate that the limit results are likely to take effect in relatively small economies. Note that the rate of crowding-out is \( r_n \equiv \left| \left. \frac{\partial}{\partial R} G^*(n, R) \right| \right. \in (0, 1) \). For utility (6), let \( m = 100, \alpha = .03, \omega = .5 \) and \( R = 5 \). Then, \( r_n = \theta_n \) and \( r_n = .90, .98, \) and .99 for \( n = 10, 50, \) and 100, respectively; that is, crowding-out is almost complete for \( n = 100 \). For utility (7), let \( m = 100, \alpha = .03, \omega = .01 \) and \( R = 5 \). Then, \( r_n = .13, .05 \) and .03 for \( n = 10, 50, \) and 100, respectively; that is, crowding-out is almost zero for \( n = 100 \).
5 Discussion

Taken at face value, Proposition 4 implies that if each donor has in mind a “benchmark” level of the public good to be provided, then crowding out should be complete in large sample data. Such a benchmark for the donor exists if altruism or concern for the charity’s output always plays a role in her giving decision (Proposition 1). Thus, any hope of explaining the incomplete crowding-out observed in empirical data would require that giving be driven purely by warm-glow and in turn, no crowding-out should occur.\footnote{This is the route taken by Ribar and Wilhelm (2002).} Such an all-or-nothing crowding-out hypothesis does not, however, reconcile theory and evidence on charitable giving.

An alternative interpretation of Proposition 4 is that a theory based only on the donor side is inadequate and the fundraiser side should be an integral part of it. Such added realism opens up a new channel of crowding-out due to the fundraiser behavior. Andreoni and Payne (2011a,b) confirm this view: while estimating 73\% to 100\% total crowding-out, they attribute almost all to reduced fundraising. Thus, a significant crowding-out need not be inconsistent with a strong warm-glow motive for donors.\footnote{Complementing these studies is the theoretical investigation of Name-Correa and Yildirim (2013).}

Another interpretation of Proposition 4 is that the theory should account for informational asymmetries. For one, donors may be uncertain about the size of government aid and thus unable to tailor their donations to the grant. Such uncertainty is, however, unlikely to explain the incomplete crowding-out if the charity itself is perfectly informed of the grant and can costlessly disclose it, perhaps through its website.\footnote{An important assumption here is that the charity cannot forge the grant information, which seems reasonable, as most grants are awarded as written contracts.} The reason is that the charity that receives little grant will have an incentive to disclose this information to boost private giving. This means that any time the grant information is not disclosed, donors are likely to believe that there is generous government support. This belief then leads to full disclosure of the grant at all levels, much like in the literature on signaling a verifiable quality, e.g., Grossman (1981). The full disclosure may, however, not be possible if there is a nontrivial disclosure cost and/or it is too costly for donors to process the grant information, in which case the uncertainty in grant size may play a role in the degree of crowding-out.

Perhaps an equally important source of informational asymmetry is that individuals
are unsure about the quality of the charity, and a government grant can provide a valuable signal – a point first made by Vesterlund (2003).\textsuperscript{18} This signaling probably works best in settings where donors evaluate the charity’s qualifications similarly; that is, they at least agree on what constitutes a “good” charity. It is, however, possible that people may have diverse views. Krasteva and Yildirim (2013) consider such a setting where donors acquire costly information about their private valuations of a charity. These authors show that a government grant in this case can cause “classical” crowding-out as well as informational crowding-out, which leaves donors less informed – a donor who is likely to contribute a smaller amount is also less likely to be curious about the charity’s project.

6 Conclusion

In this paper, we have offered a full equilibrium characterization of warm-glow giving. The important features of this characterization are that it systematically identifies individual incentives to give and it is conducive to comparative statics. It is also open to future extensions, as partly demonstrated by including government grants here and by including fundraising in Name-Correa and Yildirim (2013). Our analysis has heavily utilized individuals’ dropout amounts, which reflect their incentives to contribute. Future empirical research may focus on eliciting these dropout amounts when it is difficult to elicit preferences and the access to income data is limited.

7 Appendix

**Proof of Proposition 1.** To ease notation, let \( z = G_{-i} \) in this proof. Define \( F_i(z) = f^i(m_i + z, z) - z \). Since \( f^i_a + f^i_w < 1 \), clearly \( F_i(z) < 0 \). Moreover, since \( F(0) = f^i(m_i, 0) > 0 \), there is a unique solution to \( F_i(z) = 0 \) if and only if \( F_i(z) \leq 0 \) for some \( z < \infty \). We consider two cases. If \( \lim_{z \to \infty} f^i(m_i + z, z) < \infty \), there obviously exists such a \( z < \infty \). Suppose \( \lim_{z \to \infty} f^i(m_i + z, z) = \infty \). Employing l’Hospital’s rule, \( \lim_{z \to \infty} \frac{f^i(m_i + z, z)}{z} = \lim_{z \to \infty} \frac{f^i_a + f^i_w}{1} = \theta_i < 1 \). Then, because \( F_i(z) = z \left[ \frac{f^i(m_i + z, z)}{z} - 1 \right] \), there must again exist some \( z < \infty \) for which \( F_i(z) < 0 \).

\textsuperscript{18}Indeed, several studies have found evidence of such government signaling, e.g., Khanna and Sandler (2000) for the UK charities; Payne (2001) for academic research institutions; and Hautel (2012) for younger charities in the U.S.
To prove part (b), suppose $G_i^0 = \infty$. Then, $\theta_i = 1$ by part (a), and thus $\lim_{G_i \to \infty} \frac{d}{dG_i} \bar{g}_i = 0$ by (2).

To prove part (c), suppose $u^i$ is homothetic. It can be verified that $f^i(m_i + z, z)$ is then homogenous of degree 1 and in turn, $f^i_a$ and $f^i_w$ are each homogenous of degree 0 in $(m_i, z)$. The latter reveals that $f^i_a(m_i + z, z) = f^i_a(\frac{m_i}{z} + 1, 1)$ and $f^i_w(m_i + z, z) = f^i_w(\frac{m_i}{z} + 1, 1)$. Therefore, $\theta_i = f^i_a(1, 1) + f^i_w(1, 1)$. Next, homogeneity of $f^i$ implies that $f^i(m_i + z, z) = zf^i(\frac{m_i}{z} + 1, 1)$. Moreover, by Euler’s equation, we have $f^i = f^i_a \times (m_i + z) + f^i_w \times z$ and thus

$$f^i(\frac{m_i}{z} + 1, 1) = f^i_a(\frac{m_i}{z} + 1, 1) \times (\frac{m_i}{z} + 1) + f^i_w(\frac{m_i}{z} + 1, 1).$$

As $z \to \infty$, clearly $f^i(1, 1) = f^i_a(1, 1) + f^i_w(1, 1)$ and thus $\theta_i = f^i(1, 1)$, as stated. To complete the proof, note that $F(z) = z[f^i(\frac{m_i}{z} + 1, 1) - 1]$. Suppose $G_i^0 < \infty$ but, to the contrary, $\theta_i = f^i(1, 1) = 1$. Using l’Hospital’s rule, $\lim_{z \to \infty} F(z) = \lim_{z \to \infty} f^i(\frac{m_i}{z} + 1, 1) - 1 = f^i_a(1, 1)m_i$, which is strictly positive and contradicts $G_i^0 < \infty$. Hence, $\theta_i < 1$. Next suppose $\theta_i = 1$. Then, $G_i^0 = \infty$, which means person $i$ contributes for any $z$, and by the same limit argument, as $z \to \infty$, $\bar{g}_i \to f^i_a(1, 1)m_i$, as recorded. To see that this would be the contribution by a pure warm-glow giver, recall that such a person would satisfy: $f^i_a + f^i_w = 1$ for all $z \geq 0$. Homogeneity would then imply $F(z) = f^i_a(\frac{m_i}{z} + 1, 1) \times m_i$, and $\lim_{z \to \infty} F(z) = f^i_a(1, 1)m_i$.

\[ \blacksquare \]

**Proof of Proposition 3.** Using (10), Facts 1-4 easily extend by replacing $G$ with $\bar{G}$. To see Fact 5, suppose $\bar{G}$ is an equilibrium in $C_i$. Then, $g_j = \bar{G} - \phi^j(\bar{G}, m_j)$ for $j \in C_i$. Summing over all $j \in C_i$ yields $\Delta_i(\bar{G}, m) - R = 0$. Conversely, suppose $\Delta_i(\bar{G}, m) - R = 0$, and let $g_j = \bar{G} - \phi^j(\bar{G}, m_j)$. Since $\bar{G} < G_i^0$, we have $g_j > 0$ and this profile constitutes an equilibrium. Parts (a) and (b) mimic parts (b) and (c) of Proposition 2, respectively.

To prove part (c), suppose $G^0_i > R' > R''$. From part (a), we have $\emptyset \neq C' \subseteq C''$ or $1 \leq k' \leq k''$. Suppose, to the contrary, that $\bar{G}'''' \geq \bar{G}''$. Then, by Fact 1, $k''' \leq k'$ and thus $k'' = k' = k$. This means that $\Delta_k(\bar{G}'''', m) = R'$ and $\Delta_k(\bar{G}''''', m) = R''$ by part (b), which, since $\Delta_k(G, m)$ is strictly increasing in $G$, imply $\bar{G}''''' < \bar{G}'''$ — a contradiction. Hence, $\bar{G}''''' < \bar{G}'''$. From part (b), this implies $g^i_'' < g^i_'''$ for $i = 1, \ldots, k'$, and since $k' \leq k''$, we have $G''' < G'''$. \[ \blacksquare \]
References


