2005

# On the Endogeneity of Cournot-Nash and Stackelberg Equilibria: Games of Accumulation

Richard Romano, *University of Florida*
Huseyin Yildirim, *Duke University*

Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Journal of Economic Theory 120 (2005) 73–107
JOURNAL OF Economic Theory
http://www.elsevier.com/locate/jet

# On the endogeneity of Cournot–Nash and Stackelberg equilibria: games of accumulation

Richard Romano[a] and Huseyin Yildirim[b,*]

[a] *Department of Economics, University of Florida, Gainesville, FL 32611, USA*
[b] *Department of Economics, Duke University, Durham, NC 27708, USA*

Received 11 October 2002; final version received 10 December 2003

**Abstract**

We characterize equilibria of games with two properties: (i) Agents have the opportunity to adjust their strategic variable after their initial choices and before payoffs occur; but (ii) they can only add to their initial amounts. The equilibrium set consists of just the Cournot–Nash outcome, one or both Stackelberg outcomes, or a continuum of points including the Cournot–Nash outcome and one or both Stackelberg outcomes. A simple theorem that uses agents' standard one-period reaction functions and the one-period Cournot–Nash and Stackelberg equilibria delineates the equilibrium set. Applications include contribution, oligopoly, and rent-seeking games.


*JEL classification:* C72; C73; L13; H41

*Keywords:* Accumulation games; Cournot–Nash outcome; Stackelberg outcome

## 1. Introduction

In a variety of settings, agents repeatedly interact and take irreversible actions before payoffs accrue. For instance, donors can make multiple non-refundable contributions to a public good, lobbies repeatedly engage in rent-seeking activities to influence a policy decision,[1] and duopolists can add to their previous stock of output

*Corresponding author. Fax: +1-919-684-8974.
*E-mail addresses:* romanor@notes.cba.ufl.edu (R. Romano), yildirh@econ.duke.edu (H. Yildirim).

[1] For instance, in the U.S., interest groups in different industries can repeatedly make campaign contributions to congressmen before a policy decision will be made at a predetermined date. Very soon after contributions are made, their amounts, the recipient congressmen, and the timing of contributions become public information. This information is posted at www.opensecrets.org.

0022-0531/$ - see front matter 
doi:10.1016/j.jet.2003.12.005

before the market clears. Such games have two main features: (i) Before payoffs occur, agents have multiple opportunities to vary their strategic variable and to observe their opponent's most recent strategy choice; but (ii) they can only accumulate their strategic variable over time.

With some exceptions discussed below, studies of such games have assumed that agents make their choices once and that they interact either in a standard Cournot–Nash or Stackelberg fashion. While these modeling approaches provide valuable insights into the nature of agents' choices and the equilibrium outcomes, a more realistic specification of such games should embody the two elements discussed above. Our objectives in this paper are to determine the consequences of the possibility of "strategic accumulation" for a large set of games, and to examine the implications in a variety of applications. The contribution of this paper is twofold: On the technical side, we are able to solve this set of games in a unified manner thus allowing us to highlight the common themes; and, on the application side, we show how some predictions of previously analyzed models might change dramatically once we account for the dynamics and irreversibility of initial actions. As a byproduct, our study also allows us to identify the environments where leadership roles arise endogenously.

A brief preview of our main findings and the organization of our paper are as follows. We present the model in Section 2. Two agents are present whose preferences and strategy spaces are common knowledge. Agents simultaneously make initial choices, and, after these are observed, simultaneously choose whether to increase their strategic variable. Payoffs depend on the accumulated values.

In Section 3, we characterize the equilibrium set. We show that the equilibrium set can be delineated using the standard one-period reaction functions and the standard Cournot–Nash and Stackelberg outcomes. This characterization provides a convenient program for identifying the equilibrium possibilities in different scenarios.

Next we focus on when the Cournot–Nash outcome is the unique equilibrium. *The* necessary and sufficient condition is simply that each agent's standard Stackelberg-leader choice is less than his Cournot–Nash amount. This finding provides insight into the nature of the accumulation game. An example with this outcome is the standard model of private contributions to a public good where agents would like to free ride. A standard Stackelberg leader would free ride by committing to a low contribution—below the Cournot–Nash amount—knowing that this would induce a relatively high contribution by the follower. If, however, the Stackelberg leader could contribute again along with the "follower," then the leader's incentive to do so would lead back to the Cournot–Nash outcome. This intuition holds generally in this case thereby ruling out all but the Cournot–Nash outcome.

In other settings, equilibria in the accumulation game are equivalent to one or both of the standard Stackelberg outcomes. An example is duopoly quantity competition by producers of complements. Here initial choice of the standard Stackelberg leader's amount constitutes a credible commitment to maintaining that output because it exceeds the Cournot–Nash quantity (and this initial choice is an equilibrium strategy). The other possibility is to have a continuum of equilibria

along one or both agents' (standard) reaction functions between the Cournot–Nash and Stackelberg outcomes. Here equilibria with "partial leadership" arise. Both agents take their actions in the initial period, the "partial follower" making a commitment that limits the cost of being a follower in these cases.

In Section 4, we provide some specific applications. In Section 5, we examine the effects of discounting which is not an element of the basic model. The noteworthy finding here is that (even slight) discounting eliminates the possibility of the continuum of equilibria, but two modified Stackelberg and the second-period Cournot–Nash outcomes remain as the only possible equilibria. The outcome that prevails in equilibrium however depends on the underlying game structure as well as the discount rate.

Settings also exist where agents can only reduce their earlier choices over time. Duopolists competing in price may be bound to no higher than preannounced prices to keep customer goodwill and/or to avoid antitrust scrutiny. Political candidates announcing preferred tax rates may face prohibitive political costs of then favoring higher rates, but not so for lower rates. Not surprisingly, our techniques and results for the accumulation game can be applied to the "decumulation game" by an appropriate change of variables, which we demonstrate in Section 6. We also consider an extension to an arbitrary number of periods in Section 6, where we show that, with no discounting, adding more periods to the accumulation game changes neither the equilibrium set nor agents' payoffs. Finally, we conclude in Section 7. The appendix contains all proofs.

Before proceeding, we relate and distinguish our paper from closely related previous work. Saloner [18] solves the duopoly output game of producers of a homogeneous product with two production periods, which is a special case of the problem we study. Pal [16] extends Saloner's analysis by introducing cost changes over time.[2] Our analysis of discounting is close to his, though ours applies to a much wider range of cases and thus yields novel insights. We further discuss this point in Section 5. More recently, Henkel [10] examines the value of partial commitment by the first-mover. Our model differs from his in that we allow both agents to move initially and then both agents can revise their initial decisions. In Romano and Yildirim [17], using a fairly general utility function, we analyze a two-period contribution game to a public good to determine the role of announcements in fund-raising activities. One version of the contribution game we studied is a special case of the problem analyzed here. Admati and Perry [1] investigate the conditions under which two agents can complete a jointly valued discrete project when they take turns making contributions toward its completion. Our model can be applied to a similar problem but with continuous public good and where each party can contribute each period.[3] Marx and Matthews [15] consider a more general model along the lines of

[2] Also, building on Saloner's work, Maggi [14] examines the equilibrium sizes of firms when there is demand uncertainty and early investment to gain leadership is possible.

[3] Varian [23] also considers sequential contributions to a continuous public good assuming each agent can contribute only once, thus facilitating Stackelberg leadership. Our model complements this analysis by highlighting the commitment problem when multiple contributions are allowed. We discuss this point further below.

Admati and Perry, including allowing agents to contribute in any period. They obtain more "positive" results regarding the likely completion of the project and attribute this primarily to the change in timing of the game.[4] As we discuss below, a similar intuition arises in our two-period model with continuous payoff functions. Hamilton and Slutsky [8,9], and Van Damme and Hurkens [22] analyze endogenous timing games having firms choose not only how much to produce but also in which period to produce. Our analysis complements this literature by allowing positive production in each period.

## 2. The model

Two agents, $i = 1, 2$, take continuous actions $y_i^t$ simultaneously in each of two periods, $t = 1, 2$, before payoffs accrue.[5] The first-period choices are observed before the second-period choices are made. Let $Y_i \equiv y_i^1 + y_i^2$ denote the cumulative value of agent $i$'s strategic variable. The total action is bounded and agents cannot make negative choices: $y_i^1 \in [0, I_i] \equiv F_i$ and $y_i^2 \in [0, I_i - y_i^1]$. The lower bound of zero on the latter set characterizes the accumulation game.[6]

Agents receive payoffs at the end of the second period, with payoff or utility function: $U^i = U^i(Y_i, Y_j)$. Thus, we assume the timing of actions is irrelevant to the payoff function. This is at least a good approximation when payoffs dwarf time costs, as when the period of interaction is short. Also, for applications where actions correspond to one-sided binding commitments, as sometimes with an announced political position, this is the appropriate assumption. In any case, we show in Section 5 that the fundamental results are unchanged with the introduction of discounting. Both utility functions and the action spaces are common knowledge, and Subgame Perfect Nash Equilibrium in pure strategies is the equilibrium concept we adopt.

Our analytical approach employs the standard one-period reaction functions and Cournot–Nash and Stackelberg outcomes for the total actions $(Y_1, Y_2)$. Thus, define the reaction function of agent $i$ as:

$$f^i(Y_j) = \arg\max_{Y_i} \; U^i(Y_i, Y_j). \tag{1}$$

Let $G_0$ indicate the one-period Cournot–Nash game and $G_i$ indicate the standard Stackelberg game where agent $i$ leads. We write the Cournot–Nash outcome as $(Y_i(G_0), Y_j(G_0))$, which occurs at the intersection of the reaction functions.

[4] Gale [7] and Lockwood and Thomas [13] analyze versions of infinitely repeated games of accumulation. Our model differs from their models in two significant ways: First, we consider finitely repeated games. More importantly, while Gale restricts attention to games with positive spillovers between agents and Lockwood and Thomas study only a version of the repeated prisoners' dilemma game, we impose no such restrictions and thus consider more applications. We emphasize that predictions vary widely with the properties of the payoff functions.

[5] Where it is obvious by context, assume that $i, j = 1, 2$, $i \neq j$ and that $t = 1, 2$.

[6] If the lower bound on $y_i^2$ were $-y_i^1$, then the problem would be equivalent to the one-period game, making any initial action meaningless.

Similarly, we write the Stackelberg outcome where $i$ is the leader as $(Y_i(G_i), Y_j(G_i))$, i.e.,

$$Y_i(G_i) \equiv \arg\max_{Y_i} U^i(Y_i, f^j(Y_i)) \quad \text{and} \quad Y_j(G_i) \equiv f^j(Y_i(G_i)). \tag{2}$$

We focus here on sufficiently well-behaved games for our analysis and make the following assumptions that hold in many applications.

**Assumption 1.** (a) Both $U^i(Y_i, Y_j)$ and $U^i(Y_i, f^j(Y_i))$ are twice continuously differentiable, and strictly quasi-concave in $(Y_i, Y_j)$ and $Y_i$ respectively.

(b) $f^i(Y_j)$ is strictly monotonic for both agents.

(c) The Cournot–Nash and each Stackelberg equilibrium in the one-period game are unique and interior.

Several remarks are in order. First, the assumptions on $U^i(Y_i, Y_j)$ imply that $f^i$ is a continuous and differentiable function. Furthermore, since the strategy set $F_i$ is a nonempty compact convex set, these assumptions also imply the existence of a pure strategy Cournot–Nash equilibrium in the one-period game (see, e.g., [6, Theorem 1.2]). Second, the assumptions on $U^i(Y_i, f^j(Y_i))$ imply that there is a unique Stackelberg outcome for each agent.[7] The strict quasi-concavity of $U^i(Y_i, f^j(Y_i))$ also means that a Stackelberg leader's payoff increases monotonically if we move along the follower's reaction function from any point toward the Stackelberg outcome. Third, while we require the monotonicity of reaction functions in part (b), we show by example in Section 4 that it is not always needed for our results. Note however that the monotonicity of reaction functions does not guarantee the uniqueness of the one-period Cournot–Nash equilibrium, which we assume in part (c).[8]

Based on our assumptions above, we identify 10 qualitatively unique cases as defined by the slopes of the agents' reaction functions and whether each agent's payoff increases or decreases in his rival's strategic variable.[9] For example, duopoly output setters that produce complements have upward sloping reaction functions, each with their payoff increasing in their rival's output. Another case is illustrated by the standard model of voluntary contributions to a public good (see (3) below). Again each agent has a payoff that increases in the other agent's strategic variable, but now with downward sloping reaction functions. We will see that the nature of

[7] While the uniqueness of each Stackelberg equilibrium is implied by part (a) of Assumption 1, we state it in part (c) for convenience.

[8] See, e.g., Tirole [20, p. 226] for a discussion of multiplicity of Cournot–Nash equilibrium, and sufficient conditions to ensure uniqueness. One such condition is that the derivatives of reaction functions are each less than 1 around the intersection points.

[9] Three cases have each agent with upward sloping reaction function, one with each agent's payoff increasing in the rival's strategic variable, another with each agent's payoff decreasing in the rival's strategic variable, and the third with one agent's payoff increasing and the other agent's payoff decreasing in the rival's choice. Analogously, three cases exist that have each agent with downward sloping reaction function, and four cases exist with each agent having opposite slopes of their reaction functions.

equilibria varies across these cases. All 10 cases have important applications as further discussed below.

Before proceeding to the main analysis, we record the following preliminary finding which compares the Cournot–Nash and Stackelberg outcomes.

**Proposition 1.** *Agent i's Stackelberg-leader amount is greater than* (*less than*) (*the same as*) *his Cournot–Nash amount if and only if* $\left.\frac{\partial U^i}{\partial Y_j}\frac{\partial f^j}{\partial Y_i}\right|_{G_0}$ *is positive* (*negative*) (*zero*).

All proofs are contained in the appendix.

## 3. The accumulation game

To determine the set of equilibria, we start with the second period. Upon observing $(y_i^1, y_j^1)$ and conjecturing $Y_j = y_j^1 + y_j^2$, agent $i$ solves the following second-period program.

$$[P] \quad \max_{Y_i} \quad U^i(Y_i, Y_j)$$
$$\text{s.t.}$$
$$y_i^1 \leqslant Y_i \leqslant I_i$$

Together with (1), the solution to $[P]$ is

$$Y_i = \max\{y_i^1, f^i(Y_j)\}.$$

This observation leads us to the following lemma that determines the equilibrium strategies in the second period.

**Lemma 1.** *The following strategies constitute the unique continuation equilibrium strategies in the second period.*

$$y_i^2(y_i^1, y_j^1) = \left\{\begin{array}{ll} 0 & \textit{if } y_i^1 \geqslant f^i(y_j^1) \textit{ and } y_j^1 \geqslant f^j(y_i^1), \\ Y_i(G_0) - y_i^1 & \textit{if } y_i^1 \leqslant Y_i(G_0) \textit{ and } y_j^1 \leqslant Y_j(G_0), \\ 0 & \textit{if } y_i^1 \geqslant Y_i(G_0) \textit{ and } y_j^1 \leqslant f^j(y_i^1), \\ f^i(y_j^1) - y_i^1 & \textit{if } y_j^1 \geqslant Y_j(G_0) \textit{ and } y_i^1 \leqslant f^i(y_j^1). \end{array}\right\}$$

The strategies in Lemma 1 were shown by Saloner [18] to be the equilibrium continuation strategies in his analysis of homogenous good duopolists with two production periods. The appendix shows that these constitute the *unique* continuation equilibrium strategies in the more general setting here. Saloner [18] does not point out that this strategy is unique in his paper, perhaps taking this as clear.

To present the main finding of the paper, we define the following outcome sets:

$$S_1 \equiv \{(Y_1, Y_2) \in F_1 \times F_2 \text{ such that } Y_i \geqslant f^i(Y_j)\}.$$

$$S_2 \equiv \{(Y_1, Y_2) \in F_1 \times F_2 \text{ such that } Y_i = f^i(Y_j) \text{ for at least one agent}\}.$$

$$S_3 \equiv \{(Y_1, Y_2) \in F_1 \times F_2 \text{ such that } Y_i \leqslant Y_i(G_i) \text{ whenever } Y_i \neq f^i(Y_j)\}.$$

$$S_4 \equiv \{(Y_1, Y_2) \in F_1 \times F_2 \text{ such that if } Y_i = f^i(Y_j), \\ \text{then either } Y_j \geqslant Y_j(G_j) \text{ or } Y_i \geqslant Y_i(G_j)\}.$$

$$S_5 \equiv \left\{ \begin{array}{c} (Y_1, Y_2) \in F_1 \times F_2 \text{ such that if } Y_i = f^i(Y_j), \text{ then } U^i(Y_i, Y_j) \geqslant \\ \max_{\tilde{Y}_i} U^i(\tilde{Y}_i, f^j(\tilde{Y}_i)) \text{ s.t. } f^j(\tilde{Y}_i) \geqslant Y_j \text{ and } \tilde{Y}_i \geqslant f^i(f^j(\tilde{Y}_i)) \end{array} \right\},$$

where $i, j = 1, 2$; $i \neq j$ in each set. Let $S$ denote their intersection: $S \equiv S_1 \cap S_2 \cap S_3 \cap S_4 \cap S_5$. Together with Lemma 1, the following theorem provides a convenient program for finding equilibrium outcomes in a variety of settings.

**Theorem 1.** *A $(Y_i, Y_j)$ pair is an equilibrium outcome if and only if it is in $S$.*

The sets that delineate $S$ are based on the usual definitions of reaction functions and the one-shot Cournot–Nash and Stackelberg outcomes so Theorem 1 is easy to apply. Note that due to Assumption 1 $S$ is never empty, so Theorem 1 implies existence as well.[10] We illustrate specific applications in Section 4.

In general, equilibria described in Theorem 1 will be a subset of the outcomes of those in the Cournot–Nash and Stackelberg models and including points on the reaction functions in between. When not the Cournot–Nash outcome, equilibrium will entail an element of leadership by one agent. Even so, one can see from the applications in the next section and from the Proof of Theorem 1 that all actions can occur in the first period for all equilibrium outcomes including the Stackelberg ones, unless the Stackelberg outcomes are the only equilibria. Hence, observing agents taking actions initially and doing nothing later need not imply that they are playing the one-period Cournot–Nash game.

Next we investigate the settings for which the Cournot–Nash outcome is the unique equilibrium. Understanding these cases is important in two ways: First, these are the cases where the irreversibility of early actions is not binding in equilibrium. That is, the same Cournot–Nash outcome would arise if agents could costlessly decrease as well as increase their previous actions in the second period. Second, these are the cases where no agent is able to exercise leadership. We provide the necessary and sufficient conditions for uniqueness of the Cournot–Nash outcome in

[10] Note, too, that the condition in $S_5$ is rarely constraining. For instance, as we show in Observation B1 in the appendix, when both reaction functions are downward-sloping, the constraint set for the maximization in $S_5$ is a singleton with $\tilde{Y}_i = Y_i(G_0)$, and thus the condition is automatically satisfied. One interesting case in which $S_5$ is binding occurs when firms produce complementary products (see Section 4 for details) and they are sufficiently asymmetric that one Stackelberg outcome Pareto dominates the other. In such a case, $S_5$ rules out the Pareto-dominated Stackelberg outcome as an equilibrium in the accumulation game, yielding the remaining Stackelberg outcome as the unique equilibrium outcome.

**Proposition 2.** *The one-period Cournot–Nash outcome*, $(Y_i(G_0), Y_j(G_0))$, *is the unique equilibrium outcome if and only if each agent's Stackelberg-leader amount is less than or equal to his corresponding Cournot–Nash amount*: $Y_i(G_i) \leqslant Y_i(G_0)$ *for* $i = 1, 2$.

Proposition 2 is most easily interpreted by its sufficiency part. Note first that if agent $i$ were to engender an outcome with his leadership, he would take a first period action less than or equal to his one-period Stackelberg amount, $Y_i(G_i)$, in equilibrium as required by $S_3$. The condition $Y_i(G_i) \leqslant Y_i(G_0)$ would then require that agent $i$ be below his reaction function. However, such a low first period action would also give agent $i$ an incentive to add to it in the second period, destroying his leadership commitment. Proposition 2 simply asserts that if the condition $Y_i(G_i) \leqslant Y_i(G_0)$ applies to both agents, then neither can exercise a leadership role. Furthermore, Proposition 1 above helps identify the cases for which these conditions are satisfied. We now provide specific applications.

## 4. Applications

### 4.1. Private provision of public goods

Consider the standard model of private contributions to a public good where agent $i$ $(i = 1, 2)$ allocates his income, $I_i$, between the numeraire consumption, $x_i$, and the private contribution, $Y_i$, to a public good. The utility function is given by

$$U^i = U^i(x_i, Y_i + Y_j); \tag{3}$$

where $U^i$ is increasing in its arguments.[11] Refer to Fig. 1. Under mild restrictions, both agents have downward sloping reaction functions. Since each agent benefits from the other agent's contribution, $\dfrac{\partial U^i}{\partial Y_j} > 0$, and Proposition 1 implies that $Y_i(G_i) < Y_i(G_0)$. Intuitively, if agent $i$ could commit to a lower contribution level than $Y_i(G_0)$, then he could gain since this would induce a larger contribution from agent $j$. While one can easily conclude from Proposition 2 that the Cournot–Nash outcome is the unique outcome here, one can also apply Theorem 1. Note that $S_1 \cap S_2 \cap S_3$ yields only point $G_0$, which satisfies $S_4$ as well. The constraint set on the maximization in $S_5$ contains only point $G_0$ (see footnote 10), so the inequality in $S_5$ is satisfied with equality. The Proof of Theorem 1 shows that $y_i^1 = Y_i(G_0)$, $y_j^1 = Y_j(G_0)$, and $y_i^2 = y_j^2 = 0$ make up an equilibrium.[12]

[11] Substituting $(I_i - Y_i)$ for $x_i$ from the budget constraint, observe that utility may be written as a function of $(Y_i, Y_j)$. Hence, the assumption of our general model is satisfied. See Bergstrom, Blume, and Varian [3] for an analysis of this voluntary contribution game.

[12] In fact, there is a continuum of equilibria here in that any $y_i^1 \leqslant Y_i(G_0)$ followed by $y_i^2 = Y_i(G_0) - y_i^1$ for both agents constitutes an equilibrium. However, all equilibria yield the same Cournot–Nash outcome so this multiplicity is not very important.

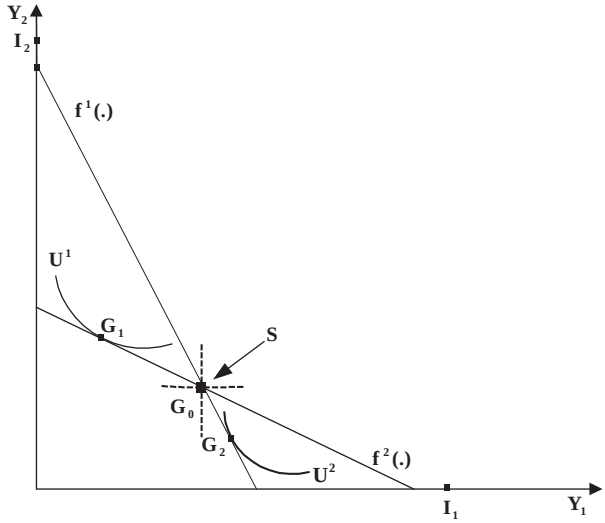


Fig. 1. Standard public good game.

To gain intuition, suppose that agent $i$ could commit to contributing $Y_i(G_i)$ in the initial period and to contributing nothing in the second period. Then agent $j$ can do no better than to choose the corresponding follower amount, $Y_j(G_i)$, either in the first period or in the second period. But this is not an equilibrium in the accumulation game because agent $i$ would prefer to increase his contribution in the second period, i.e., $f^i(Y_j) > Y_i(G_i)$ as seen in Fig. 1. This argument rules out only the Stackelberg outcome. However, if agent $i$ chooses $Y_i$ not on his reaction function, then the equilibrium condition in $S_3$ of Theorem 1 requires that $Y_i \leqslant Y_i(G_i)$. Since agent $i$ cannot commit to maintaining $Y_i(G_i)$, it is not surprising that he cannot commit to maintaining any lower amount. Both agents must then be on their reaction functions in equilibrium.

In the standard public good game above, Varian [23] shows that the total equilibrium contribution in a Stackelberg game where agents contribute only once is less than the one-period Cournot–Nash total. The dynamic element in the Stackelberg game exacerbates the free-rider problem. Our analysis highlights the difficulty of committing to leadership when multiple contributions are feasible. In fact, if the leader has no mechanism to commit not to increase his contribution later, a first-mover "advantage" vanishes in equilibrium. This helps alleviate the additional free-rider problem from sequential moves. This intuition provides a perspective as to why Marx and Mathews [15] find a more "positive" result regarding the completion of a joint project than do Admati and Perry [1] as we noted in the Introduction.

Now consider another version of contribution games and suppose that two politicians or business leaders contribute to a public good and are concerned mainly about their relative contribution to gain voter or customer goodwill. Let their utility

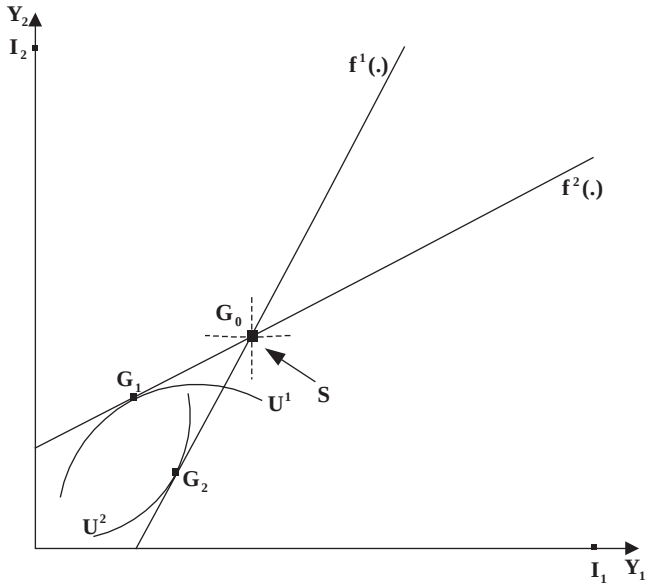


Fig. 2. Contribution game with prestige.

functions be given by

$$U^i = x_i(k_i + Y_i - Y_j), \tag{4}$$

where $x_i = I_i - Y_i$ is numeraire consumption, $I_i$ is agent $i$'s (exogenous) income, and $k_i$ is a parameter on $(\max\{0, I_j - I_i\}, I_i)$.[13] Refer to Fig. 2. Each agent has upward sloping reaction function and dislikes the other's contribution: $\dfrac{\partial U^i}{\partial Y_j} < 0$. Proposition 1 then implies $Y_i(G_i) < Y_i(G_0)$ for each agent $i$ and so Proposition 2 applies. Here, a Stackelberg leader would contribute less than the Cournot–Nash amount to soften the competition in the contribution game. By a similar argument to that above, such a first-period contribution is not a credible commitment in the accumulation game, leading to the Cournot–Nash outcome as the unique equilibrium outcome.

### 4.2. Differentiated product duopolists

Consider the following model first proposed by Dixit [4] and also analyzed by Singh and Vives [19]. Duopolists produce differentiated products with inverse demand function for firm $i$:

$$p_i = \alpha_i - \beta_i q_i - \gamma q_j, \quad i, j = 1, 2 \text{ and } i \neq j; \tag{5}$$

with obvious notation and where $\alpha_i, \beta_i, (\beta_1\beta_2 - \gamma^2)$, and $(\alpha_i\beta_j - \alpha_j\gamma)$ are all assumed positive. Products are substitutes (complements) if $\gamma$ is positive (negative). Duopolist $i$

[13] The example also requires that $2I_i > I_j$.

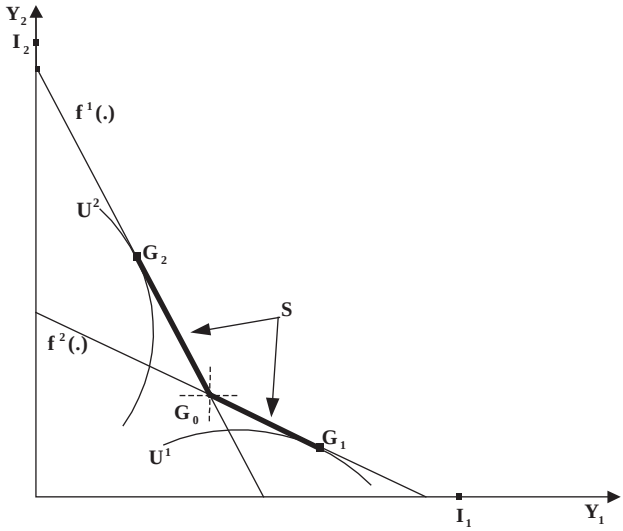


Fig. 3. Quantity competition with substitutes.

has constant average cost $m_i$, so profits are given by: $\Pi^i = (p_i - m_i)q_i$. Quantities are the strategic variables. Reaction functions are given by:

$$f^i(q_j) = \frac{\alpha_i - m_i}{2\beta_i} - \frac{\gamma}{2\beta_i} q_j. \tag{6}$$

With a few parameter restrictions, the problem is well-behaved with all the concavity and uniqueness properties satisfying Assumption 1 above.[14]

Consider first the accumulation game when products are substitutes, i.e., $\gamma > 0$. This case coincides with Saloner's [18] model and is depicted in Fig. 3. Applying Theorem 1, one can see that $S_1 \cap S_2 \cap S_3$ consists of points on each reaction function between $G_0$ and $G_i$, $i = 1, 2$. The requirements in $S_4$ and $S_5$ are not binding, yielding the following set of equilibrium outcomes: $S = \{(q_i, q_j)$ such that $q_i = f^i(q_j)$ and $q_i(G_0) \leqslant q_i \leqslant q_i(G_i),\ i,j = 1,2,\ i \neq j\}$.

As shown by Saloner, the equilibrium set is comprised of a continuum of points along the reaction functions between and including the Cournot–Nash and Stackelberg outcomes. As discussed further below, with the exception of the Stackelberg outcomes, both agents necessarily make first-period choices in $S$ and choose zero in the second period.

The case with complements, i.e., $\gamma < 0$, is depicted in Fig. 4. Applying Theorem 1 to the case of symmetric or nearly symmetric agents, one finds that the set $S_1 \cap S_2 \cap S_3$ consists of points on each reaction function between $G_0$ and $G_i$, $i = 1, 2$. The

[14] For example, in the symmetric case $\beta > \alpha - m > 0$ is sufficient. The upper bounds on the quantities can be set arbitrarily high.

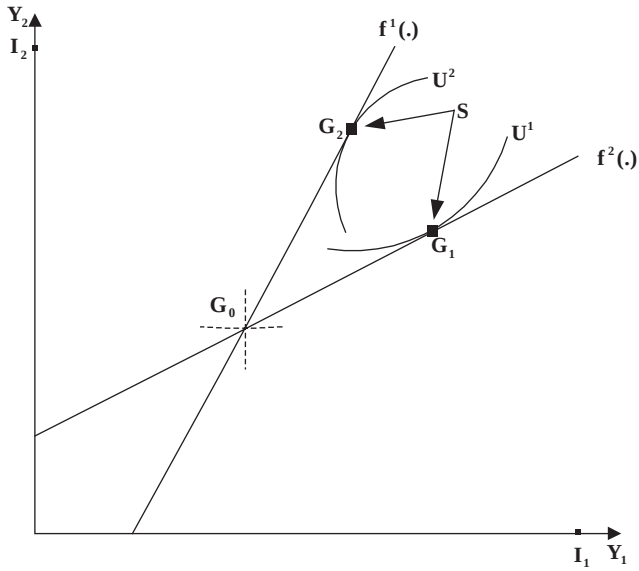


Fig. 4. Quantity competition with complements.

requirements of $S_4$ reduce the candidate equilibrium set to just the two Stackelberg points, $G_1$ and $G_2$. In Fig. 4, both $G_1$ and $G_2$ satisfy $S_5$, since both agents prefer following to leading (see also footnote 10). Thus, only the Stackelberg outcomes are equilibria. The Proof of Theorem 1 (in the appendix) shows that $G_i$, $i = 1, 2$, is an equilibrium with $y_i^1 = Y_i(G_i)$, $y_i^2 = y_j^1 = 0$, and $y_j^2 = Y_j(G_i)$.[15,16]

In both the cases of substitutes and complements, producing the Stackelberg output the first period—while the other duopolist produces zero or a low amount—is an equilibrium strategy with a credible commitment to not produce more later. Given duopolist $j$ produces say zero in the first period, engendering the Stackelberg outcome $G_i$ is the best that duopolist $i$ can do. With $q_j^1 = 0$, agent $i$ prefers $q_i^1 = q_i(G_i)$; any higher $q_i^1$ would induce a continuation equilibrium further away from $G_i$ on $j$'s reaction function. Given $q_i^1 = q_i(G_i)$, $j$ can do no better than produce nothing in the first period with then an outcome of $G_i$. In the case of complements (Fig. 4), both agents making initial choices below the Cournot–Nash outputs would lead to the latter in the continuation equilibrium. Either agent prefers to increase output the first period and effect his leadership outcome. If leadership in the sense of ending up on the other's reaction function is to result, it is best to choose the Stackelberg

[15] $G_i$ also arises as an equilibrium with the same choices by agent $i$ and a set of "low" choices $y_j^1$ followed by $y_j^2 = Y_j(G_i)$-$y_j^1$. This multiplicity is again not very important since the final outcomes, including payoffs, are invariant.

[16] Near symmetry simply guarantees that $S_5$ will not rule out one Stackelberg equilibrium.

leadership amount initially. Hence, only the Stackelberg equilibria arise in the case of complements.

In the case of substitutes (Fig. 3), other equilibria arise with "partial-" or "limited-leadership." Here agents do not like to follow, i.e., their payoffs rise moving from their Stackelberg follower's point along their reaction function to the Cournot–Nash point. By committing initially to an output in this range, the partial follower engenders the corresponding point on his reaction function as the equilibrium. The partial leader does best by choosing in the first period the corresponding output (with both choosing zero in the second-period continuation equilibrium). Given the partial leader's first-period choice, the partial follower is actually indifferent to choosing any output level up to the point on his reaction function, the continuation equilibrium at the same point on the partial follower's reaction function in any case. However, choosing less than the level on his reaction function would allow the partial leader to increase output the first period and move toward his Stackelberg leadership point in the continuation equilibrium. Both making choices on the partial follower's reaction function in the first period constitute the only equilibrium choices with partial leadership in the two-period game, the partial follower's equilibrium choice curtailing the effects of the other agent's leadership.

### 4.3. A rent-seeking model

Consider the following stylized rent-seeking model first developed by Tullock [21]. Two risk-neutral parties have opposed interests over a binary decision of a policy maker and take actions to influence that decision. Examples of such decisions include awarding of monopoly rights or government contracts, or passing of disputed legislation. Rent-seeking activities might take the form of political lobbying, bribes, or campaign contributions to political candidates. Party $i$ attaches a positive value equal to $I_i$ if the decision is in its favor and zero otherwise. The likelihood of party $i$'s winning is given by:

$$P_i(Y_1, Y_2) = \frac{Y_i}{Y_1 + Y_2} \text{ if } Y_1 > 0 \text{ or } Y_2 > 0, \text{ and } P_i(0,0) = 1/2, \tag{7}$$

where $Y_i \geqslant 0$ and denotes $i$'s rent seeking expenditures or effort. Thus party $i$ has payoff function:

$$U^i(Y_1, Y_2) = \frac{Y_i}{Y_1 + Y_2} I_i - Y_i. \tag{8}$$

From here, party $i$'s reaction function can be found as:[17]

$$f^i(Y_j) = \left\{ \begin{array}{ll} (I_i Y_j)^{1/2} - Y_j & \text{if } Y_j \in (0, I_i], \\ 0 & \text{if } Y_j > I_i. \end{array} \right\} \tag{9}$$

[17] $f^i(0)$ is not defined in this model. Specifying $P_i(Y_i, 0) = 1$ for $Y_i \geqslant \varepsilon$, for any small $\varepsilon$, resolves this issue without affecting any results.

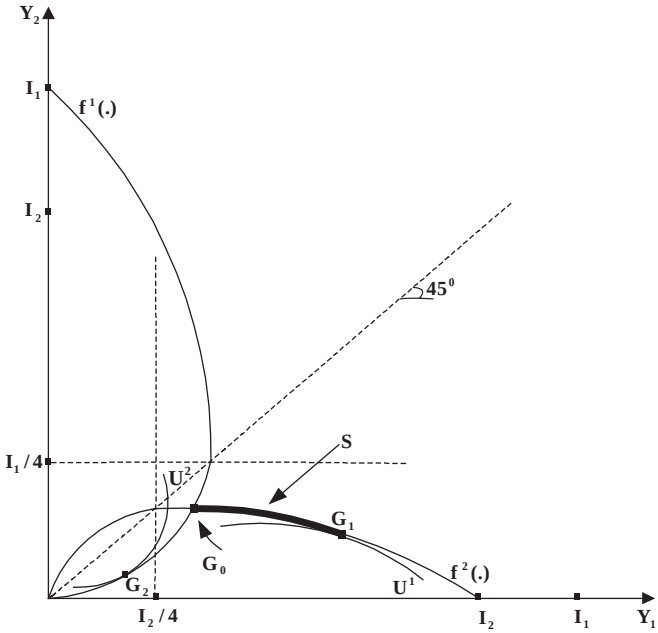


Fig. 5. Rent-seeking game.

Although most analyses of this model have used the simultaneous-move assumption and focused on the Cournot–Nash equilibrium, Linster [12] for one examines the Stackelberg alternative and compares the resulting outcomes. Fig. 5 depicts an example with asymmetric parties $(I_1 > I_2)$. Observe that $f^i(Y_j)$ is increasing for $Y_j \in (0, I_i/4)$ and decreasing for $Y_j \in (I_i/4, I_i)$, with maximum of $I_i/4$ when $Y_j = I_i/4$. Hence, $f^i(.)$ is nonmonotonic, violating part (b) of Assumption 1 above. Even so, we will show below that our results hold. The Cournot–Nash and Stackelberg equilibria exist and are unique. Assuming interior solutions in the Stackelberg cases (see below):

$$Y_i(G_0) = \frac{I_i^2 I_j}{(I_1 + I_2)^2}, \quad Y_i(G_i) = \frac{I_i^2}{4I_j}, \quad \text{and} \quad Y_j(G_i) = \frac{I_i}{2} - \frac{I_i^2}{4I_j}. \tag{10}$$

When $I_1 = I_2$ the Cournot–Nash and Stackelberg equilibria coincide. We analyze the more interesting case with $I_1 > I_2$ depicted in Fig. 5. In the Cournot–Nash equilibrium, since $P_1(.) > 1/2$, we call party 1 the favorite, and 2 the "underdog" using Dixit's [5] terminology.[18] Although it would not undermine our results, we avoid corner Stackelberg outcomes by assuming $I_2 > I_1/2$. Before proceeding, note the following ordering for our asymmetric case:

$$I_2/4 < Y_1(G_0) < Y_1(G_1) \text{ and } Y_2(G_2) < Y_2(G_0) < I_1/4. \tag{11}$$

[18] Dixit [5] analyzes an alternative specification of the rent-seeking game.

Now consider the two-period accumulation game applied to this model. Although the reaction functions are non-monotonic, Theorem 1 continues to hold. We sketch the argument. Observe that if the game is played in the space $(Y_1, Y_2) \in [I_2/4, I_1] \times [0, I_1/4]$, the monotonicity of the reaction functions would hold and Theorem 1 could be applied (also redefining agent 1's strategic variable so it has lower bound of zero). The game on the restricted strategy set requires $y_1^1 \geqslant I_2/4$ and $Y_2 \leqslant I_1/4$ (rather than $Y_2 \leqslant I_2$). The latter restrictions do not change the play of the unrestricted game.

Consider $Y_2 \leqslant I_1/4$. First we argue that $y_2^1 = I_1/4$ is always a better play for party 2 than any $y_2^1 > I_1/4$. By drawing party 2's implied second-period reaction functions for any $y_2^1 \geqslant I_1/4$, one can see that party 2 would commit himself to $y_2^2 = 0$ for any of these choices, and the equilibrium would have $(Y_1, Y_2) = (f^1(y_2^1), y_2^1)$. Party 2 is better off at $(f^1(I_1/4), I_1/4)$ than at any other of these points, implying that $y_2^1 > I_1/4$ is never an equilibrium choice.[19] Hence, the constraint $y_2^1 \leqslant I_1/4$ is innocuous. Similarly, party 2 would never want to increase $Y_2$ above $I_1/4$ in the second period.[20]

Requiring that party 1 choose at least $I_2/4$ in the first period is also harmless. If $y_2^1 \in [0, Y_2(G_0)]$, the continuation equilibrium is at $G_0$ for any $y_1^1 \in [0, I_2/4]$. Similarly, for any $y_2^1 \in [Y_2(G_0), I_1/4]$, the continuation equilibrium is at $f^1(y_2^1)$ for any $y_1^1 \in [0, I_2/4]$. Hence, requiring that $y_1^1$ be at least $I_2/4$ does not affect the equilibrium set.

Applying Theorem 1 then, the equilibrium set is given by:

$$S = \{(Y_1, Y_2) \text{ is such that } Y_2 = f^2(Y_1) \text{ and } Y_1(G_0) \leqslant Y_1 \leqslant Y_1(G_1)\}. \tag{12}$$

First, observe that the Stackelberg outcome where the "underdog" leads cannot be sustained as an equilibrium. In the case depicted in Fig. 5, if the underdog were to choose his Stackelberg leadership amount the first period, then the equilibrium would have the "favorite" choose his Stackelberg *leadership* amount the first period too, and second-period choices would lead to the favorite's Stackelberg-leadership outcome. The underdog cannot lead because his incentive is to increase effort in the second period. This is in sharp contrast to Baik and Shogren [2] and Leininger [11]. They find that the underdog-leadership outcome is the unique equilibrium if each party can take action in only one of two periods, and they initially and publicly commit to their period of action.[21] If a commitment to taking action only once is infeasible, then the outcome is very different.

The continuum of equilibria that arises in this case of the accumulation game is similar to that which arises in Saloner's problem. In the next section, we will see that discounting eliminates the possibility of a continuum, while preserving either the Cournot–Nash or one or both (modified) leadership outcomes as equilibria.

[19] This follows because $U^2(f^1(Y_2), Y_2)$ is concave in $Y_2$ in this problem.

[20] We should probably note that constraining $Y_2$ more tightly to not exceed $I_2/4$ is also innocuous. The argument goes through for either restriction, the key being that 2's standard reaction function is unconstrained.

[21] This is an application of "the observable delay game" of Hamilton and Slutsky [8].

We find in the rent-seeking example that the possibility of early actions works to the advantage of the favorite in the sense that this introduces equilibria with his leadership. More generally, we have shown that whether equilibrium has leadership or not is endogenous to the setting.

## 5. Discounting

Until now we have assumed that the cost of action remains constant across periods, i.e., there is no discounting. This is appropriate when initial actions constitute only committed minima or, as an approximation, when period lengths are short. In such cases, only strategic considerations determine when agents take actions in equilibrium. If there is discounting, however, the previous analysis needs to be modified. Let $r \geqslant 0$ be the discount rate. To allow for (possible) income effects, we explicitly introduce the numeraire good consumption, $x_i$, into the utility function, denoted here $U^i(Y_i, Y_j) \equiv \hat{U}^i(x_i, Y_i, Y_j)$, and assume without loss of generality that $\hat{U}_1(.) > 0$. In the second period, agent $i$ solves

$$
\begin{aligned}
[P_d] \quad & \max_{y_i^2} \quad \hat{U}^i(x_i, Y_i, Y_j) \\
& \text{s.t.} \\
& \qquad 0 \leqslant y_i^2 \leqslant I_i - (1+r)y_i^1, \\
& \qquad x_i + (1+r)y_i^1 + y_i^2 = I_i,
\end{aligned}
$$

where $I_i$ is agent $i$'s second-period income.[22]

Defining the adjusted income as $\tilde{I}_i = I_i - ry_i^1$, the program $[P_d]$ can be rewritten as

$$
\begin{aligned}
[P'_d] \quad & \max_{Y_i} \quad \hat{U}^i(\tilde{I}_i - Y_i, Y_i, Y_j) \\
& \text{s.t.} \\
& \qquad y_i^1 \leqslant Y_i \leqslant \tilde{I}_i
\end{aligned}
$$

Note that $[P'_d]$ is equivalent to $[P]$ above except that the former utilizes the adjusted income. Thus, the solution to $[P'_d]$ is

$$
Y_i = \max\{y_i^1, f^i(Y_j \mid \tilde{I}_i)\}, \tag{13}
$$

where $f^i(Y_j \mid \tilde{I}_i)$ denotes agent $i$'s one-period reaction function, conditional on the adjusted income.

Given (5.1) and letting $Y_i(G_0 \mid \tilde{I}_i, \tilde{I}_j)$ denote the one-period Cournot–Nash outcome conditional on the adjusted incomes, we can state the following variant of Lemma 1 for the discounting case:

[22] We assume for simplicity first period income is zero. This is, however, without loss of generality since we can think of second-period income as the value of total income in period two if there is income in each period.

**Lemma 2.** *The following strategies constitute the unique continuation equilibrium in the second period.*

$$y_i^2(y_i^1, y_j^1) = \left\{ \begin{array}{ll} 0 & \text{if } y_i^1 \geqslant f^i(y_j^1 \mid y_i^1) \text{ and } y_j^1 \geqslant f^j(y_i^1 \mid y_j^1), \\ Y_i(G_0 \mid y_i^1, y_j^1) - y_i^1 & \text{if } y_i^1 \leqslant Y_i(G_0 \mid y_i^1, y_j^1) \text{ and } y_j^1 \leqslant Y_j(G_0 \mid y_i^1, y_j^1), \\ 0 & \text{if } y_i^1 \geqslant Y_i(G_0 \mid y_i^1, y_j^1) \text{ and } y_j^1 \leqslant f^j(y_i^1 \mid y_j^1), \\ f^i(y_j^1 \mid y_i^1) - y_i^1 & \text{if } y_j^1 \geqslant Y_j(G_0 \mid y_i^1, y_j^1) \text{ and } y_i^1 \leqslant f^i(y_j^1 \mid y_i^1), \end{array} \right\}$$

*where we find it more convenient to make the dependence on the choices* $(y_i^1, y_j^1)$ *explicit, by suppressing the exogenous values* $(I_i, I_j, r)$ *and writing* $f^i(Y_j \mid \tilde{I}_i) \equiv [f^i(Y_j \mid y_i^1)$, *and* $Y_i(G_0 \mid \tilde{I}_i, \tilde{I}_j) \equiv [Y_i(G_0 \mid y_i^1, y_j^1)$.

Two remarks are in order here. First, $[P'_d]$ reduces to $[P]$ for $r = 0$. Second, for $r > 0$, in general, two additional effects come into play when agents take early actions: (1) there is the intertemporal substitution effect, as taking an early action is now costlier; and (2) there is the income effect, as taking an early action reduces the adjusted income, which may in turn shift the one-period reaction function. The latter effect is not present, however, in settings where agents have quasi-linear utility functions: $\hat{U}^i(x_i, Y_i, Y_j) = \alpha_i x_i + \Phi^i(Y_i, Y_j)$ for some $\alpha_i > 0$. This is because, in such settings, the one-period reaction function and thus the Cournot–Nash outcome are independent of the adjusted income. While it is conceivable that agents might possess quasi-linear utilities in many interesting applications, such utilities are typical for firms in duopoly games.[23] In what follows, we allow for income effects, but place an assumption on their sign, which is to be satisfied in many (if not most) cases, including cases with no income effects.

To motivate Assumption 2 below, in $[P'_d]$, let $V^i(y_i^1, Y_j)$ be agent $i$'s continuation equilibrium utility given his first-period choice and $j$'s accumulated amount. Consider now the following local comparative static: Suppose that agent $i$ increases $y_i^1$ slightly in cases where the continuation equilibrium would begin and stay at the Cournot–Nash outcome. That is, even after the change in $y_i^1$, the conditions in Lemma 2 that $y_i^1 \leqslant Y_i(G_0 \mid y_i^1, y_j^1)$ and $y_j^1 \leqslant Y_j(G_0 \mid y_i^1, y_j^1)$ are satisfied. Given $Y_j = Y_j(G_0 \mid y_i^1, y_j^1)$ and applying the Envelope Theorem to $[P'_d]$ at the Cournot–Nash outcome yields

$$\left. \frac{\partial V^i(y_i^1, Y_j(G_0 \mid y_i^1, y_j^1))}{\partial y_i^1} \right|_{G_0} = -r\hat{U}_1^i(.) + \hat{U}_3^i(.) \frac{\partial Y_j(G_0 \mid y_i^1, y_j^1)}{\partial y_i^1} - \lambda_1 - r\lambda_2, \qquad (14)$$

[23] For instance, consider an output game with constant marginal cost and one-period profit function: $\prod^i = P^i(Q_i, Q_j)Q_i - Q_i$; where $Q_i$ is measured so that marginal cost is one and $P^i(.)$ is inverse demand. Defining $x_i \equiv I_i - Q_i$ as the numeraire good for arbitrary $I_i$, one can re-write the profit function as $\hat{\Pi}^i = x_i + P^i(Q_i, Q_j)Q_i - I_i$, yielding our model with quasi-linear utility function.

where $\lambda_1$ and $\lambda_2$ are nonnegative Lagrange multipliers for the constraints, $y_i^1 \leqslant Y_i$ and $Y_i \leqslant \tilde{I}_i$ in $[P'_d]$, respectively. Given our assumption that the continuation equilibrium is at the Cournot–Nash outcome, neither constraint binds implying $\lambda_1 = \lambda_2 = 0$.

The first term on the r.h.s. of (14) represents the negative substitution effect. The second term comes from the income effect, as an increase in $y_i^1$ reduces $i$'s adjusted income. As observed above, when agents have quasi-linear utility functions, there is no income effect and thus the second term vanishes. This implies that the expression in (14) has a negative sign in such cases. Intuitively, these are the cases where the discounting introduces only the cost allocation incentive across periods. When there is an income effect however, the sign of (14) will continue to be negative unless this effect is positive and sufficiently large. For simplicity, we make the following assumption, the first part holding trivially in cases without income effects and holding as well in many cases with income effects:

**Assumption 2.** (a) If there are positive income effects on the r.h.s. of (14), the substitution effect is sufficiently negative to render $\left.\dfrac{\partial V^i(.)}{\partial y_i^1}\right|_{G_0} < 0$.

(b) If there are income effects, then goods $(x_i, Y_i)$ are weakly normal.

Part (a) of Assumption 2 implies that if agent $i$ knows that the equilibrium will be at a Cournot–Nash point, then he will shift all his action to the second and less costly period. As we will see below, this puts additional burden on being a leader.[24] The normality assumption in part (b) implies that the one-period reaction function shifts downward with a decrease in the adjusted income.

Theorem 2 presents the main result of this section. For its statement, let $Y_i(G_i \mid r)$ denote agent $i$'s Stackelberg-leader amount where $i$'s action costs are $(1+r)\,Y_i$ and $j$ takes all actions in the second period: $Y_i(G_i \mid r) = \arg\max_{Y_i} \hat{U}^i(I_i - (1+r)\,Y_i, Y_i, f^j(Y_i \mid 0))$.

**Theorem 2.** *If Assumption* 2 *holds, then a* $(Y_i, Y_j)$ *equilibrium pair must satisfy one of the following*:

I: $\{(y_i^1 = y_j^1 = 0), (y_i^2 = Y_i(G_0 \mid 0, 0), y_j^2 = Y_j(G_0 \mid 0, 0)\}$

II: $\{(y_i^1 = Y_i(G_i \mid r), y_j^1 = 0), (y_i^2 = 0, y_j^2 = f^j(y_i^1 \mid 0))\}$

Theorem 2 implies that any equilibrium must either be the Cournot–Nash equilibrium with all actions in the second period, or a variant of a Stackelberg equilibrium with marginal action cost of $1+r$. Since $y_j^1 = 0$ in both types of equilibria, agent $i$ must be indifferent between the Cournot–Nash equilibrium and the $r$-dependent leadership equilibrium if both types are to arise, obviously implying *either* Type I or Type II equilibria arise generically.[25] The Proof of Theorem 2 shows

[24] When Assumption 2 is violated, agent $i$ would like to take his action early on. This case can be analyzed using similar techniques.

[25] Existence is easy to show given our restrictions to well-behaved cases.

that a necessary condition for Type II equilibrium is that $y_i^1 > Y_i(G_0 \mid y_i^1, 0)$ for $y_i^1 = Y_i(G_i \mid r)$. To make a credible commitment to being the leader, agent $i$ has to take sufficiently large initial action. This reinforces our previous finding in Proposition 2 that gaining leadership requires some minimal initial commitment. In fact, we can state the following corollary to Proposition 2:[26]

**Corollary 1.** *If Assumption* 2 *holds and* $Y_i(G_i \mid r) \leqslant Y_i(G_0 \mid 0, 0)$ *for both agents, then the unique equilibrium is the Cournot–Nash outcome with all actions in the second period.*

Theorem 2 also implies that both agents' taking positive actions early on cannot be part of an equilibrium. The reason why one agent might take early action when it is more costly is to gain a leadership advantage. Given that one agent does so, it is best for the other to follow by shifting all his action to the less costly period. This rules out the possibility of the continuum of equilibria that we sometimes encountered in the no-discounting case. In cases like Saloner's, all actions must be taken in the first period (except in the pure Stackelberg outcomes). In such cases, given the "partial" leader commits to his action, the follower is actually indifferent about when to take action, as there is no cost difference across periods. However, this indifference breaks down with discounting and the follower would postpone all his action to the second and less costly period. This permits at most one equilibrium with player $i$ leading.

Which of the three possible equilibria can prevail depends on the structure of the specific game as well as the discount rate. Given agent $j$ does nothing in the first period, agent $i$ would decide whether to engender the Cournot–Nash outcome in the second period by taking no early action, or to engender the Stackelberg outcome where he leads. For instance, it is clear that if the first-period action is sufficiently costly, this will take away all the benefits of leadership and the Cournot–Nash outcome with actions in the second period will prevail. Being the Stackelberg leader pays off only if the discount rate is small enough. Regardless of how small the discount rate is, however, no agent may attempt to lead since the leadership amount may be insufficient to commit the agent to no future action as without discounting (see Corollary 1 above).

Our analysis of discounting builds on the insightful paper by Pal [16], who introduces an intertemporal cost differential of production into Saloner's homogenous good duopoly game. Like us, he also notes the disappearance of the continuum of equilibria, and characterizes the three possible equilibria described in Theorem 2 for Saloner's setting. Our analysis, in addition to applying to a larger set of cases including those entailing income effects, highlights the importance of the underlying game structure in predicting the equilibrium outcome as well as the role of the discount rate. Now we illustrate these points in two of our previous applications in Section 4.

[26] We thank a referee for this observation.

Consider the symmetric version of the differentiated product duopoly game in Section 4 now with discounting. That is, suppose the first-period production costs $(1+r)m$ dollars per unit while the cost of production in the second period is $m$ dollars per unit. Since there are no income effects here, Assumption 2 holds trivially. Applying Theorem 2, suppose firm 1 produces nothing in the first period, implying that firm 2 has two options to consider: (a) It can engender the Cournot–Nash outcome by producing only in the second period. That is, we have:

$$q_2(G_0) = \frac{\alpha - m}{2\beta + \gamma} \quad \text{and} \quad \Pi^2(G_0) = \frac{\beta(\alpha - m)^2}{2(2\beta^2 - \gamma^2)}. \tag{15}$$

(b) It can engender the Stackelberg outcome by producing in the first period. This yields

$$q_2(G_2 \mid r) = \frac{(\alpha - m)(2\beta - \gamma) - 2\beta m r}{2(2\beta^2 - \gamma^2)}$$
$$\text{and} \quad \Pi^2(G_2 \mid r) = \frac{(\alpha - m)^2(2\beta - \gamma)^2 - 4\beta^2 m^2 r^2}{8\beta(2\beta^2 - \gamma^2)}. \tag{16}$$

Comparing the payoffs, in equilibrium firm 2 would like to lead if and only if $r < r^*$, where $r^* = \dfrac{(\alpha - m)\gamma^2}{2\beta(2\beta + \gamma)m}$. However, firm 2 must also satisfy the condition that $q_2(G_2 \mid r) > q_2(G_0)$ to be the leader as discussed above. This condition is satisfied for $r < r^*$. When the discount rate is sufficiently small, either Stackelberg equilibrium arises. For $r > r^*$, firms produce only in the second period resulting in the Cournot–Nash outcome.[27] For $r = r^*$, all three equilibria are possible. Recalling the results above with no discounting, we find that discounting eliminates the continuum in the case of substitutes, and, generally, makes the Cournot–Nash outcome more likely. From (16), we see that when the Stackelberg outcome arises, the leader produces less due to the increased cost of producing early.

Now consider the standard model of public good provision, where agents have Cobb–Douglas utility functions:

$$U^i(x_i, Y_i + Y_j) = x_i(Y_i + Y_j). \tag{17}$$

The one-period reaction functions are given by

$$f^i(Y_j) = \frac{I_i - Y_j}{2}, \tag{18}$$

where $I_i$ is agent $i$'s (second-period) income. Also, the one-period Cournot–Nash equilibrium is

$$Y_1(G_0) = \frac{2I_1 - I_2}{3} \quad \text{and} \quad Y_2(G_0) = \frac{2I_2 - I_1}{3}. \tag{19}$$

To guarantee this equilibrium is interior, we assume $I_1/2 < I_2 < 2I_1$.

[27] When the goods are substitutes, i.e., $\gamma > 0$, our results coincide with Pal [16].

Turning to the accumulation game with discounting, let $I_i$ denote the value of income in the second period. It is easy to see that while the income effect is positive in this example, Assumption 2 still holds. As we argued in Section 4 with $r = 0$, $Y_i(G_i \mid r) \leqslant Y_i(G_0 \mid 0, 0)$ in the standard model of public good provision. Furthermore, since $Y_i(G_i \mid r)$ is decreasing in $r$, we also have $Y_i(G_i \mid r) \leqslant Y_i(G_0 \mid 0, 0)$ for any $r > 0$. Thus, one can appeal to the Corollary 1 and conclude that the unique equilibrium is the second period Cournot–Nash outcome. Intuitively, if an agent cannot commit to a high enough initial action in the no-discounting setting to exercise leadership, then the same agent will not be able to do so when the initial action is costlier. It is worth noting that unlike the previous example, the underlying game structure of this setting is such that regardless of the discount rate, no Stackelberg outcome arises in equilibrium.

## 6. Extensions

### 6.1. The decumulation game

In some two-period settings, agents' first-period decisions may bind them to a maximum final value of their strategic variable. Here decumulation is the only strategic option. For example, two competing political candidates who announce favored tax rates may find themselves effectively bound to supporting no higher rates during a campaign. They might revise their initially announced tax positions downward, while changing platform to support a higher rate would be the kiss of death. Another conceivable example is duopolists competing in prices who can pre-announce price. While setting a lower price before transactions take place is an option, increasing price above the pre-announced price may alienate customers and/or invite antitrust scrutiny. The implied "decumulation game" can be readily analyzed within our framework by just redefining strategies and applying results from the accumulation game. We illustrate our point by an example.

Consider again the differentiated duopoly model analyzed in Section 4 above. However, we now assume that firms engage in price competition, and write the demand functions by inverting (5) as

$$q_i = a_i - b_i p_i + c p_j, \tag{20}$$

where we let $\delta = \beta_1\beta_2 - \gamma^2, a_i = (\alpha_i\beta_j - \alpha_j\gamma)/\delta, b_i = \beta_j/\delta$, and $c = \gamma/\delta$. Note that $a_i$ and $b_i$ are positive due to the assumptions made in Section 4. Duopolist $i$'s profit function continues to be $\Pi^i = (p_i - m_i)q_i$, and his reaction function is

$$f^i(p_j) = \frac{a_i + b_i m_i}{2b_i} + \frac{c}{2b_i} p_j. \tag{21}$$

Suppose that duopolists can pre-announce their prices on $[0, \bar{P}_i]$ and then engage in price competition where the only strategic option is to reduce the pre-announced price. Here we assume $\bar{P}_i$ is high enough to be nonconstraining. Now we make the following change of variables: $\hat{p}_i^t \equiv -p_i^t$ where $i$, $t = 1, 2$, which further implies from

(20) and (21) that $\hat{q}_i \equiv a_i + b_i\hat{p}_i - c\hat{p}_j$, $\hat{\Pi}^i \equiv (-\hat{p}_i - m_i)\hat{q}_i$, and

$$\hat{f}(\hat{p}_j) \equiv -\frac{a_i + b_i m_i}{2b_i} + \frac{c}{2b_i}\hat{p}_j. \tag{22}$$

Note that the converted model of price competition with actions $\hat{p}_i^1 \in [-\bar{P}_i, 0]$, $\hat{p}_i^2 \in [\hat{p}_i^1, 0]$ and payoffs $\hat{\Pi}_i$ is the accumulation game played on $[-\bar{P}_i, 0] \times [-\bar{P}_j, 0]$. This game also yields unique Cournot–Nash and Stackelberg equilibria, as well as satisfying Assumption 1. Furthermore, since $\partial\hat{\Pi}^i/\partial\hat{p}_j = -c(-\hat{p}_i - m_i)$, Proposition 1 implies that $\hat{p}_i(G_i) \leqslant \hat{p}_i(G_0)$ for both firms. Applying Theorem 1 or Proposition 2, this further implies that the Cournot–Nash outcome is the unique equilibrium of the modified game. Moreover, by converting the variables back, we conclude that the Cournot–Nash outcome is also the unique equilibrium of the original decumulation game. Interestingly, unlike the quantity competition, no firm can exercise leadership with price competition regardless of whether the products are substitutes or complements, i.e., regardless of the sign of $c$. Letting $Y_i$ denote the price of duopolist $i$, Fig. 3 depicts the (we think) more interesting case of complements. Any attempts at leadership would fail. If, for example, duopolist 1 set $p_1^1 = p_1(G_1)$, then duopolist 2 can engender $G_0$—which 2 prefers to $G_1$—by choosing any $p_2^1 \geqslant p_2(G_0)$.[28]

### 6.2. Arbitrary number of periods

Our analysis up to this point has assumed that agents have two periods to accumulate their strategic variables. While this two-period framework has provided valuable insights into the nature of accumulation games, an important question is whether or not the number of periods has any significant impact on agents' equilibrium actions and payoffs. To address this question, we extend the basic model with no discounting presented in Section 3 and let $T \geqslant 2$ denote the number of periods. Here we assume that (pure-strategy and subgame-perfect) equilibria exist in every subgame and further that these continuation equilibrium sets are continuous in the state variables. The following is the main result of this section:

**Proposition 3.** *Suppose there is no discounting. If a $(Y_i, Y_j)$ pair is an equilibrium outcome in $T$ periods, then it is also an equilibrium outcome in two periods.*

While we are unable to prove points in $S$ necessarily arise as equilibrium outcomes generally, Proposition 3 narrows the search for equilibria in applications. Consider, for example, the standard case of contributing to a public good depicted in Fig. 1. Recall that $S$ is a singleton, the Cournot–Nash point. It is not difficult to confirm that this outcome arises as an equilibrium when $T = 3$ by using the techniques in the analysis of the two-period problem to show agent $i$ can do no better than choose

[28] In the present (decumulation) application, the Cournot–Nash outcome is the unique equilibrium outcome as we have noted, so $S$ in Fig. 3 should be ignored.

$y_i^1 = 0$ given that $y_j^1 = 0$. More generally, Proposition 3 implies that increasing the number of periods does not create new types of equilibria.

## 7. Concluding remarks

In a number of settings, payoffs depend on the total or final values of agents' strategic variables and agents have multiple opportunities to increase them. Examples are contribution games, rent-seeking games, and a number of duopoly games. We have examined in some detail the two-period, two-player version of this game. With no discounting, we provide a simple program for identifying the equilibrium set that can be applied in a variety of settings. Potential equilibria have outcomes corresponding to the standard Cournot–Nash or Stackelberg equilibria, or sometimes involve more limited leadership. For leadership to arise, it is necessary and sufficient that the Stackelberg-leader action is sufficiently high to commit the leader to no future action. Frequently, only the Cournot–Nash or one or both Stackelberg equivalents arise. We show further how the results extend when there is discounting.

We have also considered two extensions to our basic model. First, we let agents have only the strategic option of decumulating their initial choices. While we have demonstrated that this case is essentially an accumulation game with the appropriate change of variables, we note that the actual equilibrium outcomes of a decumulation game can be markedly different from those of an accumulation game in the original strategic variables. Second, we let agents have more than two periods to accumulate their strategic variables within the no-discounting setup. We show that equilibrium outcomes of the latter game must also be equilibrium outcomes of the two-period game.

Our framework is open to other promising extensions. For one, our analysis can be adapted to cases where one agent can only accumulate his strategic variable while the other agent can only decumulate her strategic variable. The other obvious and important extension is to more than two agents. This is quite complicated because there are as many "standard" equilibria as there are agents, $N$. In addition to the multi-agent Cournot–Nash equilibrium, there are $N$ Stackelberg equilibria, i.e., any number up to $N$ could take action first, with the remaining agents moving second. These extensions await future research.

## Acknowledgments

We thank two anonymous referees, the editors of this journal, Jonathan Hamilton, Tracy Lewis, Steve Slutsky, and participants of 2000 Public Choice Society Meetings and Duke Public Economics Workshop for thoughtful comments and suggestions.

## Appendix A

**Proof of Proposition 1.** Assume $\left.\frac{\partial U^i}{\partial Y_j}\frac{\partial f^j}{\partial Y_i}\right|_{G_0} > 0$. In the Cournot–Nash equilibrium $(G_0)$, both agents are on their reaction functions so that

$$\frac{\partial U^i}{\partial Y_i} = 0, \quad i = 1, 2. \tag{A.1}$$

In the Stackelberg equilibrium where $i$ leads $(G_i)$, $j$ is on his reaction function whereas $i$ satisfies the following first-order condition:

$$\frac{\partial U^i}{\partial Y_i} + \frac{\partial U^i}{\partial Y_j}\frac{\partial f^j}{\partial Y_i} = 0. \tag{A.2}$$

If we evaluate (A.2) at $G_0$, then the first term vanishes by (A.1). Thus, the strict quasi-concavity of $U^i(Y_i, f^j(Y_i))$ in $Y_i$ and $\left.\frac{\partial U^i}{\partial Y_j}\frac{\partial f^j}{\partial Y_i}\right|_{G_0} > 0$ imply that $Y_i(G_i) > Y_i(G_0)$.

Using an analogous argument, the results when $\left.\frac{\partial U^i}{\partial Y_j}\frac{\partial f^j}{\partial Y_i}\right|_{G_0}$ is negative or zero easily follow. □

**Proof of Lemma 1.** First we establish another lemma.

**Lemma A.1.** *Suppose that there is a unique interior Cournot–Nash equilibrium in the one-period game* (*as we have already assumed*). *Also suppose that* $Y_j = f^j(Y_i)$ *for a feasible* $(Y_i, Y_j)$, *i.e., on* $[0, I_i] \times [0, I_j]$. *Then* $Y_i < Y_i(G_0)$ *if and only if* $Y_i < f^i(Y_j)$.

**Proof.** Define the functions $f(x) \equiv f^i(f^j(x))$ and $F(x) \equiv f(x) - x$. Note that the Cournot–Nash equilibrium $(Y_i(G_0), Y_j(G_0))$ is such that $Y_i(G_0)$ is a fixed point of $f(.)$, and $Y_i(G_0) = f^i(Y_j(G_0))$. Also note that $F(Y_i(G_0)) = 0$. Since $Y_i \in [0, I_i]$, we have $F(I_i) \leqslant 0$ and $F(0) \geqslant 0$.

($\Rightarrow$) Suppose for some feasible $(Y_i, Y_j)$ pair we have $Y_j = f^j(Y_i)$ and $Y_i < Y_i(G_0)$. Suppose, however, that $Y_i \geqslant f^i(Y_j)$. That is, $Y_i \geqslant f(Y_i)$, or $F(Y_i) \leqslant 0$. Since $F(0) \geqslant 0$ and $F(.)$ is continuous, from the Intermediate Value Theorem, there exists some $\tilde{Y}_i \in [0, Y_i]$ such that $F(\tilde{Y}_i) = 0$. However, then $(\tilde{Y}_i, \tilde{Y}_j) \neq (Y_i(G_0), Y_j(G_0))$ where $\tilde{Y}_i$ and $\tilde{Y}_j = f^j(\tilde{Y}_i)$ is another Cournot–Nash equilibrium, which contradicts the uniqueness assumption.

($\Leftarrow$) Suppose now that for some feasible $(Y_i, Y_j)$ pair we have $Y_j = f^j(Y_i)$ and $Y_i < f^i(Y_j)$. However, suppose $Y_i \geqslant Y_i(G_0)$. Here $Y_i < f(Y_i)$, or $F(Y_i) > 0$. Since $F(I_i) \leqslant 0$ and $F(.)$ is continuous, there exists some $\tilde{Y}_i \in (Y_i, I_i]$ such that $F(\tilde{Y}_i) = 0$. However, then $(\tilde{Y}_i, \tilde{Y}_j) \neq (Y_i(G_0), Y_j(G_0))$ where $\tilde{Y}_i$ and $\tilde{Y}_j = f^j(\tilde{Y}_i)$ is another equilibrium, again contradicting uniqueness. □

First we show the strategies in Lemma 1 are equilibrium strategies, and then we show that they are unique. Consider the second period strategies and suppose $y_i^1 \geqslant f^i(y_j^1)$ and $y_j^1 \geqslant f^j(y_i^1)$. Given that $y_j^2 = 0$, $y_i^2 = 0$ since $U^i(.)$ is quasi-concave in $(Y_i, Y_j)$. Now suppose $y_i^1 \leqslant Y_i(G_0)$ and $y_j^1 \leqslant Y_j(G_0)$. Further suppose $y_j^2 = Y_j(G_0) - y_j^1$ is given. This implies $Y_j = Y_j(G_0)$. Since $Y_i(G_0) - y_i^1 \geqslant 0$, and by definition agent $i$'s best response to $Y_j(G_0)$ is $Y_i(G_0)$, we have $y_i^2 = Y_i(G_0) - y_i^1$.

In the third case of Lemma 1, suppose that $y_i^1 \geqslant Y_i(G_0)$ and $y_j^1 \leqslant f^j(y_i^1)$. Further suppose that $y_j^2 = f^j(y_i^1) - y_j^1$ is given, implying that $Y_j = f^j(y_i^1)$. From Lemma A.1, then $y_i^1 \geqslant f^i(Y_j)$. This implies that $y_i^2 = 0$ is the best response due to quasi-concavity of $U^i(.)$ in $(Y_i, Y_j)$. Given $y_i^2 = 0$, obviously $y_j^2 = f^j(y_i^1) - y_j^1$ is the best response for agent $j$. Finally, consider the last case of the second-period strategies. Given $y_j^2 = 0$, since $y_i^1 \leqslant f^i(y_j^1)$, $y_i^2 = f^i(y_j^1) - y_i^1$ is the best response for agent $i$. Now given $y_i^2 = f^i(y_j^1) - y_i^1$, i.e., $Y_i = f^i(y_j^1)$, $y_j^2 = 0$ is the best response for agent $j$ as for agent $i$ in the previous case.

Uniqueness can be seen as follows. By quasi-concavity of $U^i(Y_i, Y_j)$, the second-period reaction of agent $i$ satisfies:

$$\hat{y}_i^2(Y_j; y_i^1) = \begin{cases} f^i(Y_j) - y_i^1 & \text{if } y_i^1 \leqslant f^i(Y_j), \\ 0 & \text{if } y_i^1 \geqslant f^i(Y_j), \end{cases} \tag{A.3}$$

where it is convenient to write $\hat{y}_i^2$ in terms of $Y_j$ (rather than $y_j^2$). Now write agent $i$'s second-period reaction function in terms of his total $Y_i$:

$$\hat{f}^i(Y_j; y_i^1) \equiv \hat{y}_i^2 + y_i^1 = \max\{f^i(Y_j), y_i^1\},$$

the latter equality by (A.3). Second-period equilibrium is at the intersection of $\hat{f}^i$ and $\hat{f}^j$. Given $f^i$ and $f^j$ have a unique intersection, so too do $\hat{f}^i$ and $\hat{f}^j$. This can be seen easily in two steps. Relative to $f^i$, $\hat{f}^i$ is "shifted out" over a range to a constant value. Given monotonicity of $f^j$, clearly $\hat{f}^i$ and $f^j$ have a unique intersection. Now "shift out" $f^j$ to $\hat{f}^j$, with a unique intersection of $\hat{f}^i$ and $\hat{f}^j$ by the same logic. □

**Proof of Theorem 1.** We proceed by first showing conditions $S_i$, $i = 1, 2, \ldots, 5$ are necessary for equilibrium.

($\Rightarrow$) Suppose that $(Y_i, Y_j)$ is an equilibrium outcome. We show that it satisfies the conditions of $S_i$, $i = 1, 2, \ldots, 5$ respectively.

($S_1$): The conditions of $S_1$ trivially hold by (A.3).

($S_2$): Suppose the equilibrium pair is not in $S_2$. Then, given the point is in $S_1$, $Y_i > f^i(Y_j)$ for both agents. Eq. (A.3) implies and $y_i^2 = y_j^2 = 0$ and so $y_i^1 > f^i(y_j^1)$ for each agent. However, in the first period, given $j$'s contribution agent $i$ would be better off by reducing his amount to $f^i(y_j^1)$, a contradiction.

($S_3$): Given $Y_i \neq f^i(Y_j)$, it must be that $Y_i > f^i(Y_j)$ since the pair is in $S_1$. From (A.3), it must also be that $y_i^2 = 0$ and thus $y_i^1 = Y_i$. Since $Y_i > f^i(Y_j)$, we have

$Y_j = f^j(Y_i)$ since $(Y_i, Y_j)$ is in $S_2$. Now we argue that because $y_i^1 > Y_i(G_i)$, agent $i$ could increase his utility by marginally reducing his first period choice.

If $y_j^1$ is such that following the marginal reduction in $y_i^1$, agent $j$ can choose $y_j^2$ such that $Y_j = f^j(Y_i)$, then agent $i$ is better off due to the quasi-concavity of $U^i(Y_i, f^j(Y_i))$ in $Y_i$. If, however, $y_j^1$ is such that following the marginal reduction in $y_i^1$, agent $j$ cannot choose $y_j^2$ such that $Y_j = f^j(Y_i)$, i.e., if $y_j^1 > f^j(Y_i)$, then $Y_j$ would be unchanged. In the latter case, agent $i$ is better off since $y_i^1 > f^i(Y_j)$ and $U^i(Y_i, Y_j)$ is quasi-concave.

$(S_4)$: Suppose that $(Y_i, Y_j)$ is an equilibrium outcome with $Y_j = f^j(Y_i)$. Suppose, however, that $Y_i < Y_i(G_i)$ and $Y_j < Y_j(G_i)$. Since the pair is in $S_1$, we have $Y_i \geqslant f^i(Y_j)$. Thus from Lemma A.1, $Y_i \geqslant Y_i(G_0)$, which implies $Y_i(G_0) < Y_i(G_i)$. Given agent $j$'s first-period strategy, agent $i$ can engender $(Y_i(G_i),\ Y_j(G_i))$ as an equilibrium outcome where he would be better off.

To see this, let $y_i^1 = Y_i(G_i)$. Then since $Y_i(G_i) > Y_i(G_0)$ and $y_j^1 \leqslant Y_j < Y_j(G_i) = f^j(y_i^1)$, agent $i$'s second-period strategy dictates $y_i^2 = 0$ (by Lemma A.1). Again since $y_j^1 < Y_j(G_i)$ and $Y_i(G_i) > Y_i(G_0)$, by Lemma 1, $y_j^2 = f^j(y_i^1) - y_j^1 = Y_j(G_i) - y_j^1$. Thus $Y_j = Y_j(G_i)$.

$(S_5)$: Suppose that $(Y_i, Y_j)$ is an equilibrium outcome with $Y_i = f^i(Y_j)$. Suppose, however, that $U^i(Y_i, Y_j) < \max_{\tilde{Y}_i} U^i(\tilde{Y}_i, f^j(\tilde{Y}_i))$ subject to $f^j(\tilde{Y}_i) \geqslant Y_j$ and $\tilde{Y}_i \geqslant f^i(f^j(\tilde{Y}_i))$, where we are then assuming the constraint set is not empty. (If the constraint set is empty, then $(S_5)$ is automatically satisfied.) Let $\tilde{Y}_i^*$ denote the solution to the latter optimization problem. We now show that agent $i$ engenders $(\tilde{Y}_i^*, f^j(\tilde{Y}_i^*))$ in equilibrium by choosing $\tilde{y}_i^1 = \tilde{Y}_i^*$. The second-period equilibrium would have $\tilde{y}_j^2 = f^j(\tilde{Y}_i^*) - \tilde{y}_j^1$ and $y_i^2 = 0$ as we now confirm. Taking $\tilde{y}_i^2 = 0$ as given and using the fact that $\tilde{y}_j^1 \leqslant Y_j$, we know by the first constraint on the above maximization that $f^j(\tilde{Y}_i^*) \geqslant y_j^1$. Using (A.3), this implies that $\tilde{y}_j^2 = f^j(\tilde{Y}_i^*) - \tilde{y}_j^1$. Now take the latter, i.e., agent $j$'s second-period strategy as given. The second constraint on the above maximization problem along $\tilde{y}_i^1 = \tilde{Y}_i^*$ implies $\tilde{y}_i^1 \geqslant f^i(f^j(\tilde{y}_i^1))$, which by Lemma A.1, implies $\tilde{y}_i^1 \geqslant Y_i(G_0)$. This and $\tilde{y}_j^1 \leqslant f^j(\tilde{y}_i^1)$ imply $\tilde{y}_i^2 = 0$ by Lemma 1. Hence, $\tilde{y}_i^1 = \tilde{Y}_i^*$ does engender $(\tilde{Y}_i^*, f^j(\tilde{Y}_i^*))$ in equilibrium, implying the necessity of $(S_5)$.

$(\Leftarrow)$ Take any $(Y_i, Y_j)$ in $S$. There are two cases:

*Case* 1: $Y_i = f^i(Y_j)$ and $Y_j = f^j(Y_i)$.

In this case, of course, $Y_i = Y_i(G_0)$ and $Y_j = Y_j(G_0)$. Claim 1 below shows that if $S_4$ is satisfied for both agents, then $y_i^1 = Y_i(G_0)$, $y_j^1 = Y_j(G_0)$, and $y_i^2 = y_j^2 = 0$ make up an equilibrium.

*Case* 2: $Y_i = f^i(Y_j)$ and $Y_j \neq f^j(Y_i)$.

It must be that $Y_j > f^j(Y_i)$ and $Y_j \leqslant Y_j(G_j)$ since the pair is in $S_1$ and $S_3$, respectively. Also, since $Y_i = f^i(Y_j)$, being in $S_4$ implies either $Y_j \geqslant Y_j(G_j)$ or $Y_i \geqslant Y_i(G_j)$. From Lemma A.1, we have $Y_j > Y_j(G_0)$. There are two possibilities.

Suppose that $Y_j < Y_j(G_j)$. Then $Y_i \geqslant Y_i(G_j)$ due to being in $S_4$. Claim 2 below shows that such a pair is supported as an equilibrium outcome so long as $S_5$ is satisfied. Now consider $Y_j = Y_j(G_j)$. This means $Y_i = f^i(Y_j) = Y_i(G_j)$. It is straightforward to show that when $S_5$ is satisfied, this pair can be supported as an equilibrium by the second-period strategies together with $y_j^1 = Y_j(G_j)$ and $y_i^1 = 0$.

Given Claims 1 and 2 below, the proof is complete. □

**Claim 1.** $(Y_i(G_0), Y_j(G_0))$ *is an equilibrium outcome under the following cases*:

*Case* 1: $Y_i(G_0) > Y_i(G_i)$ and $Y_i(G_0) > Y_i(G_j)$,i.e., at least one agent's Cournot–Nash strategy exceeds both his Stackelberg leader's and follower's strategy.

**Proof.** Suppose $y_j^1 = Y_j(G_0)$ is given. If $y_i^1 > Y_i(G_0)$, then $y_i^2 = 0$ by Lemma 1, i.e., either the first or third cases must be satisfied. In this case, there are two possibilities. If $y_j^1 \geqslant f^j(y_i^1)$, then $y_j^2 = 0$ again by Lemma 1. However, $y_i^1 = Y_i(G_0)$ is better for agent $i$ due to quasi-concavity of $U^i(.)$ in $(Y_i, Y_j)$. (Both agents would continue to choose zero in the second period.) On the other hand, if $y_j^1 < f^j(y_i^1)$, then $y_j^2 = f^j(y_i^1) - y_j^1$ by Lemma 1. Since the resulting outcome is on agent $j$'s reaction function and $Y_i > Y_i(G_0) > Y_i(G_i)$, agent $i$ is worse off than $y_i^1 = Y_i(G_0)$ due to strict quasi-concavity of $U^i(Y_i, f^j(Y_i))$ in $Y_i$. Thus, $y_i^1 = Y_i(G_0)$ is always a better response than $y_i^1 > Y_i(G_0)$ and would lead to the outcome $(Y_i(G_0), Y_j(G_0)$.

Now consider the possibility of $y_i^1 \leqslant Y_i(G_0)$. Then, $y_j^2 = 0$ and $y_i^2 = Y_i(G_0) - y_i^1$ by Lemma 1. Thus, the resulting outcome is $Y_i = Y_i(G_0)$. As a result, given $y_i^1 = Y_j(G_0), y_i^1 = Y_i(G_0)$ is a best response for agent $i$.

Suppose now that $y_i^1 = Y_i(G_0)$ is given. Since $Y_i(G_0) = f^i(Y_j(G_0)) > f^i(Y_j(G_j)) = Y_i(G_j)$ by hypothesis, we analyze two cases regarding the monotonicity of $f^i(.)$. If $f^i(.)$ is upward sloping, then we must have $Y_j(G_j) < Y_j(G_0)$, where $y_j^1 = Y_j(G_0)$ would be a best response for agent $j$ from the above discussion for agent $i$. (We only used the condition $Y_i(G_0) > Y_i(G_i)$ there.) If, on the other hand, $f^i(.)$ is downward sloping, then $Y_j(G_0) < Y_j(G_j)$. Consider $y_j^1 < Y_j(G_0)$. Then, $y_i^2 = 0$ and $y_j^2 = Y_j(G_0) - y_j^1$ by Lemma 1, implying $Y_i = Y_i(G_0)$ and $Y_j = Y_j(G_0)$. Thus, $y_j^1 = Y_j(G_0)$ yields the same payoff for $j$. Now consider $y_j^1 > Y_j(G_0)$. Then $y_i^1 > f^i(y_j^1)$ and $y_j^1 > Y_j(G_0) = f^j(y_i^1)$, implying that $y_i^2 = y_j^2 = 0$ by Lemma 1. However, $y_j^1 = Y_j(G_0)$ is a better response. Thus, given $y_i^1 = Y_i(G_0), y_j^1 = Y_j(G_0)$ is a best response for agent $j$.

Overall, since, $y_i^1 = Y_i(G_0)$ and $y_j^1 = Y_j(G_0)$ are equilibrium strategies with second-period strategies $y_i^2 = y_j^2 = 0, (Y_i(G_0), Y_j(G_0))$ is an equilibrium outcome.

*Case* 2: $Y_j(G_0) > Y_j(G_i)$ and $Y_i(G_0) > Y_i(G_j)$, i.e., both Cournot–Nash strategies exceed their Stackelberg followers' strategies.

**Proof.** We have $Y_j(G_0) = f^j(Y_i(G_0)) > f^j(Y_i(G_i)) = Y_j(G_i)$ and $Y_i(G_0) = f^i(Y_j(G_0)) > f^i(Y_j(G_j)) = Y_i(G_j)$. If both reaction functions are upward sloping, it must be that $Y_i(G_0) > Y_i(G_i)$ and $Y_j(G_0) > Y_j(G_j)$. In this case, the conditions of Case 1 are satisfied for both agents, and the same argument applies. Again, then, $y_i^1 = Y_i(G_0)$ and $y_j^1 = Y_j(G_0)$ together with $y_i^2 = y_j^2 = 0$ make up an equilibrium with the totals of $(Y_i(G_0),\ Y_j(G_0))$.

If $f^i(.)$ is upward sloping and $f^j(.)$ is downward sloping, then we have $Y_j(G_0) > Y_j(G_j)$, $Y_i(G_0) < Y_i(G_i)$, and the conditions of Case 1 are satisfied for agent $j$. Again $(Y_i(G_0),\ Y_j(G_0))$ is an equilibrium outcome. It is also easy to show that when both reaction functions are downward sloping, the Cournot–Nash outcome is supported as the equilibrium again with $(y_i^1, y_j^1) = (Y_i(G_0), Y_j(G_0))$ and $(y_i^2, y_j^2) = (0, 0)$. (Either agent reducing their first-period strategy would lead to the same outcome. If an agent increase his first-period strategy, $(y_i^2, y_j^2) = (0, 0)$ continues to hold, and that agent would be worse off.) □

**Claim 2.** *$(Y_i,\ Y_j)$ is an equilibrium outcome if $Y_i = f^i(Y_j)$ and $Y_j(G_0) < Y_j < Y_j(G_j)$, $Y_i \geqslant Y_i(G_j)$, and $U^i(Y_i, Y_j) \geqslant \max_{\tilde{Y}_i} U^i(\tilde{Y}_i, f^j(\tilde{Y}_i))$ subject to $f^j(\tilde{Y}_i) \geqslant Y_j$ and $\tilde{Y}_i \geqslant f^i(f^j(\tilde{Y}_i))$.*

**Proof.** First note that since $Y_i = f^i(Y_j)$ and $Y_j(G_0) < Y_j$, we have $Y_j > f^j(Y_i)$ from Lemma A.1. Now observe that since $Y_j(G_j) > Y_j(G_0)$, from Proposition 1 we have $\left(\dfrac{\partial U^j}{\partial Y_i}\dfrac{\partial f^i}{\partial Y_j}\right)$ must be positive. Suppose both $\dfrac{\partial U^j}{\partial Y_i}$ and $\dfrac{\partial f^i}{\partial Y_j}$ are positive. Then, since $Y_j < Y_j(G_j)$, $Y_i = f^i(Y_j) < f^i(Y_j(G_j)) = Y_i(G_j)$, a contradiction. Thus, both derivatives must be negative. This immediately implies that $Y_i < Y_i(G_0)$. Now we show that $y_i^1 = Y_i, y_j^1 = Y_j$ together with the second-period strategies in Lemma 1 constitute an equilibrium.

Suppose $y_i^1 = Y_i$ is given and suppose $y_j^1 = Y_j$ is not a best response for agent $j$. If $y_j^1 \leqslant Y_j(G_0)$, then the second-period strategies dictate $y_i^2 = Y_i(G_0) - Y_i$ and $y_j^2 = Y_j(G_0) - y_j^1$, which is a worse outcome for agent $j$ than $(Y_i,\ Y_j)$ due to strict quasi-concavity of $U^j(Y_j, f^i(Y_j))$ in $Y_j$. If, however, $Y_j(G_0) < y_j^1 < Y_j$, then $Y_i < f^i(y_j^1)$. In this case, $y_j^2 = 0$ and $y_i^2 = f^i(y_j^1) - Y_i$, which is again a worse outcome for agent $j$ for the same reason.

Assume instead that $y_j^1 > Y_j$. Then, $y_j^1 > f^j(Y_i)$ and $y_i^1 > f^i(y_j^1)$. Thus, $y_i^2 = y_j^2 = 0$. However, agent $j$ is worse off due to quasi-concavity of $U^j(.)$ in $(Y_i,\ Y_j)$. Thus, $y_j^1 = Y_j$ is a best response.

Now we take $y_j^1 = Y_j$ as given and show $y_i^1 = Y_i$ is a best response. Consider any $y_i^1 < Y_i$. Then $y_i^1 < f^i(y_j^1)$ and we are given $y_j^1 = Y_j > Y_j(G_0)$. Lemma 1 implies $y_i^2 = f^i(y_j^1) - y_i^1$ and $y_j^2 = 0$, leading to the same outcome $(Y_i,\ Y_j)$. Hence, no $y_i^1 < Y_i$ is strictly better for agent $i$.

For alternatives with $y_i^1 > Y_i$, $y_i^1 > f^i(y_j^1)$, and there are two cases. If $f^j(Y_i)$ is decreasing, then $y_j^1 > f^j(y_i^1)$ since we know $y_j^1 = Y_j > f^j(Y_i)$. Here, $(y_i^2, y_j^2) = (0,0)$ from Lemma 1, and agent $i$ is worse off by quasi-concavity of $U^i(Y_i, Y_j)$.

If, for $y_i^1 > Y_i$, $f^j(Y_i)$ is increasing, it could still be that $y_j^1 > f^j(y_i^1)$ and the same argument applies. Suppose, however, that $y_j^1 \leqslant f^j(y_i^1)$. Since $f^i(.)$ is downward sloping (from above) and $f^j(.)$ is upward sloping, $y_i^1 > f^i(y_j^1)$ and $y_j^1 \leqslant f^j(y_i^1)$ imply that $y_i^1 > Y_i(G_0)$. Using Lemma 1, then, $y_j^1 = f^j(y_i^1) - y_j^1$ and $y_j^2 = 0$. By the (last) condition on agent $i$'s utility function in Claim 2, agent $i$ is no better off. Hence, $y_i^1 = Y_i$ is a best response.

Since for these first-period strategies Lemma 1 implies $(y_i^2, y_j^2) = (0,0)$, such that $(Y_i, Y_j)$ indeed constitutes an equilibrium. □

**Proof of Proposition 2.** ($\Rightarrow$) Suppose that $(Y_i(G_0), Y_j(G_0))$ is the unique equilibrium outcome. Then, it must be in $S$, in particular in $S_4$. Since $Y_j(G_0) = f^j(Y_i(G_0))$, we must have either

$$Y_i(G_0) \geqslant Y_i(G_i) \text{ or } Y_j(G_0) \geqslant Y_j(G_i). \tag{A.4}$$

Analogously, since $Y_i(G_0) = f^i(Y_j(G_0))$, either

$$Y_j(G_0) \geqslant Y_j(G_j) \text{ or } Y_i(G_0) \geqslant Y_i(G_j). \tag{A.5}$$

Now we develop a contradiction to the uniqueness of the above equilibrium under the presumption that $Y_i(G_i) > Y_i(G_0)$ for at least one agent. There are two cases:

*Case* 1: $Y_i(G_i) > Y_i(G_0)$ and $Y_j(G_0) > Y_j(G_j)$.

Since $Y_i(G_i) > Y_i(G_0)$, to satisfy (A.4) we must have $Y_j(G_0) > Y_j(G_j)$ or $Y_j(G_0) = f^j(Y_i(G_0)) \geqslant Y_j(G_i) = f^j(Y_i(G_i))$, which implies $f^j(.)$ is downward sloping.

Note that $(Y_i(G_i), Y_j(G_i))$ is in $S_1 \cap S_2 \cap S_3 \cap S_4$. If we can show that it also satisfies $S_5$, then by Theorem 1 it is an equilibrium, a contradiction. Satisfaction of $S_5$ requires that

$$\begin{aligned} &U^j(Y_j(G_i), Y_i(G_i)) \geqslant \max_{\tilde{Y}_j} U^j(\tilde{Y}_j, f^i(\tilde{Y}_j)) \\ &\text{s.t. } f^i(\tilde{Y}_j) \geqslant Y_i(G_i) \text{ and } \tilde{Y}_j \geqslant f^j(f^i(\tilde{Y}_j)). \end{aligned} \tag{A.6}$$

To satisfy $Y_j(G_0) > Y_j(G_j)$, Proposition 1 requires that $\dfrac{\partial U^j}{\partial Y_i}\dfrac{\partial f^i}{\partial Y_j} < 0$.

If $\dfrac{\partial U^j}{\partial Y_i} < 0$, the first constraint on the maximization in (A.6) implies $S_5$ is satisfied. (Using also that $j$ is on his reaction function at the outcome $(Y_i(G_i), Y_j(G_i))$.

If $\dfrac{\partial U^j}{\partial Y_i} > 0$, then $\dfrac{\partial f^i}{\partial Y_j} < 0$, so both reaction functions are downward sloping. Here the constraint set on the maximization in $S_5$ is empty, so condition $S_5$ is satisfied trivially. One can see this by drawing the graph with two downward sloping reaction functions and unique Cournot–Nash equilibrium. More formally, suppose that

$(\tilde{Y}_i, \tilde{Y}_j)$ is in the constraint set of the maximization in (A.6). Since $\tilde{Y}_i = f^i(\tilde{Y}_j)$ and $\tilde{Y}_j \geqslant f^j(\tilde{Y}_i)$, from Lemma A.1, we have $\tilde{Y}_j \geqslant Y_j(G_0)$. Then, $\tilde{Y}_i = f^i(\tilde{Y}_j) \leqslant f^i(Y_j(G_0)) = Y_i(G_0) < Y_i(G_i) \leqslant \tilde{Y}_i$, the first inequality since $f^i$ is downward sloping, the second inequality as it characterizes Case 1, and the last inequality by the constraint set. Hence, we have a contradiction. We have shown that equilibrium would not be unique given $Y_i(G_i) > Y_i(G_0)$ in Case 1, so $Y_i(G_i) \leqslant Y_i(G_0)$ is a necessary condition for uniqueness of the Cournot–Nash outcome.

*Case* 2: $Y_i(G_i) > Y_i(G_0)$ and $Y_j(G_0) < Y_j(G_j)$.

To satisfy (A.5), $Y_i(G_0) \geqslant Y_i(G_j)$. Similar arguments to those in Case 1 show that $f^i$ is downward sloping. Again, similar arguments show that the constraint set on the maximization in $S_5$ is empty. Again, this implies that $(Y_i(G_i),\ Y_j(G_i))$ is an equilibrium, contradicting uniqueness.

($\Leftarrow$) Suppose that $Y_i(G_i) < Y_i(G_0)$ for both agents. Similar arguments of Case 1 of Claim 1 above show that $(Y_i(G_0), Y_j(G_0))$ can be obtained as an equilibrium.

Now suppose that there exists a feasible $(Y_i,\ Y_j) \neq (Y_i(G_0),\ Y_j(G_0))$ that is also an equilibrium. Then it must be in $S$. This implies w.l.o.g. $Y_i = f^i(Y_j)$ and $Y_j \neq f^j(Y_i)$. This also implies that $Y_j < Y_j(G_0)$, which follows from having $Y_j \leqslant Y_j(G_j)$ and $Y_j(G_j) < Y_j(G_0)$ by hypothesis. Since agent $i$ is on his reaction function, Lemma A.1 reveals $Y_j < f^j(Y_i)$. However, this contradicts $Y_j > f^j(Y_i)$ which must hold since the pair is in $S$. □

**Proof of Theorem 2.** We first start recording four lemmas.

**Lemma A.2.** *In equilibrium, if* $y_i^1 \in [0, Y_i(G_0 \mid y_i^1, y_j^1)]$, *then* $y_j^1 \notin (0, Y_j(G_0 \mid y_i^1, y_j^1)]$.

**Proof.** Suppose not. That is, suppose $y_i^1 \in [0, Y_i(G_0 \mid y_i^1, y_j^1)]$ and $y_j^1 \in (0, Y_j(G_0 \mid y_i^1, y_j^1)]$. Then, the second-period strategies in Lemma 2 imply that $y_i^2 = Y_i(G_0 \mid y_i^1, y_j^1) - y_i^1$ and $y_j^2 = Y_j(G_0 \mid y_i^1, y_j^1) - y_j^1$. Suppose agent $j$ makes an arbitrarily small reduction in $y_j^1$, i.e., $y_j^{1*} = y_j^1 - \varepsilon$. Since $\dfrac{dY_j(G_0 \mid y_i^1, y_j^1)}{dy_j^1} \leqslant 0$ by part (b) of Assumption 2, we have $Y_j(G_0 \mid y_i^1, y_j^{1*}) > Y_j(G_0 \mid y_i^1, y_j^1)$, which implies $y_j^{1*} \in (0, Y_j(G_0 \mid y_i^1, y_j^{1*})]$. However, there are two possibilities for agent $i$. If $\dfrac{dY_i(G_0 \mid y_i^1, y_j^1)}{dy_j^1} \leqslant 0$, then $y_i^{1*} \in [0, Y_i(G_0 \mid y_i^1, y_j^{1*})]$. In this case, Lemma 2 dictates $y_i^2 = Y_i(G_0 \mid y_i^1, y_j^{1*}) - y_i^1$ and $y_j^{2*} = Y_j(G_0 \mid y_i^1, y_j^{1*}) - y_j^{1*}$. That is, the resulting outcome would be at the new Cournot–Nash equilibrium. Due to part (a) of Assumption 2, this small reduction strictly benefits agent $j$, contradicting the equilibrium hypothesis.

If, however, $\dfrac{dY_i(G_0 \mid y_i^1, y_j^1)}{dy_j^1} > 0$ and $y_i^1$ is such that $y_i^1 \notin [0, Y_i(G_0 \mid y_i^1, y_j^{1*})]$, i.e., $Y_i(G_0 \mid y_i^1, y_j^{1*}) < y_i^1 < Y_i(G_0 \mid y_i^1, y_j^1)$, then, again from Lemma 2, $y_i^2 = 0$ and

$y_j^{2*} = f^j(y_i^1 \mid y_j^{1*}) - y_j^{1*}$. As $y_j^{1*} \to y_j^1$, i.e., $\varepsilon \to 0$, $y_i^1 \to Y_i(G_0 \mid y_i^1, y_j^1)$ due to the Sandwich Theorem. Therefore, for arbitrarily small reductions in $y_j^1$, the outcome is arbitrarily close to the Cournot–Nash outcome. However, agent $j$ would strictly benefit from shifting his action towards the second period due to part (a) of Assumption 2. □

**Lemma A.3.** *In equilibrium, if* $y_i^1 > Y_i(G_0 \mid y_i^1, y_j^1)$, *then* $y_j^1 \notin (0, Y_j(G_0 \mid y_i^1, y_j^1)]$.

**Proof.** Suppose not. That is, suppose in equilibrium $y_i^1 > Y_i(G_0 \mid y_i^1, y_j^1)$ but $y_j^1 \in (0, Y_j(G_0 \mid y_i^1, y_j^1)]$. Lemma 2 implies $y_i^2 = 0$. However, given $y_i^1$, agent $j$ could choose $y_j^{1*} = y_j^1 - \varepsilon$ for an arbitrarily small $\varepsilon > 0$ so that $y_i^1 > Y_i(G_0 \mid y_i^1, y_j^{1*})$. Moreover, since $\dfrac{dY_j(G_0 \mid y_i^1, y_j^1)}{dy_j^1} < 0$ from part (b) of Assumption 2, and thus $y_j^{1*} \in (0, Y_j(G_0 \mid y_i^1, y_j^{1*})]$, we still have $y_i^2 = 0$. This means, due to Assumption 2, agent $j$ strictly benefits from this small reduction in the first period, contradicting the equilibrium hypothesis. Hence, $y_j^1 \notin (0, Y_j(G_0 \mid y_i^1, y_j^1)]$. □

**Lemma A.4.** *Both* $y_i^1 > Y_i(G_0 \mid y_i^1, y_j^1)$ *and* $y_j^1 > Y_j(G_0 \mid y_i^1, y_j^1)$ *cannot be part of an equilibrium.*

**Proof.** Suppose it can. Then, it is clear from Lemma 2 that at least one agent, say $i$, has $y_i^2 = 0$. Applying the same reasoning in Lemma A.3, we reach a contradiction. □

**Lemma A.5.** *In equilibrium, if* $y_i^1 > Y_i(G_0 \mid y_i^1, y_j^1)$, *then* $y_j^1 = 0$.

**Proof.** Directly follows from Lemmas A.3 and A.4 above. □

Now we turn to the proof of Theorem 2.

Suppose, in equilibrium, $y_i^1 \in (0, Y_i(G_0 \mid y_i^1, y_j^1)]$. From Lemma A.2, we must have either $y_j^1 = 0$ or $y_j^1 > Y_j(G_0 \mid y_i^1, y_j^1)$. If $y_j^1 = 0$, then $y_i^2 = Y_i(G_0 \mid y_i^1, y_j^1) - y_i^1$ and $y_j^2 = Y_j(G_0 \mid y_i^1, y_j^1)$ due to Lemma 2. That is, agents are at the Cournot–Nash outcome. However, from Assumption 2, agent $i$ could improve his utility by at least marginally shifting his total towards the second period. If, on the other hand, $y_j^1 > Y_j(G_0 \mid y_i^1, y_j^1)$, then Lemma A.5 implies $y_i^1 = 0$, a contradiction to the hypothesis. Thus, $y_i^1$ is either zero or greater than $Y_i(G_0 \mid y_i^1, y_j^1)$. Since Lemma A.5 implies that both $y_i^1$ and $y_j^1$ greater than the Cournot–Nash amounts cannot be part of an equilibrium. Furthermore, since, when $y_i^1 > Y_i(G_0 \mid y_i^1, y_j^1)$, Lemma A.5 implies $y_j^1 = 0$, agent $i$ becomes the Stackelberg leader and chooses $Y_i(G_i \mid r)$ as defined in the text. This leaves us the cases:

I: $\{(y_i^1 = y_j^1 = 0), (y_i^2 = Y_i(G_0 \mid 0, 0), y_j^2 = Y_j(G_0 \mid 0, 0)\}$

II: $\{(y_i^1 = Y_i(G_i \mid r), y_j^1 = 0), (y_i^2 = 0, y_j^2 = f^j(y_i^1 \mid 0))\}$.

**Proof of Corollary 1.** Note that for type II equilibrium to arise, one needs to have $y_i^1 = Y_i(G_i \mid r)$, and $y_i^1 > Y_i(G_0 \mid y_i^1, 0)$. Since from part (b) of Assumption 2, $Y_i(G_0 \mid y_i^1,\ 0)$ is decreasing in $y_i^1$, type II equilibrium requires that $Y_i(G_i \mid r) > Y_i(G_0 \mid 0, 0)$. Thus, if no agent's first period action satisfies this condition, then the equilibrium must be of type I. □

## Appendix B

This appendix contains the proof of Proposition 3. We show below that an equilibrium outcome in the T-period game must be in the set $S$. Let $Y_i^t$ be agent $i$'s accumulated amount *at the end of period t*. For convenience, also let $Y_i \equiv Y_i^T$.

**Lemma B.1.** *If* $(Y_i, Y_j)$ *is an equilibrium outcome, then* $y_i^T = \max\{0, f^i(Y_j) - Y_i^{T-1}\}$.

**Proof.** Using the similar arguments in the two-period game, the result follows. □

**Corollary B.1.** *If* $(Y_i,\ Y_j)$ *is an equilibrium outcome, then* $Y_i = f^i(Y_j)$, *hence is in* $S_1$.

**Proof.** This easily follows from Lemma B.1. □

**Lemma B.2.** *If* $(Y_i,\ Y_j)$ *is an equilibrium outcome, then* $Y_i = f^i(Y_j)$ *for at least one agent, hence is in* $S_2$.

**Proof.** Take an equilibrium outcome $(Y_i,\ Y_j)$, and suppose, on the contrary, $Y_i \neq f^i(Y_j)$ for both agents. Then, $Y_i > f^i(Y_j)$ from Corollary B.1. Furthermore, Lemma B.1 implies $y_i^T = 0$ for both agents. If $y_i^{T-1} > 0$, then agent $i$ could choose $\hat{y}_i^{T-1} = y_i^{T-1} - \varepsilon$, and given that agents use the same last period strategies in period $T$ as in the two-period case, could engender $(Y_i - \varepsilon, Y_j)$ as the continuation outcome. However, since by hypothesis, $Y_i > f^i(Y_j)$, he would, then be better off, contradicting the equilibrium assumption. Thus, we must have $y_i^{T-1} = 0$ for both agents. For continuation equilibria beginning in any period $t$, we assume continuity of the equilibrium outcome set $(\hat{Y}_i, \hat{Y}_j)$ in the state variables $(Y_i^{t-1},\ Y_j^{t-1})$.

Now suppose $y_i^{T-2} > 0$ and let agent $i$ choose $\hat{y}_i^{T-2} = y_i^{T-2} - \varepsilon$. This engenders a continuation outcome $(\hat{Y}_i, \hat{Y}_j)$ such that $\hat{Y}_i > f^i(\hat{Y}_j)$ for both agents. Using the same argument above for periods $T-1$ and $T$, we conclude that $\hat{y}_i^{T-1} = \hat{y}_i^T = 0$ for both agents. Thus, $\hat{Y}_i = Y_i - \varepsilon$ and $\hat{Y}_j = Y_j$. However, given $Y_i > f^i(Y_j)$, agent $i$ is strictly better off by choosing $\hat{y}_i^{T-2}$, contradicting the equilibrium assumption. Hence, $y_i^{T-2} = 0$. Using the exact arguments, one can show inductively that $y_i^t = 0$ for $t \in \{1, 2, \ldots, T\}$ for both agents. However, this contradicts $Y_i > f^i(Y_j)$. Hence, $Y_i = f^i(Y_j)$ for at least one agent. □

**Lemma B.3.** *If* $(Y_i,\ Y_j)$ *is an equilibrium outcome and* $Y_i \neq f^i(Y_j)$, *then* $Y_i \leqslant Y_i(G_i)$, *hence is in* $S_3$.

**Proof.** Take $(Y_i, Y_j)$ is an equilibrium outcome and $Y_i \neq f^i(Y_j)$. From Corollary B.1, this implies $Y_i > f^i(Y_j)$. Furthermore, from Lemma B.2, we have $Y_j = f^j(Y_i)$. Now, by way of contradiction, suppose $Y_i > Y_i(G_i)$. Since $Y_i > f^i(Y_j)$, Lemma B.1 implies $y_i^T = 0$. If $y_i^{T-1} > 0$, then given that agents use the same equilibrium strategies in period $T$ as in the two-period case, one can use the same arguments as in the two-period setting and reach a contradiction to $y_i^{T-1} > 0$ being part of an equilibrium path. Thus, we must have $y_i^{T-1} = 0$. Now let $t_0 \in \{1, 2, \ldots, T-2\}$ be the last period in which $y_i^{t_0} > 0$, i.e., with the $y_i^t = 0$ for $t \in \{t_0 + 1, \ldots, T\}$. Suppose agent $i$ reduces his period $t_0$ choice by $\varepsilon$, i.e., $\hat{y}_i^{t_0} = y_i^{t_0} - \varepsilon$. This engenders a continuation equilibrium such that $\hat{Y}_i > f^i(\hat{Y}_j)$ by continuity of the continuation equilibrium outcome set. If the continuation equilibrium is also such that $\hat{Y}_j > f^j(\hat{Y}_i)$, then a similar argument as in Lemma B.2 above implies that $\hat{y}_i^t = \hat{y}_j^t = 0$ for $t \in \{t_0 + 1, \ldots, T\}$, which in turn implies that $\hat{Y}_i = Y_i - \varepsilon$ and $\hat{Y}_j = Y_j$. However, this means agent $i$ is better off by choosing $\hat{y}_i^{t_0} = y_i^{t_0} - \varepsilon$ due to the strict quasiconcavity of $U^i(.)$ in $(Y_i, Y_j)$. If, on the other hand, the continuation equilibrium is also such that $\hat{Y}_j = f^j(\hat{Y}_i)$, then agent $i$ is again better off due to the strict quasiconcavity of $U^i(.)$ along $j$'s reaction function and the hypothesis $Y_i > Y_i(G_i)$. Finally, note that $\hat{Y}_j < f^j(\hat{Y}_i)$ cannot be part of a continuation equilibrium due to Lemma B.1 above. Thus, $y_i^t = 0$ for $t \in \{1, 2, \ldots, T\}$, implying that $Y_i = 0$. This, however, contradicts $Y_i > f^i(Y_j)$. □

**Lemma B.4.** *If $(Y_i, Y_j)$ is an equilibrium outcome and $Y_j = f^j(Y_i)$, then $Y_i = Y_i(G_i)$ or $Y_j = Y_j(G_i)$, hence is in $S_4$.*

**Proof.** (The proof closely follows the proof of the necessity of $S_4$ in the two-period game.) Suppose $(Y_i, Y_j)$ is an equilibrium outcome and $Y_j = f^j(Y_i)$. However, suppose, on the contrary, that $Y_i < Y_i(G_i)$ and $Y_j < Y_j(G_i)$. Since $Y_j = f^j(Y_i)$ and $Y_i = f^i(Y_j)$ from Corollary B.1, Lemma A.1 implies $Y_i = Y_i(G_0)$, which in turn implies $Y_i(G_0) < Y_i(G_i)$. Now consider period $t = T - 1$. Since, by definition, $Y_j^{T-1} \leqslant Y_j$ and $Y_j^{T-1} < Y_j(G_i)$. We shall argue next that in period $T - 1$, agent $i$ could engender $(Y_i(G_i), Y_j(G_i))$ as an equilibrium outcome, and be better off.

To see this, first note that given $(Y_i^{T-1}, Y_j^{T-1})$ the equilibrium strategies in Lemma 1 continues to be the unique continuation equilibrium in the $T \geqslant 2$ games. Now, given $Y_j^{T-1}$, let agent $i$ choose $\hat{y}_i^{T-1} = Y_i(G_i) - Y_i^{T-1}$ so that $\hat{Y}_i^{T-1} = \hat{y}_i^{T-1} + Y_i^{T-1} = Y_i(G_i)$. Then, since $Y_i(G_0) < Y_i(G_i)$ and $Y_j^{T-1} < Y_j(G_i) = f^j(\hat{Y}_i^{T-1})$, the last period strategies dictate that $\hat{y}_i^T = 0$, and $\hat{y}_j^T = Y_j(G_i) - Y_j^{T-1}$ so that $\hat{Y}_j = Y_j(G_i)$, completing the proof. □

**Lemma B.5.** *If $(Y_i, Y_j)$ is an equilibrium outcome and $Y_i = f^i(Y_j)$, then $U^i(Y_i, Y_j) \geqslant \max_{\tilde{Y}_i} U^i(\tilde{Y}_i, f^j(\tilde{Y}_i))$ s.t. $f^j(\tilde{Y}_i) \geqslant Y_j$ and $\tilde{Y}_i \geqslant f^i(f^j(\tilde{Y}_i))$, hence is in $S_5$.*

**Proof.** (The proof closely follows the proof of the necessity of $S_5$ in the two-period case.) Suppose $(Y_i,\ Y_j)$ is an equilibrium outcome and $Y_i = f^i(Y_j)$. Note that if the constraint set is empty, then the assertion in Lemma B.5 holds trivially. Thus, we assume the set is nonempty. Let $\tilde{Y}_i^*$ denote the solution to the maximization problem. In Observation 1 below, we show that either $\tilde{Y}_i^* = Y_i(G_0)$ or $Y_i \leqslant \tilde{Y}_i^*$. Suppose $\tilde{Y}_i^* = Y_i(G_0)$. Then, $Y_j \leqslant Y_j(G_0)$ from the first constraint of the maximization. Furthermore, since $Y_i = f^i(Y_j)$, Lemma A.1 implies that $Y_j \leqslant f^j(Y_i)$. Given that $(Y_i,\ Y_j)$ is an equilibrium outcome, we also have $Y_j = f^j(Y_i)$. This means $Y_j = f^j(Y_i)$. Since $Y_i = f^i(Y_j)$ by hypothesis, this also means $(Y_i,\ Y_j) = (Y_i(G_0),\ Y_j(G_0))$. Together with $\tilde{Y}_i^* = Y_i(G_0)$, the inequality in Lemma B.5 then holds with equality.

Next suppose $Y_i \leqslant \tilde{Y}_i^*$, but, on the contrary, that $U^i(Y_i, Y_j) < U^i(\tilde{Y}_i^*, f^j(\tilde{Y}_i^*))$. We demonstrate now that agent $i$ could engender $(\tilde{Y}_i^*, f^j(\tilde{Y}_i^*))$ and be strictly better off than the outcome $(Y_i,\ Y_j)$. Consider period $T-1$, and note that $Y_i^{T-1} \leqslant Y_i \leqslant \tilde{Y}_i^*$. Given $Y_j^{T-1}$, let agent $i$ choose $\hat{y}_i^{T-1} = \tilde{Y}_i^* - Y_i^{T-1}$. Since, by definition, $\tilde{Y}_i^* \geqslant f^i(f^j(\tilde{Y}_i^*))$, Lemma A.1 implies $\tilde{Y}_i^* \geqslant Y_i(G_0)$. Furthermore, from the constraint set, we also know $Y_j^{T-1} \leqslant Y_j \leqslant f^j(\tilde{Y}_i^*)$. The last period strategies then dictate that $\hat{y}_i^T = 0$ and $\hat{y}_j^T = f^j(\tilde{Y}_i^*) - Y_j^{T-1}$, yielding $(\tilde{Y}_i^*, f^j(\tilde{Y}_i^*))$ as an equilibrium. However, since, by hypothesis, $U^i(Y_i, Y_j) < U^i(\tilde{Y}_i^*, f^j(\tilde{Y}_i^*))$, agent $i$ has a strict incentive to deviate in $(Y_i,\ Y_j)$, contradicting the equilibrium assumption. □

**Observation B.1.** *Let $(Y_i, Y_j)$ is an equilibrium outcome and $Y_i = f^i(Y_j)$. Furthermore, let* $\tilde{Y}_i^* = \arg\max_{\tilde{Y}_i} U^i(\tilde{Y}_i, f^j(\tilde{Y}_i))$ s.t. $f^j(\tilde{Y}_i) \geqslant Y_j$ *and* $\tilde{Y}_i \geqslant f^i(f^j(\tilde{Y}_i))$. *Then, either* $\tilde{Y}_i^* = Y_i(G_0)$ *or* $Y_i \leqslant \tilde{Y}_i^*$.

**Proof.** We consider three cases depending on whether reaction functions are increasing or decreasing.

- $f^i(.)$ is increasing: Then, using the constraints successively, we have $\tilde{Y}_i^* \geqslant f^i(f^j(\tilde{Y}_i^*)) \geqslant f^i(Y_j) = Y_i$, which means $Y_i \leqslant \tilde{Y}_i^*$.
- $f^i(.)$ is decreasing: In this case, we need to consider the slope of $f^j(.)$ as well.

(1) $f^j(.)$ *is increasing*: Using the first constraint, we have $f^i(f^j(\tilde{Y}_i^*)) \leqslant f^i(Y_j)$. Furthermore, given that $(Y_i,\ Y_j)$ is an equilibrium outcome, Corollary B.1 above implies $Y_j \geqslant f^j(Y_i)$. Since $f^j(.)$ is decreasing, this further implies $f^i(Y_j) \leqslant f^i(f^j(Y_i))$. Overall, we then have $f^i(f^j(\tilde{Y}_i^*)) \leqslant f^i(Y_j) \leqslant f^i(f^j(Y_i))$. Since $f^i(.)$ is decreasing, this implies) $f^j(\tilde{Y}_i^*) \geqslant f^j(Y_i)$, which further reveals $Y_i \leqslant \tilde{Y}_i^*$, since $f^j(.)$ is increasing.

(2) $f^j(.)$ *is decreasing*: When both reaction functions are decreasing, the constraint set contains a single point at which $\tilde{Y}_i = Y_i(G_0)$, and thus $\tilde{Y}_i^* = Y_i(G_0)$. To see this, first note that since $Y_i = f^i(Y_j)$ and $Y_j \geqslant f^j(Y_i)$, Lemma A.1 implies that $Y_j \geqslant Y_j(G_0)$. Furthermore, since $f^i(.)$ is decreasing, this implies $Y_i \leqslant Y_i(G_0)$. Now, using the first constraint and $Y_j \geqslant f^j(Y_i)$, we have $f^j(\tilde{Y}_i) \geqslant f^j(Y_i)$, which implies that $\tilde{Y}_i \leqslant Y_i$, where

$\tilde{Y}_i$ is an arbitrary point in the set. Thus, we have $\tilde{Y}_i \leqslant Y_i(G_0)$. Moreover, together with Lemma A.1, the second constraint implies that $\tilde{Y}_i \geqslant Y_i(G_0)$. Hence, the only point in the constraint set is $\tilde{Y}_i = Y_i(G_0)$, which also implies $\tilde{Y}_i^* = Y_i(G_0)$.

Overall then, we either have $\tilde{Y}_i^* = Y_i(G_0)$ or $Y_i \leqslant \tilde{Y}_i^*$. □

## References


[1] A. Admati, M. Perry, Joint projects without commitment, Rev. Econ. Stud. 58 (1991) 259–276.
[2] K.H. Baik, J.F. Shogren, Strategic behavior in contests: comment, Amer. Econ. Rev. 82 (1992) 359–362.
[3] T. Bergstrom, L. Blume, H. Varian, On the private provision of public goods, J. Public Econ. 29 (1986) 25–49.
[4] A. Dixit, A model of duopoly suggesting a theory of entry barriers, Bell J. Econ. 10 (1979) 20–32.
[5] A. Dixit, Strategic behavior in contests, Amer. Econ. Rev. 77 (1987) 891–898.
[6] D. Fudenberg, J. Tirole, Game Theory, MIT Press, Cambridge, MA, 1991.
[7] D. Gale, Monotone games with positive spillovers, Games Econ. Behav. 37 (2001) 295–320.
[8] J. Hamilton, S. Slutsky, Endogenous timing in duopoly games: Stackelberg or Cournot equilibria, Games Econ. Behav. 2 (1990) 29–46.
[9] J. Hamilton, S. Slutsky, Endogenizing the order of moves in matrix games, Theory Dec. 34 (1993) 47–62.
[10] J. Henkel, The 1.5th mover advantage, RAND J. Econ. 33 (2002) 156–170.
[11] W. Leininger, More efficient rent-seeking: a Munchhausen solution, Public Choice 75 (1993) 43–62.
[12] B. Linster, Stackelberg rent-seeking, Public Choice 77 (1993) 307–321.
[13] B. Lockwood, J. Thomas, Gradualism and irreversibility, Rev. Econ. Stud. 69 (2002) 339–357.
[14] G. Maggi, Endogenous leadership in a new market, RAND J. Econ. 27 (1996) 641–659.
[15] L. Marx, S. Matthews, Dynamic voluntary contribution to a public project, Rev. Econ. Stud. 67 (2000) 327–358.
[16] D. Pal, Cournot duopoly with two production periods and cost differentials, J. Econ. Theory 55 (1991) 441–448.
[17] R. Romano, H. Yildirim, Why charities announce donations: a positive perspective, J. Public Econ. 81 (2001) 423–447.
[18] G. Saloner, Cournot duopoly with two production periods, J. Econ. Theory 42 (1987) 183–187.
[19] N. Singh, X. Vives, Price and quantity competition in a differentiated duopoly, RAND J. Econ. 15 (1984) 546–554.
[20] J. Tirole, The Theory of Industrial Organization, MIT Press, Cambridge, MA, 1988.
[21] G. Tullock, Efficient rent-seeking, in: J. Buchanan, R. Tollison, G. Tullock, (Eds.), Toward a Theory of the Rent-seeking Society, Texas A&M University Press, College Station, TX, 1980.
[22] E. Van Damme, S. Hurkens, Commitment robust equilibria and endogenous timing, Games Econ. Behav. 15 (1996) 290–311.
[23] H. Varian, Sequential contributions to public goods, J. Public Econ. 53 (1994) 165–186.