Rational Energy Balance Method to Nonlinear Oscillators with Cubic Term

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In this paper, a novel approach is proposed for solving the nonlinear problems based on the collocation and energy balance methods (EBMs). Rational approximation is employed as an initial guess and then it is combined with EBM and collocation method for solving nonlinear oscillators with cubic term. Obtained frequency amplitude relationship is compared with exact numerical solution and subsequently, a very excellent accuracy will be revealed. According to the numerical comparisons, this method provides high accuracy with 0.03% relative error for Duffing equation with strong nonlinearity.

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in the second-order of approximation. Furthermore, achieved results are compared with other types of modified EBMs and the second-order of harmonic balance method. It is demonstrated that the new proposed method has the highest accuracy in comparison with different approaches such as modified EBMs and the second-order of harmonic balance method.

Keywords: Duffing oscillators; rational approximation; energy balance method.

1. Introduction

A number of techniques have been already suggested for solving nonlinear equations. Homotopy analysis method (HAM) [18], variational iteration method [17] (VIM), Adomian decomposition method (ADM) [3] and perturbation methods [22] are the prominent and the potent approaches that have been employed for solving several nonlinear problems [1, 2, 4, 7, 19, 24, 32, 33]. In addition, Hamiltonian approach (HA) [16], energy balance method (EBM) [12], frequency amplitude formulation (FAF) [14], max-min approach (MMA) [15], variational approach (VA) [13] can be mentioned as novel ways that have been utilized for evaluating lots of sophisticated nonlinear equations [5, 8, 11, 25–31, 34]. HA, EBM, FAF, MMA and VA are so straightforward in comparison with HAM, VIM, ADM and perturbation methods and they have been modified by several researchers for extracting analytical solutions of the nonlinear problems. For instance, Younesian et al. [31] have amended EBM using Petrov–Galerkin method. Moreover, HA has been modified by Yildirim et al. [27]. In addition, Durmaz and Kaya [10] have modified EBM using Petrov–Galerkin approach and collocation method [9]. In this paper, EBM is modified using rational approximation. Rational approximation have been originally proposed by Mickens [20] for analyzing of nonlinear problems and then, this novel idea has been employed by several researchers for solving diverse forms of nonlinear equations [6, 21]. In the present study, the aforementioned idea is used and expanded with combining EBM. It is proved, this new modification of EBM has the highest accuracy in comparison with other kinds of its modifications. Furthermore, achieved result of developed method is numerically compared with second-order of harmonic balance method [23].

2. Classical Energy Balance Method

In this part, EBM is succinctly illustrated. First, we consider Eq. (2.1) as a generalized nonlinear oscillator and next according to the He’s procedure, variational and Hamiltonian functions are constructed in Eqs. (2.3) and (2.4), respectively.

$$\ddot{x} + f(x) = 0$$  \hspace{1cm} (2.1)

with the following initial conditions:

$$x(0) = A, \quad \dot{x}(0) = 0$$  \hspace{1cm} (2.2)
variational and Hamiltonian functions can be constructed as:

\begin{equation}
J(x) = \int_{0}^{T} \left[ -\frac{1}{2} \dot{x}^2 + F(x) \right] dt,
\end{equation}

where \( F(x) = \int f(x) dx \) and \( T = \frac{2\pi}{\omega} \):

\begin{equation}
H(x) = \frac{1}{2} \dot{x}^2 + F(x)
\end{equation}

and based on He’s method, the residual function is defined as

\begin{equation}
R(t) = H(x) - H(x(0)).
\end{equation}

In the classical procedure of EBM, \( A \cos \omega t \) is assumed as an initial guess for Eq. (2.1) and it is inserted in Eq. (2.5). In addition, \( \omega t = \frac{\pi}{4} \) is used as location point and subsequently the following algebraic equation is obtained:

\begin{equation}
R(t, \omega) = (\omega t - \frac{\pi}{4}) \frac{1}{2} A^2 \omega^2 \sin^2 \omega t + F(A \cos \omega t) - F(A) = 0.
\end{equation}

3. Rational Energy Balance Method

In this section, first, rational approximation is briefly introduced and then this approximation is combined with classical EBM. As illustrated in Sec. 1 of the paper, the following function is exerted as an approximation for the analyzing of the nonlinear problems by Mickens [20].

\begin{equation}
x_N^M = \sum_{n=0}^{N} \frac{A_n \cos(2n+1)\theta + B_n \sin(2n+1)\theta}{1 + \sum_{m=1}^{M} C_m \cos(2m)\theta + D_m \sin(2m)\theta},
\end{equation}

in which

\begin{equation}
\theta = \omega t,
\end{equation}

where \( \omega \) is the frequency of the oscillation, \( \{A_n, B_n, C_m, D_m\} \) are a set of constants and \( M \) and \( N \) are integers [20]. As defined by Mickens, the first approximation of Eq. (3.1) reaches the following function:

\begin{equation}
x_0^0(t) = A_0 \cos \theta.
\end{equation}

And the second approximations can be defined as [20]:

\begin{equation}
x_0^1(t) = A_0 \cos \theta + A_1 \cos 3\theta,
\end{equation}

\begin{equation}
x_1^1(t) = \frac{A_0 \cos \theta}{1 + C_1 \cos 2\theta}
\end{equation}

Eq. (3.4) has been already used by several researchers for modifying diverse kinds of analytical techniques [9, 10, 27]. In this study, Eq. (3.5) is exerted as a second approximation and it is already combined with EBM. By substituting Eq. (3.5)
into Eq. (2.5), Eq. (3.6) is obtained as an original residual function for Eq. (2.1).

\[
R(t) = \frac{1}{2} \left( \frac{A_0 \omega \sin(\omega t)(1 + C_1 \cos(2\omega t) - 2A_0C_1\omega \cos(\omega t) \sin(2\omega t))}{(1 + C_1 \cos(2\omega t))^2} \right)^2 \\
+ F \left( \frac{A_0 \cos \omega t}{1 + C_1 \cos 2\omega t} \right) - F(A).
\] (3.6)

For finding \( A_0, C_1, \omega \), two location points are employed based on the collocation method.

According to the collocation method, Eqs. (3.7)–(3.8) are defined as residual functions that must be solved.

\[
R(t, \omega) = \left( \omega t \to \frac{\pi}{6} \right) \frac{1}{2} \left( \frac{A_0 \omega \sin(\omega t)(1 + C_1 \cos(2\omega t) - 2A_0C_1\omega \cos(\omega t) \sin(2\omega t))}{(1 + C_1 \cos(2\omega t))^2} \right)^2 \\
+ F \left( \frac{A_0 \cos \omega t}{1 + C_1 \cos 2\omega t} \right) - F(A) = 0.
\] (3.7)

And

\[
R(t, \omega) = \left( \omega t \to \frac{\pi}{3} \right) \frac{1}{2} \left( \frac{A_0 \omega \sin(\omega t)(1 + C_1 \cos(2\omega t) - 2A_0C_1\omega \cos(\omega t) \sin(2\omega t))}{(1 + C_1 \cos(2\omega t))^2} \right)^2 \\
+ F \left( \frac{A_0 \cos \omega t}{1 + C_1 \cos 2\omega t} \right) - F(A) = 0.
\] (3.8)

Furthermore, based on the initial conditions, the third equation for the finding the unknown parameters of Eq. (3.6) is:

\[
\frac{A_0}{1 + C_1} = A.
\] (3.9)

By simultaneously solving Eqs. (3.7)–(3.9), the unknown parameters of Eq. (3.6) is easily obtained.

4. Applications

In this part, rational EBM is utilized for extracting analytical solution of the non-linear oscillator with cubic terms, namely the duffing equation:

\[
\ddot{x} + x + \varepsilon x^3 = 0.
\] (4.1)

Based on Eq. (2.3), the variational function can be defined as:

\[
J(x) = \int_0^T \left( -\frac{1}{2} \dot{x}^2 + x^2 + \frac{1}{4} \varepsilon x^4 \right) dt.
\] (4.2)
And the Hamiltonian can be shown by:

\[ H(x) = \frac{1}{2} \dot{x}^2 + x^2 + \frac{1}{4} \varepsilon x^4 = \frac{1}{2} A^2 + \frac{1}{4} \varepsilon A^4. \]  

(4.3)

According to the He’s classical EBM [12], the first approximate frequency-amplitude relationship of Eq. (4.1) is equal to

\[ \omega = \sqrt{1 + \frac{3}{4} \varepsilon A^2}. \]  

(4.4)

By employing Eq. (3.5) as a second approximation and based on Eqs. (3.7) and (3.8), a coupled algebraic equation is obtained as:

\[
\begin{align*}
2\omega^2(A_0(2 + C_1) - 6A_0C_1)^2 + 6A_0^2(2 + C_1)^2 \\
+ 9\varepsilon A_0^3 - (2A^2 + \varepsilon A^4)(2 + C_1)^4 &= 0, \\
6\omega^2(A_0(2 - C_1) - 2A_0C_1)^2 + 2A_0^2(2 - C_1)^2 \\
+ \varepsilon A_0^3 - (2A^2 + \varepsilon A^4)(2 - C_1)^4 &= 0.
\end{align*}
\]  

(4.5)

(4.6)

By simultaneously solving Eqs. (3.9), (4.5)–(4.6), natural frequencies and other unknown parameters can be readily achieved.

5. Discussion and Numerical Results

For evaluating the correctness of the modified version of EBM by rational approximation, it is numerically compared with exact solutions. For a large range of amplitude, obtained natural frequencies of Eq. (4.1) by means of REBM are tabulated in Table 1. In accordance with this table, it can be concluded, REBM gives very

![Table 1. Comparison of obtained results of different kinds of analytical approaches for Eq. (4.1).](image)

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outstanding results in the second approximation for the Duffing equation. Moreover, obtained natural frequencies of diverse kinds of EBMs and harmonic balance approach have been tabulated in the mentioned table. As seen from Table 1, REBM has the highest accuracy among all of the represented methods in Table 1. According to Table 1, REBM reaches to very exceptional accuracy for very strong nonlinearity in comparison with the other methods. For example, when $\varepsilon A^2 = 5000$, relative errors of SEBM – C, SEBM – GP, SHBM and REBM are 0.77%, 0.59%, 0.06783% and 0.0301%, correspondingly. It is apparent from the abovementioned conclusion that REBM are so potent in comparison with other forms of modified EBM. Besides, it can be concluded that REBM is more effective than SHBM for solving the Duffing equation.

6. Conclusion
In this study, a novel technique was explained based on the collocation method, EBM and rational approximation. Initially, classical EBM was illustrated and then rational approximation was briefly explicated. In continuation, by combination of the EBM and the rational approximation, a novel modified version of the EBM was proposed and subsequently it was applied to the well-known Duffing equation. Furthermore, to examine accuracy of the proposed method, natural frequencies were tabulated and compared with exact numerical solutions. From the tabulated frequencies, it can be concluded that the modified method is so potent for solving nonlinear problems. In addition, the modified method was compared with diverse types of EBM and second-order of the harmonic balance method. According to Table 1 of this paper, accuracy of these techniques can be sorted as:

SEBM – C < SEBM – GP < SHBM < REBM.

Future research can examine accuracy of REBM to the other kinds of nonlinear oscillators. Furthermore, it can be a very interesting idea to find a way for employing rational elliptic function [35] and combining it with classical EBM for solving nonlinear problems.

References


