Solid angle subtended by a rectangular right pyramid at its apex by HCR

Harish Chandra Rajpoot, HCR
Solid angle subtended by a rectangular right pyramid (solid/hollow) at its apex

(Application of HCR’s Theory of Polygon)

Mr Harish Chandra Rajpoot

Master of Technology, IIT Delhi

Introduction: Here we are to derive the formula for finding out the solid angle subtended by a rectangular right pyramid at its apex by using formula of solid angle subtended by a rectangular plane at any point lying on the perpendicular passing through its centre which has already been derived in HCR’s Theory of Polygon. The solid angle subtended by a rectangular right pyramid will be derived in terms of apex angles $\alpha$ & $\beta$ (i.e. angles between two pairs of consecutive lateral edges meeting at the apex of rectangular right pyramid).

Derivation: Let there be a right pyramid (solid or hollow) with apex point ‘P’ & base as a rectangle ABCD such that the angles between two pairs of consecutive lateral edges PA & PB and PB & PC are $\alpha$ and $\beta$ respectively (as shown in the figure).

Now, drop the perpendiculars PO & PM on the rectangular base ABCD & side AB respectively & join the diagonals AC & BD of rectangular base ABCD (as shown by dotted lines in fig-1). Let $a$ be the length of each of four equal lateral edges PA, PB, PC & PD of right pyramid.

In right $\Delta PMA$, (see triangular face APB of right pyramid in fig-1)

$$\sin \angle APM = \frac{AM}{AP} \Rightarrow \sin \frac{\alpha}{2} = \frac{AM}{a}$$

$$AM = a \sin \frac{\alpha}{2}$$

$\therefore AB = CD = 2AM = 2a \sin \frac{\alpha}{2}$

Similarly, in isosceles $\Delta PBC$, it can be proved by dropping a perpendicular from apex P to the side BC,

$$BC = AD = 2a \sin \frac{\beta}{2}$$

Using Pythagorean theorem in right $\Delta ABC$,

$$AC^2 = AB^2 + BC^2 = \left(2a \sin \frac{\alpha}{2}\right)^2 + \left(2a \sin \frac{\beta}{2}\right)^2 = 4a^2 \left(\sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2}\right)$$

$$AC = \sqrt{4a^2 \left(\sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2}\right)} = 2a \sqrt{\sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2}}$$

$$AO = \frac{AC}{2} = a \sqrt{\frac{\sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2}}{2}} = \frac{a}{\sqrt{2}} $$

Using Pythagorean theorem in right $\Delta POA$ (see above fig-1),

$$PA^2 = AO^2 + PO^2 \Rightarrow PO = \sqrt{PA^2 - AO^2}$$
\[ PO = \sqrt{a^2 - \left( a \sqrt{\sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2}} \right)^2} \]

\[ = a \sqrt{1 - \sin^2 \frac{\alpha}{2} - \sin^2 \frac{\beta}{2}} \]

\[ = a \sqrt{\cos^2 \frac{\alpha}{2} - \sin^2 \frac{\beta}{2}} \]

Now, the solid angle (\(\omega_{\text{pyramid}}\)) subtended by the rectangular right pyramid at its apex \(P\) will be equal to the solid angle (\(\omega_{\text{rectangle}}\)) subtended by rectangle \(ABCD\) of length and width \(l\) & \(b\), at the apex \(P\) lying a normal height \(h\) from the centre ‘O’ which is given by the general formula of HCR’s Theory of Polygon as follows

\[ \omega_{\text{rectangle}} = 4\sin^{-1} \left( \frac{lb}{\sqrt{(l^2 + 4h^2)(b^2 + 4h^2)}} \right) \]

Now, setting the value of normal height \(h = PO\), length \(l = AB\) & width \(b = BC\) in the above general formula of solid angle, we get the solid angle subtended by the rectangular right pyramid at its apex

\[ \omega_{\text{pyramid}} = 4\sin^{-1} \left( \frac{(AB)(BC)}{\sqrt{(AB)^2 + 4(PO)^2}(BC)^2 + 4(PO)^2}} \right) \]

\[ = 4\sin^{-1} \left( \frac{2a \sin \frac{\alpha}{2}}{\sqrt{\left(2a \sin \frac{\alpha}{2}\right)^2 + 4 \left(a \sqrt{\cos^2 \frac{\alpha}{2} - \sin^2 \frac{\beta}{2}}\right)^2}} \right) \]

\[ = 4\sin^{-1} \left( \frac{4a^2 \sin \frac{\alpha}{2} \sin \frac{\beta}{2}}{\sqrt{4(a^2)^2 \left(\sin^2 \frac{\alpha}{2} + \cos^2 \frac{\alpha}{2} - \sin^2 \frac{\beta}{2}\right) \left(\sin^2 \frac{\beta}{2} + \cos^2 \frac{\alpha}{2} - \sin^2 \frac{\beta}{2}\right)}} \right) \]

\[ = 4\sin^{-1} \left( \frac{\sin \frac{\alpha}{2} \sin \frac{\beta}{2}}{\sqrt{1 - \sin^2 \frac{\beta}{2}} \left(\cos^2 \frac{\alpha}{2}\right)} \right) \]

\[ = 4\sin^{-1} \left( \frac{\sin \frac{\alpha}{2} \sin \frac{\beta}{2}}{\cos \frac{\alpha}{2} \cos \frac{\beta}{2}} \right) \]

\[ = 4\sin^{-1} \left( \tan \frac{\alpha}{2} \tan \frac{\beta}{2} \right) \]
Hence, the solid angle \( \omega \) subtended at the apex by any right pyramid with a rectangular base & the apex angles \( \alpha \) & \( \beta \) (i.e. angles between two pairs of consecutive lateral edges meeting at apex), is given by the following formula

\[
\omega = 4\sin^{-1}\left(\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right)
\]

Where, \( 0 < \alpha + \beta < \pi \)

**Important deduction:** The solid angle subtended at the apex by a right pyramid with a square base & the apex angle \( \alpha \) is obtained by substituting \( \beta = \alpha \) in above general equation of solid angle, we get

\[
\omega = 4\sin^{-1}\left(\tan\frac{\alpha}{2}\tan\frac{\alpha}{2}\right)
\]

\[
\omega = 4\sin^{-1}\left(\tan^2\frac{\alpha}{2}\right)
\]

Where, \( 0 < \alpha < \frac{\pi}{2} \)

The above value of solid angle subtended at apex by a square right pyramid can also be proved by substituting \( n = \) number of sides of square base = 4 in general formula of solid angle subtended at apex by a regular \( n \)-gonal right pyramid (which has been derived in the paper ‘solid angle subtended by a regular \( n \)-gonal right pyramid at its apex’ by the author), given as follows

\[
\omega = 2\pi - 2n\sin^{-1}\left(\cos\frac{\pi}{n}\sqrt{\tan^2\frac{\pi}{n} - \tan^2\frac{\alpha}{2}}\right)
\]

\[
= 2\pi - 2(4)\sin^{-1}\left(\cos\frac{\pi}{4}\sqrt{\tan^2\frac{\pi}{4} - \tan^2\frac{\alpha}{2}}\right) \quad \text{(setting } n = 4 \text{ for square base)}
\]

\[
= 2\pi - 8\sin^{-1}\left(\frac{1}{\sqrt{2}}\sqrt{1 - \tan^2\frac{\alpha}{2}}\right)
\]

\[
= 2\pi - 4\sin^{-1}\left(\frac{1}{\sqrt{2}}\sqrt{1 - \tan^2\frac{\alpha}{2}} \sqrt{1 - \left(\frac{1}{\sqrt{2}}\sqrt{1 - \tan^2\frac{\alpha}{2}}\right)^2}\right) \quad \text{(} 2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})\text{)}
\]

\[
= 2\pi - 4\sin^{-1}\left(\frac{1}{\sqrt{2}}\sqrt{1 - \tan^2\frac{\alpha}{2}} \frac{1}{\sqrt{2}}\sqrt{1 + \tan^2\frac{\alpha}{2}}\right)
\]

\[
= 2\pi - 4\sin^{-1}\left(\frac{1}{\sqrt{2}}\sqrt{1 - \tan^4\frac{\alpha}{2}}\right)
\]

\[
= 2\pi - 4\cos^{-1}\left(\sqrt{1 - \left(\frac{1}{\sqrt{2}}\sqrt{1 - \tan^4\frac{\alpha}{2}}\right)^2}\right)
\]

\[
= 2\pi - 4\cos^{-1}\left(\frac{1}{\sqrt{2}}\sqrt{1 + \tan^4\frac{\alpha}{2}}\right)
\]

\[
= 2\pi - 4\cos^{-1}\left(\tan^2\frac{\alpha}{2}\right)
\]
\[
= 4 \left( \frac{\pi}{2} - \cos^{-1}\left( \tan^2 \frac{\alpha}{2} \right) \right)
\]

\[
= 4 \left( \sin^{-1}\left( \tan^2 \frac{\alpha}{2} \right) \right)
\]

\[
= 4 \sin^{-1}\left( \tan^2 \frac{\alpha}{2} \right) \quad \text{(Since,} \quad \frac{\pi}{2} - \cos^{-1}x = \sin^{-1}x \text{)}
\]

\[
= 4 \sin^{-1}\left( \tan^2 \frac{\alpha}{2} \right)
\]

\text{Proved.}

\text{Note: Above articles had been derived & illustrated by Mr H.C. Rajpoot (M Tech, Production Engineering)}

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\text{Email: hcrajpoot.iitd@gmail.com}

\text{Author’s Home Page: https://notionpress.com/author/HarishChandraRajpoot}