Fall August 25, 2016

Minimum distance between any two arbitrary points on a sphere or globe given the latitudes & longitudes using vectors

Harish Chandra Rajpoot, HCR

$Minimum\ distance\ between\ any\ two\ points\ on\ a\ sphere\ or\ globe\ for\ given\ latitudes\ \&\ longitudes\ using\ vectors$

Harish Chandra Rajpoot *Aug, 2016*

M.M.M. University of Technology, Gorakhpur-273010 (UP), India

We know that the length of minor great circle arc joining any two arbitrary points on a sphere of finite radius is the minimum distance between those points. Here we are interested in finding out the minimum distance or great circle distance between any two arbitrary points on a spherical surface of finite radius (like globe) for the given values of latitudes & longitudes using vectors.

Let there any two arbitrary points $\boldsymbol{A(\phi_1, \lambda_1)}$ & $\boldsymbol{B(\phi_2, \lambda_2)}$ on the surface of sphere of radius $\boldsymbol{R}$ & centre at the point O. The **angles of latitude** $\boldsymbol{\phi_1}$ & $\boldsymbol{\phi_2}$ are measured from the equator plane (i.e. X-Y plane) & the **angles of longitude** $\boldsymbol{\lambda_1}$ & $\boldsymbol{\lambda_2}$ are measured from X-Z plane in the same (anticlockwise) direction (As shown in the figure 1). Here, we are to find out the length of **great circle arc AB** joining the given points A & B. Changing the spherical coordinates of the given point A into Cartesian coordinates as follows

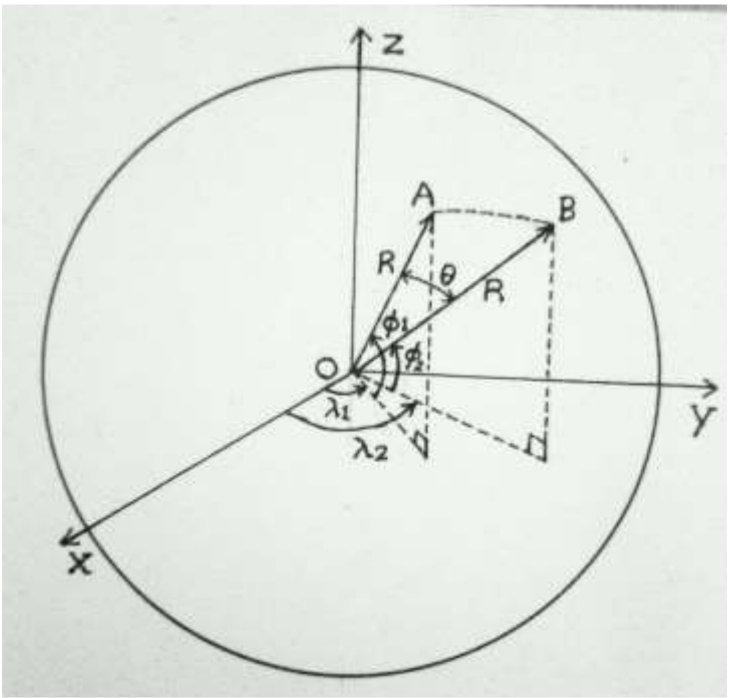

Figure 1: The two given points $A(\phi_1, \lambda_1)$ & $B(\phi_2, \lambda_2)$ lie on a spherical surface of finite radius R. The vectors $\overrightarrow{OA}$ & $\overrightarrow{OB}$ are making an angle θ

$$x = Rcos\phi_1 cos\lambda_1$$
$$y = Rcos\phi_1 sin\lambda_1$$
$$z = Rsin\phi_1$$

$$\therefore\ \boldsymbol{A(Rcos\phi_1 cos\lambda_1, Rcos\phi_1 sin\lambda_1, Rsin\phi_1)}$$

Similarly, we get the coordinates of point B

$$\boldsymbol{B(Rcos\phi_2 cos\lambda_2, Rcos\phi_2 sin\lambda_2, Rsin\phi_2)}$$

Now, join the points A & B to the centre O of the sphere to get vectors $\overrightarrow{OA}$ & $\overrightarrow{OB}$ given as follows

$$\overrightarrow{\boldsymbol{OA}} = (Rcos\phi_1 cos\lambda_1)\hat{\imath} + (Rcos\phi_1 sin\lambda_1)\hat{\jmath} + (Rsin\phi_1)\hat{k}$$

$$\overrightarrow{\boldsymbol{OB}} = (Rcos\phi_2 cos\lambda_2)\hat{\imath} + (Rcos\phi_2 sin\lambda_2)\hat{\jmath} + (Rsin\phi_2)\hat{k}$$

Now, using dot product of vectors $\overrightarrow{OA}$ & $\overrightarrow{OB}$, the angle θ between them is given as follows

$$\left(\overrightarrow{OA}\right) \cdot \left(\overrightarrow{OB}\right) = \left|\overrightarrow{OA}\right| \cdot \left|\overrightarrow{OB}\right| cos\theta$$

$$cos\theta = \frac{\left(\overrightarrow{OA}\right) \cdot \left(\overrightarrow{OB}\right)}{\left|\overrightarrow{OA}\right| \cdot \left|\overrightarrow{OB}\right|}$$

$$= \frac{\left((Rcos\phi_1 cos\lambda_1)\hat{\imath} + (Rcos\phi_1 sin\lambda_1)\hat{\jmath} + (Rsin\phi_1)\hat{k}\right) \cdot \left((Rcos\phi_2 cos\lambda_2)\hat{\imath} + (Rcos\phi_2 sin\lambda_2)\hat{\jmath} + (Rsin\phi_2)\hat{k}\right)}{R \cdot R}$$

$$= \frac{R^2(cos\phi_1 cos\phi_2 cos\lambda_1 cos\lambda_2 + cos\phi_1 cos\phi_2 sin\lambda_1 sin\lambda_2 + sin\phi_1 sin\phi_2)}{R \cdot R}$$

$$= cos\phi_1 cos\phi_2 (cos\lambda_1 cos\lambda_2 + sin\lambda_1 sin\lambda_2) + sin\phi_1 sin\phi_2$$

$$= sin\phi_1 sin\phi_2 + cos\phi_1 cos\phi_2 cos(\lambda_2 - \lambda_1)$$

$$\Rightarrow\ \theta = \cos^{-1}\left(sin\phi_1 sin\phi_2 + cos\phi_1 cos\phi_2 cos(\lambda_2 - \lambda_1)\right)$$

The great circle arcs AB is given as

$$\text{arc AB} = R\theta = R\cos^{-1}\left(sin\phi_1 sin\phi_2 + cos\phi_1 cos\phi_2 cos(\lambda_2 - \lambda_1)\right)$$

Hence, the **minimum distance between the points $\boldsymbol{A(\phi_1\,,\lambda_1)}$ & $\boldsymbol{B(\phi_2\,,\lambda_2)}$**

$$\textbf{Great circle arc AB} = \boldsymbol{R\cos^{-1}\left(sin\phi_1 sin\phi_2 + cos\phi_1 cos\phi_2 cos(\lambda_2 - \lambda_1)\right)}$$

It is obvious that the great circle distance between the points depends on the **difference of angles of longitude $\boldsymbol{\lambda_2 - \lambda_1}$** rather than the individual values of $\boldsymbol{\lambda_1}$ & $\boldsymbol{\lambda_2}$ measured from a reference plane (like prime meridian for the globe) hence if the difference of angles of longitude is $\Delta\boldsymbol{\lambda}$ then setting $\lambda_2 - \lambda_1 = \Delta\lambda$ in the above formula, we get

$$\textbf{Great circle arc AB} = \boldsymbol{R\cos^{-1}(sin\phi_1 sin\phi_2 + cos\phi_1 cos\phi_2 cos\Delta\lambda)}$$

$$\forall\quad \boldsymbol{0 \le \phi_1\,, \phi_2, \Delta\lambda \le \pi}$$

NOTE: It's worth noticing that the above formula has **symmetrical terms** i.e. if ϕ_1 & ϕ_2 are interchanged, the formula remains unchanged & hence the value of C is unchanged. It also implies that if the locations of two points for given values of latitude & longitude is interchanged, the distance between them does not change at all. Since the equator plane divides the sphere into two equal hemispheres hence the above formula is applicable to find out the minimum distance between any two arbitrary points lying on any of two hemispheres. So for the convenience, the equator plane of the sphere should be taken in such a way that the given points lie on one of the two hemispheres resulting from division of sphere by the reference equator plane.

Since the maximum value of $cos^{-1}(x)$ is π hence the max. of min. distance between two points on a sphere is

$$= R(\pi) = \boldsymbol{\pi R} = \textbf{half of the perimeter of a great circle passing through given points}$$

Case 1: If both the given points lie on the equator of the sphere then substituting $\phi_1 = \phi_2 = 0$, we get

$$\text{Great circle arc AB} = R\cos^{-1}(\sin(0)\sin(0) + \cos(0)\cos(0)cos\Delta\lambda) = R\cos^{-1}(cos\Delta\lambda) = R\Delta\lambda$$

Hence, the **minimum distance between the points lying on the equator of the sphere of radius $\boldsymbol{R}$**

$$= \boldsymbol{R\Delta\lambda} \qquad (0 \le \Delta\lambda \le \pi)$$

The above result shows that the minimum distance between the points lying on the equator of the sphere depends only on the difference of longitudes of two given points & the radius of the sphere.

If both the given points lie diametrically opposite on the equator of the sphere then substituting $\Delta\boldsymbol{\lambda} = \boldsymbol{\pi}$ in above expression, the minimum distance between such points

$$R\Delta\lambda = R(\pi) = \boldsymbol{\pi R} = \text{half of the perimeter of a great circle passing through thegiven points}$$

Case 2: If both the given points lie on a great circle arc normal to the equator of the sphere then substituting $\Delta\lambda = 0$ in the formula, we get

$$\text{Great circle arc AB} = R\cos^{-1}(sin\phi_1 sin\phi_2 + cos\phi_1 cos\phi_2 \cos(0))$$

$$= R\cos^{-1}(sin\phi_1 sin\phi_2 + cos\phi_1 cos\phi_2) = R\cos^{-1}\big(cos(\phi_2 - \phi_2)\big) = R|\phi_2 - \phi_2|$$

Hence, the **minimum distance between two points lying on a great circle arc normal to the equator of the sphere of radius $\boldsymbol{R}$**

$$= \boldsymbol{R|\phi_2 - \phi_2|} \qquad (0 \le \phi_1, \phi_2 \le \pi)$$

Illustrative Example

Consider any two arbitrary points A & B having respective angles of latitude $\boldsymbol{\phi_1 = 30^o}$ & $\boldsymbol{\phi_2 = 70^o}$ & the difference of angles of longitude $\boldsymbol{\Delta\lambda = 50^o}$ on a sphere of radius 50 cm**.** Now substituting the corresponding values in the above formula, the minimum or great circle distance between the points A & B is given as follows

$$\text{Great circle arc AB} = 50\cos^{-1}(sin30^o sin70^o + cos30^o cos70^o cos50^o)$$

$$\approx \mathbf{42.48298289}\ \boldsymbol{cm}$$

The above result also shows that the points A & B divide the perimeter $= 2\pi(50) \approx 314.1592654\ cm$ of the great circle in two great circles arcs (one is minor arc AB of length $\approx 42.48298289\ cm$ & other is major arc AB of length $\approx 271.6762825\ cm$) into a ratio $\approx {}^{42.48298289}/_{271.6762825} \approx \mathbf{5:32}$

Conclusion: It can be concluded that this formula gives the correct values of the great circle distance because there is no approximation in the formula. This is an analytic formula to compute the minimum distance between any two arbitrary points on a sphere which is equally applicable in global positioning system to calculate the geographical distance between any two points on the globe for the given latitudes & longitudes. This gives the correct values for all the distances on the tiny sphere as well as the large sphere like giant planet if the calculations are made precisely.