Derivation of the volume of tetrahedron/pyramid bounded by a given plane & the co-ordinate planes

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Volume of tetrahedron/pyramid

bounded by a given plane & the coordinate planes in 3-D space

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Introduction: Here, we are interested to find out general expression to calculate the volume of tetrahedron/pyramid bounded by a given plane & the coordinate planes (i.e. XY-plane, YZ-plane & ZX-plane) using intercept form of equation of plane in 3-D space.

Derivation of volume of tetrahedron/pyramid: Let there be any arbitrary plane having the equation in 3-D space in the intercept form as follows:

\[
\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1
\]

The above equation shows that \(a, b \& c\) are the intercepts of the plane with the coordinate axes \(x, y \& z\)-axis respectively i.e. the given plane intersects \(x\)-axis at \((a, 0, 0)\), \(y\)-axis at \((0, b, 0)\) & \(z\)-axis at \((0, 0, c)\).

Thus, solid OABC obtained is the tetrahedron/pyramid, having vertices \(O, A, B \& C\), bounded by the given plane & the coordinate planes \(X-Y, Y-Z \& Z-X\). \(\Delta OAB, \Delta OBC, \Delta OAC \& \Delta ABC\) are the triangular faces of the tetrahedron/pyramid.

Now, all the sides of \(\Delta ABC\) are calculated by using distance formula as follows:

\[
AB = \sqrt{(a-0)^2 + (0-b)^2 + (0-0)^2} = \sqrt{a^2 + b^2}
\]

\[
BC = \sqrt{(0-0)^2 + (b-0)^2 + (0-c)^2} = \sqrt{b^2 + c^2}
\]

\[
CA = \sqrt{(0-a)^2 + (0-0)^2 + (c-0)^2} = \sqrt{c^2 + a^2}
\]

Now, using cosine formula in \(\Delta ABC\)

\[
\cos \angle ABC = \frac{AB^2 + BC^2 - CA^2}{2(AB)(BC)} = \frac{(\sqrt{a^2 + b^2})^2 + (\sqrt{b^2 + c^2})^2 - (\sqrt{c^2 + a^2})^2}{2(\sqrt{a^2 + b^2})(\sqrt{b^2 + c^2})} = \frac{b^2}{(a^2 + b^2)(b^2 + c^2)}
\]

\[
\Rightarrow \sin \angle ABC = \sqrt{1 - \cos^2 \angle ABC} \tag{\forall \ 0 < \angle ABC < \pi}
\]

\[
\sin \angle ABC = \sqrt{1 - \left(\frac{b^2}{(a^2 + b^2)(b^2 + c^2)}\right)^2} = \sqrt{\frac{a^2b^2 + b^2c^2 + c^2a^2}{(a^2 + b^2)(b^2 + c^2)}}
\]

The area of triangular face ABC of tetrahedron/pyramid

\[
[\Delta ABC] = \frac{1}{2} (AB)(BC) \sin \angle ABC
\]

\[
= \frac{1}{2} \left(\sqrt{a^2 + b^2}\right) \left(\sqrt{b^2 + c^2}\right) \sqrt{\frac{a^2b^2 + b^2c^2 + c^2a^2}{(a^2 + b^2)(b^2 + c^2)}}
\]
\[ [\Delta ABC] = \frac{1}{2} \sqrt{a^2b^2 + b^2c^2 + c^2a^2} \quad \ldots \ldots \ldots \ldots \ldots (1) \]

Now, the normal distance \((H)\) of the given plane: \(\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1\) from the origin \(O(0,0,0)\) is calculated as follows
\[
H = \frac{|0 + 0 + 0 - 1|}{\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2 + \left(\frac{1}{c}\right)^2}} = \frac{1}{\sqrt{a^2b^2 + b^2c^2 + c^2a^2}} \quad \ldots \ldots \ldots \ldots (2)
\]

Hence, the volume \((V)\) of the tetrahedron/pyramid \(OABC\) bounded by the given plane: \(\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1\) & the coordinate planes is calculated by the following formula
\[
V = \frac{1}{3} (\text{Area of base})(\text{normal height}) = \frac{1}{3} ([\Delta ABC])(H)
\]

Substituting the corresponding values of \([\Delta ABC] \) & \(H\) from (1) & (2),
\[
V = \frac{1}{3} \frac{1}{2} \sqrt{a^2b^2 + b^2c^2 + c^2a^2} \left(\frac{|abc|}{\sqrt{a^2b^2 + b^2c^2 + c^2a^2}}\right) = \frac{1}{6} |abc|
\]

\[
V = \frac{1}{6} |abc| = \frac{1}{6} |(x \text{ intercept}) \times (y \text{ intercept}) \times (z \text{ intercept})|
\]

**Note:** If the equation of a given plane is \(ax + by + cz + d = 0\) then it can be written in the intercepts as follows
\[
\frac{x}{-\frac{d}{a}} + \frac{y}{-\frac{d}{b}} + \frac{z}{-\frac{d}{c}} = 1 \quad (\forall \quad a,b,c,d \in R)
\]

Hence, the volume \((V)\) of the tetrahedron/pyramid bounded by the given plane: \(ax + by + cz + d = 0\) & the coordinate planes is calculated by substituting the values of intercepts, \(a = \left(-\frac{d}{a}\right), \ b = \left(-\frac{d}{b}\right) \& c = \left(-\frac{d}{c}\right)\) in the above formula
\[
V = \frac{1}{6} \left|\left(-\frac{d}{a}\right)\left(-\frac{d}{b}\right)\left(-\frac{d}{c}\right)\right| = \frac{1}{6} \left|\frac{d^3}{abc}\right|
\]

**Conclusion:** The above are the general formulas simple & straight-forward to calculate the volume of the tetrahedron/pyramid bounded by a given plane & the coordinate planes in 3-D space. All these articles/derivations are based on the applications of simple geometry.

**Note:** Above articles had been derived & illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)

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