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Reflection of a point about a line & a plane in 2-D & 3-D co-ordinate systems

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$Reflection\ of\ a\ point\ about\ a\ line$ & $a\ plane\ in$ 3-D $space$

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Introduction: Here, we are interested to find out general expressions to calculate the co-ordinates of a point which is the reflection of a give point about a line in 2-D co-ordinate system and about a line & a plane in 3-D co-ordinate system as well as the foot of perpendicular to a line & a plane by using simple geometry.

Reflection of a point about a line in 2-D co-ordinate system: Let there be any arbitrary point say $\boldsymbol{P(x_o, y_o)}$ & a straight line AB: $\boldsymbol{y = mx + c}$ **.** Now, assume that the point $\boldsymbol{P'(x', y')}$ is **the reflection of the given point P about the given straight line AB** (See the figure 1 below) then we have the following two conditions to be satisfied

1. The mid-point M of the line joining the points $P(x_o, y_o)$ & $P'(x', y')$ must lie on the line AB
2. The line joining the points $P(x_o, y_o)$ & $P'(x', y')$ must be normal to the line AB

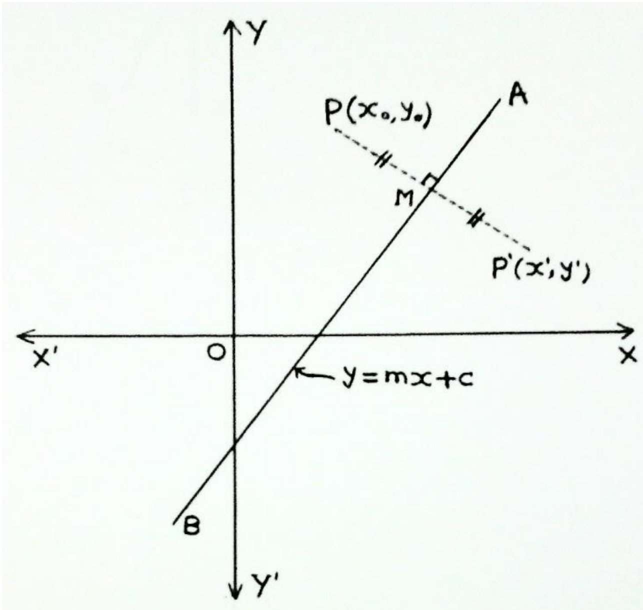

Figure 1: Point $\boldsymbol{P'(x', y')}$ is the reflection of point $\boldsymbol{P(x_o, y_o)}$ about the line AB: $\boldsymbol{y = mx + c}$

Now, we would apply both the above conditions to find out the co-ordinates of the point $P'(x', y')$. Co-ordinates of the mid-point M of the line PP' are calculated as

$$M \equiv \left(\frac{x' + x_o}{2}, \ \frac{y' + y_o}{2}\right)$$

The mid-point M (i.e. foot of perpendicular from point P to the line AB) will satisfy the equation of straight line AB as follows

$$\frac{y' + y_o}{2} = m\left(\frac{x' + x_o}{2}\right) + c$$

$$y' + y_o = m(x' + x_o) + 2c \qquad \dots \dots \dots \dots \dots (1)$$

Since, the straight lines PP' & AB are normal to each other hence we have the following condition

$$(slope\ of\ PP') \times (slope\ of\ AB) = -1$$

$$\left(\frac{y' - y_o}{x' - x_o}\right) \times (m) = -1 \ \Rightarrow \ y' - y_o = -\frac{x' - x_o}{m}$$

$$y' = y_o - \frac{x' - x_o}{m} \qquad \dots \dots \dots \dots \dots \dots \dots (2)$$

Now, substituting the value of y' from eq(2) in the eq(1), we get

$$y_o - \frac{x' - x_o}{m} + y_o = m(x' + x_o) + 2c \ \Rightarrow \ 2my_o - x' + x_o = m^2x' + m^2x_o + 2mc$$

$$(1 + m^2)x' = (1 - m^2)x_o + 2m(y_o - c) \Rightarrow x' = \frac{(1 - m^2)x_o + 2m(y_o - c)}{1 + m^2}$$

$$\therefore \ \boldsymbol{x' = \frac{(1 - m^2)x_o + 2m(y_o - c)}{1 + m^2}}$$

Substituting the value of x' in the eq(2), we get y co-ordinate as follows

$$y' = y_o - \frac{\frac{(1-m^2)x_o + 2m(y_o - c)}{1+m^2} - x_o}{m} = y_o - \frac{(1-m^2)x_o + 2m(y_o - c) - (1+m^2)x_o}{m(1+m^2)}$$

$$= y_o - \frac{2m(y_o - c) - 2m^2 x_o}{m(1+m^2)} = \frac{m(y_o + m^2 y_o - 2y_o + 2c) + 2m^2 x_o}{m(1+m^2)} = \frac{m^2 y_o - y_o + 2c + 2m x_o}{1+m^2}$$

$$= \frac{2m x_o - (1-m^2) y_o + 2c}{1+m^2}$$

$$\therefore \ \boldsymbol{y' = \frac{2m x_o - (1-m^2) y_o + 2c}{1+m^2}}$$

Hence, the point of reflection $P'(x', y')$ is given as

$$\boldsymbol{P' \equiv \left(\frac{(1-m^2)x_o + 2m(y_o - c)}{1+m^2}, \frac{2m x_o - (1-m^2) y_o + 2c}{1+m^2}\right)}$$

Foot of perpendicular: Foot of perpendicular drawn from the point $P(x_o, y_o)$ to the line AB: $y = mx + c$ can be easily determined simply by setting the values of x' & y' in the co-ordinates of point M as follows

$$M \equiv \left(\frac{x' + x_o}{2}, \frac{y' + y_o}{2}\right) \equiv \left(\frac{\frac{(1-m^2)x_o + 2m(y_o - c)}{1+m^2} + x_o}{2}, \frac{\frac{2m x_o - (1-m^2) y_o + 2c}{(1+m^2)} + y_o}{2}\right)$$

$$\therefore \ \boldsymbol{M \equiv \left(\frac{x_o + m(y_o - c)}{1+m^2}, \frac{m x_o + m^2 y_o + c}{1+m^2}\right)}$$

Note: If the line AB is passing through two points (x_1, y_1) & (x_2, y_2) then the equation of the line given as

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \ \Rightarrow \ y = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)x + \left(y_1 - \left(\frac{y_2 - y_1}{x_2 - x_1}\right)x_1\right)$$

Now, in order to find out the **point of reflection** & the **foot of perpendicular**, simply substitute the following values in above co-ordinates of point P' & M as follows

$$\boldsymbol{m = \frac{y_2 - y_1}{x_2 - x_1} \ \ \& \ \ c = y_1 - \left(\frac{y_2 - y_1}{x_2 - x_1}\right)x_1}$$

Reflection of a point about a line in 3-D co-ordinate system: Let there be any arbitrary point say $\boldsymbol{P(x_o, y_o, z_o)}$ & a straight line AB having equation

$$\boldsymbol{\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = k \ (any\ arbitrary\ constant)}$$

Now, assume that the point $\boldsymbol{P'(x', y', z')}$ is **the reflection of the given point P about the given straight line AB** (See the figure 2 below) then we have the following two conditions to be satisfied

1. The mid-point M of the line joining the points $P(x_o, y_o, z_o)$ & $P'(x', y', z')$ must lie on the line AB

2. The line joining the points $P(x_o, y_o, z_o)$ & $P'(x', y', z')$ must be normal to the line AB

Now, we would apply both the above conditions to find out the co-ordinates of the point $P'(x', y', z')$. Co-ordinates of the mid-point M of the line PP' are calculated as

$$M \equiv \left(\frac{x' + x_o}{2}, \frac{y' + y_o}{2}, \frac{z' + z_o}{2}\right)$$

The mid-point M (i.e. foot of perpendicular from point P to the line AB) will satisfy the equation of straight line AB as follows

$$\frac{x' + x_o}{2} = ka + x_1 \Rightarrow \boldsymbol{x' = 2(ka + x_1) - x_o}$$

$$\frac{y' + y_o}{2} = kb + y_1 \Rightarrow \boldsymbol{y' = 2(kb + y_1) - y_o}$$

$$\frac{z' + z_o}{2} = kc + z_1 \Rightarrow \boldsymbol{z' = 2(kc + z_1) - z_o}$$

Since, the straight lines PP' & AB are normal to each other hence we have the following **condition of normal direction ratios** of lines PP' & AB

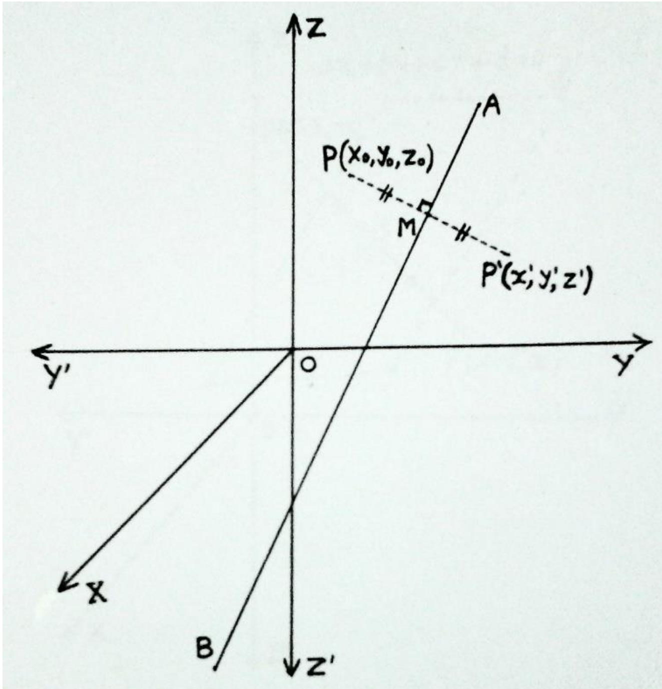

Figure 2: Point $P'(x', y', z')$ is the reflection of the point $P(x_o, y_o, z_o)$ about the line AB

$$\boldsymbol{sum\ of\ products\ of\ the\ corresponding\ direction\ ratios\ of\ the\ lines\ PP'\ \&\ AB = 0}$$

$$\Rightarrow a(x' - x_o) + b(y' - y_o) + c(z' - z_o) = 0$$

Now, setting the values of x', y' & z' in the above expression, we get

$$a(2(ka + x_1) - x_o - x_o) + b(2(kb + y_1) - y_o - y_o) + c(2(kc + z_1) - z_o - z_o) = 0$$

$$(a^2 + b^2 + c^2)k + a(x_1 - x_o) + b(y_1 - y_o) + c(z_1 - z_o) = 0$$

$$\boldsymbol{k = \frac{a(x_o - x_1) + b(y_o - y_1) + c(z_o - z_1)}{a^2 + b^2 + c^2}}$$

By substituting the value of k in the above expressions, the co-ordinates of the **point of reflection $P'(x', y', z')$** are calculated as follows

$$\boldsymbol{x' = 2a\left(\frac{a(x_o - x_1) + b(y_o - y_1) + c(z_o - z_1)}{a^2 + b^2 + c^2}\right) + 2x_1 - x_o}$$

$$\boldsymbol{y' = 2b\left(\frac{a(x_o - x_1) + b(y_o - y_1) + c(z_o - z_1)}{a^2 + b^2 + c^2}\right) + 2y_1 - y_o}$$

$$\boldsymbol{z' = 2c\left(\frac{a(x_o - x_1) + b(y_o - y_1) + c(z_o - z_1)}{a^2 + b^2 + c^2}\right) + 2z_1 - z_o}$$

Foot of perpendicular: Foot of perpendicular drawn from the point $P(x_o, y_o, z_o)$ to the line AB can be easily determined simply by setting the values of x', y' & z' in the co-ordinates of point M given as follows

$$\boldsymbol{M \equiv \left(\frac{x' + x_o}{2}, \frac{y' + y_o}{2}, \frac{z' + z_o}{2}\right)}$$

Note: If the line AB is passing through two points (x_1, y_1, z_1) & (x_2, y_2, z_2) then the equation of the line given as

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}=k\ (any\ arbitrary\ constant)$$

Now, in order to find out the **point of reflection** & the **foot of perpendicular**, simply substitute the following values of direction ratios of the lien in above co-ordinates of point P' & M as follows

$$a=x_2-x_1,\ \ b=y_2-y_1\ \ \&\ \ z=z_2-z_1$$

Reflection of a point about a plane in 3-D co-ordinate system: Let there be any arbitrary point say $\boldsymbol{P(x_o, y_o, z_o)}$ & a plane: $\boldsymbol{ax+by+cz+d=0}$

Now, assume that the point $\boldsymbol{P'(x', y', z')}$ is **the reflection of the given point P about the given plane** (See the figure 3 below) then we have the following two conditions to be satisfied

1. The mid-point M of the line joining the points $P(x_o, y_o, z_o)$ & $P'(x', y', z')$ must lie on the plane
2. The line joining the points $P(x_o, y_o, z_o)$ & $P'(x', y', z')$ must be parallel to normal to the plane

Now, we would apply both the above conditions to find out the co-ordinates of the point $P'(x', y', z')$. Co-ordinates of the mid-point M of the line PP' are calculated as

$$M\equiv\left(\frac{x'+x_o}{2},\ \frac{y'+y_o}{2},\ \frac{z'+z_o}{2}\right)$$

The mid-point M (i.e. foot of perpendicular from point P to the plane) will satisfy the equation of plane as follows

$$a\left(\frac{x'+x_o}{2}\right)+b\left(\frac{y'+y_o}{2}\right)+c\left(\frac{z'+z_o}{2}\right)+d=0$$

$$ax'+by'+cz'+ax_o+by_o+cz_o+2d=0 \qquad \ldots\ldots\ldots(3)$$

Since, the straight lines PP' & normal to the plane are parallel to each other hence we have the following **condition of parallel direction ratios** of line PP' & normal to the plane

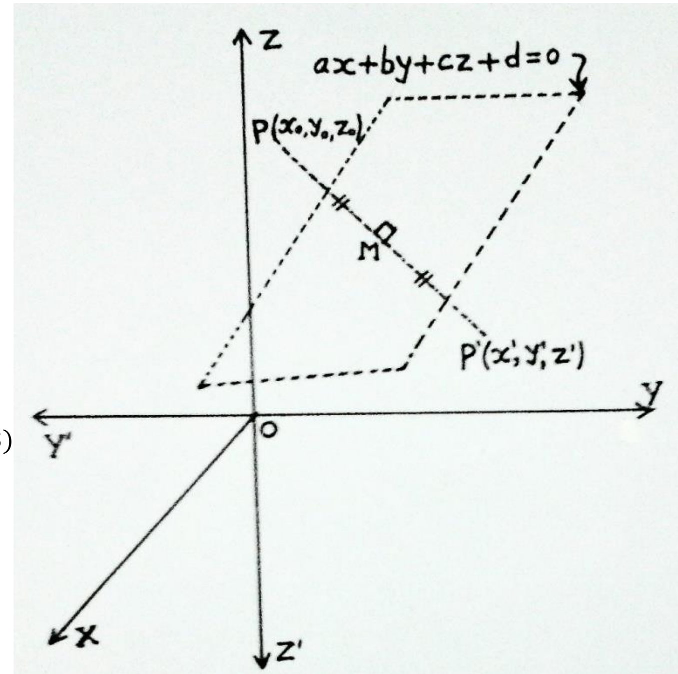

Figure 3: Point $\boldsymbol{P'(x', y', z')}$ is the reflection of the point $\boldsymbol{P(x_o, y_o, z_o)}$ about the plane: $\boldsymbol{ax+by+cz+d=0}$

$$\boldsymbol{ratios\ of\ corresponding\ direction\ ratios\ of\ two\ parallel\ lines\ are\ equal}$$

$$\Rightarrow\ \frac{\boldsymbol{x'-x_o}}{\boldsymbol{a}}=\frac{\boldsymbol{y'-y_o}}{\boldsymbol{b}}=\frac{\boldsymbol{z'-z_o}}{\boldsymbol{c}}=k\ (let\ any\ constant)$$

$$\Rightarrow\ \boldsymbol{x'=ak+x_o}\,,\qquad \boldsymbol{y'=bk+y_o}\quad \&\quad \boldsymbol{z'=ck+z_o}$$

Now, substituting the values of x', y' & z' in the equation (3), we get

$$a(ak+x_o)+b(bk+y_o)+c(ck+z_o)+ax_o+by_o+cz_o+2d=0$$

$$(a^2+b^2+c^2)k+2(ax_o+by_o+cz_o+d)=0$$

$$\boldsymbol{k}=\frac{-2(\boldsymbol{ax_o}+\boldsymbol{by_o}+\boldsymbol{cz_o}+\boldsymbol{d})}{\boldsymbol{a^2}+\boldsymbol{b^2}+\boldsymbol{c^2}}$$

By substituting the value of k in the above expressions, the co-ordinates of the **point of reflection** $\boldsymbol{P'(x', y', z')}$ are calculated as follows

$$\boldsymbol{x'}=\boldsymbol{x_o}-\frac{2\boldsymbol{a}(\boldsymbol{ax_o}+\boldsymbol{by_o}+\boldsymbol{cz_o}+\boldsymbol{d})}{\boldsymbol{a^2}+\boldsymbol{b^2}+\boldsymbol{c^2}}$$

$$\boldsymbol{y'}=\boldsymbol{y_o}-\frac{2\boldsymbol{b}(\boldsymbol{ax_o}+\boldsymbol{by_o}+\boldsymbol{cz_o}+\boldsymbol{d})}{\boldsymbol{a^2}+\boldsymbol{b^2}+\boldsymbol{c^2}}$$

$$\boldsymbol{z'}=\boldsymbol{z_o}-\frac{2\boldsymbol{c}(\boldsymbol{ax_o}+\boldsymbol{by_o}+\boldsymbol{cz_o}+\boldsymbol{d})}{\boldsymbol{a^2}+\boldsymbol{b^2}+\boldsymbol{c^2}}$$

Foot of perpendicular: Foot of perpendicular drawn from the point $P(x_o, y_o, z_o)$ to the plane can be easily determined simply by setting the values of x', y' & z' in the co-ordinates of point M given as follows

$$\boldsymbol{M}\equiv\left(\frac{\boldsymbol{x'}+\boldsymbol{x_o}}{\boldsymbol{2}}, \frac{\boldsymbol{y'}+\boldsymbol{y_o}}{\boldsymbol{2}}, \frac{\boldsymbol{z'}+\boldsymbol{z_o}}{\boldsymbol{2}}\right)$$

Conclusion: Thus, the reflection of any point about a line in 2-D co-ordinate system and about a line & a plane in 3-D co-ordinate system can be easily determined simply by applying the above procedures or by using above formula. These are also useful to determine the foot of perpendicular drawn from a point to a line or a plane in 3-D space. All these articles/derivations are based on the applications of simple geometry.

Note: *Above articles had been derived & illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)*

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