Mathematical Analysis of Great Rhombicuboctahedron (an Archimedean solid) by H.C. Rajpoot

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Introduction: A great rhombicuboctahedron is an Archimedean solid which has 12 congruent square faces, 8 congruent regular hexagonal faces & 6 congruent regular octagonal faces each having equal edge length. It has 72 edges & 48 vertices lying on a spherical surface with a certain radius. It is created/generated by expanding a truncated cube having 8 equilateral triangular faces & 6 regular octagonal faces. Thus by the expansion, each of 12 originally truncated edges changes into a square face, each of 8 triangular faces of the original solid changes into a regular hexagonal face & 6 regular octagonal faces of original solid remain unchanged i.e. octagonal faces are shifted radially. Thus a solid with 12 squares, 8 hexagonal & 6 octagonal faces, is obtained which is called great rhombicuboctahedron which is an Archimedean solid. (See figure 1), thus we have

no. of square faces = 12, no. of regular hexagonal faces = 8
no. of regular octagonal faces = 6,
no. of edges = 6(no. of edges in an octagonal face) + 2(no. of square faces) = 6(8) + 2(12) = 48 + 24 = 72
no. of vertices = 6(no. of vertices in an octagonal face) = 6(8) = 48

We would apply HCR’s Theory of Polygon to derive a mathematical relationship between radius R of the spherical surface passing through all 48 vertices & the edge length a of a great rhombicuboctahedron for calculating its important parameters such as normal distance of each face, surface area, volume etc.

Derivation of outer (circumscribed) radius \( R_o \) of great rhombicuboctahedron:

Let \( R_o \) be the radius of the spherical surface passing through all 48 vertices of a great rhombicuboctahedron with edge length \( a \) & the centre O. Consider a square face ABCD with centre \( O_1 \), regular hexagonal face EFGHIJ with centre \( O_2 \) & regular octagonal face KLMNPQRS with centre \( O_3 \) (see the figure 2 below)

Normal distance \( (H_s) \) of square face ABCD from the centre O of the great rhombicuboctahedron is calculated as follows

In right \( \Delta O_1AO \) (figure 2)

\[
OO_1 = \sqrt{(OA)^2 - (O_1A)^2} \quad \quad \quad \quad \quad (O_1A = \frac{a}{\sqrt{2}} = \text{circumscribed radius of square})
\]

\[
\Rightarrow H_s = \sqrt{(R_o)^2 - (\frac{a}{\sqrt{2}})^2} = \sqrt{\frac{2R_o^2 - a^2}{2}} \quad \quad \quad \quad \quad (I)
\]
We know that the solid angle \( \omega \) subtended by any regular polygon with each side of length \( a \) at any point lying at a distance \( H \) on the vertical axis passing through the centre of plane is given by “HCR’s Theory of Polygon” as follows

\[
\omega = 2\pi - 2n \sin^{-1} \left( \frac{2H \sin \frac{\pi}{n}}{\sqrt{4H^2 + a^2 \cot^2 \frac{\pi}{n}}} \right)
\]

Hence, by substituting the corresponding values in the above expression, we get the solid angle \( \omega_s \) subtended by each square face (ABCD) at the centre of great rhombicuboctahedron as follows

\[
\omega_s = 2\pi - 2 \times 4 \sin^{-1} \left( \frac{2 \left( \sqrt{2R_o^2 - a^2} \right) \sin \frac{\pi}{4}}{4 \left( \sqrt{2R_o^2 - a^2} \right)^2 + a^2 \cot^2 \frac{\pi}{4}} \right)
\]

\[
= 2\pi - 8 \sin^{-1} \left( \frac{\sqrt{2} \sqrt{2R_o^2 - a^2} \times 1}{\sqrt{4R_o^2 - 2a^2 + a^2 (1)^2}} \right) = 2\pi - 8 \sin^{-1} \left( \frac{\sqrt{2R_o^2 - a^2}}{\sqrt{4R_o^2 - a^2}} \right) = 2\pi - 8 \sin^{-1} \left( \frac{2R_a^2 - a^2}{4R_o^2 - a^2} \right)
\]

Let \( \frac{R_o^2}{a} = x > 1 \) (any arbitrary variable)

\[
\Rightarrow \omega_s = 2\pi - 8 \sin^{-1} \left( \frac{2 \left( \frac{R_o^2}{a} \right) - 1}{4 \left( \frac{R_o^2}{a} \right)^2 - 1} \right) = 2\pi - 8 \sin^{-1} \left( \frac{2x^2 - 1}{4x^2 - 1} \right) \quad \cdots \cdots \cdots \text{(II)}
\]
Similarly, Normal distance \((H_h)\) of regular hexagonal face EFGHIJ from the centre \(O\) of great rhombicuboctahedron is calculated as follows

In right \(\Delta E O_2 O\) (figure 2)

\[
O_2 O = \sqrt{(OE)^2 - (O_2 E)^2} \quad (O_2 E = a = \text{circumscribed radius of regular hexagon})
\]

\[
\Rightarrow H_h = \sqrt{(R_o)^2 - (a)^2} = \sqrt{R_o^2 - a^2} \quad \ldots \ldots \ldots (III)
\]

Hence, by substituting all the corresponding values, the solid angle \((\omega_h)\) subtended by each regular hexagonal face (EFGHIJ) at the centre of great rhombicuboctahedron is given as follows

\[
\omega_h = 2\pi - 2 \times 6 \sin^{-1} \left( \frac{2 \left( \sqrt{R_o^2 - a^2} \sin \frac{\pi}{6} \right)}{4 \left( \sqrt{R_o^2 - a^2} \right)^2 + a^2 \cot^2 \frac{\pi}{6}} \right)
\]

\[
= 2\pi - 12 \sin^{-1} \left( \frac{2\sqrt{R_o^2 - a^2} \times \frac{1}{2}}{4R_o^2 - 4a^2 + a^2(\sqrt{3})^2} \right) = 2\pi - 12 \sin^{-1} \left( \frac{\sqrt{R_o^2 - a^2}}{\sqrt{4R_o^2 - a^2}} \right)
\]

\[
\Rightarrow \omega_h = 2\pi - 12 \sin^{-1} \left( \frac{\sqrt{R_o^2 - a^2}}{\sqrt{4R_o^2 - a^2}} \right) = 2\pi - 12 \sin^{-1} \left( \frac{x^2 - 1}{\sqrt{4x^2 - 1}} \right) \quad \ldots \ldots \ldots (IV)
\]

Similarly, Normal distance \((H_o)\) of regular octagonal face KLMNPQRS from the centre \(O\) of great rhombicuboctahedron is calculated as follows

In right \(\Delta K O_3 O\) (figure 2)

\[
\Rightarrow O_3 O = \sqrt{(OE)^2 - (O_3 K)^2}
\]

\[
O_3 K = \text{circumscribed radius of regular octagon} = \frac{a}{2} \cosec 22.5^\circ = \frac{a}{2} \sqrt{4 + 2\sqrt{2}}
\]

\[
\Rightarrow H_o = \sqrt{(R_o)^2 - \left( \frac{a}{2} \sqrt{4 + 2\sqrt{2}} \right)^2} = \sqrt{\frac{4R_o}{4} - \left( \frac{4 + 2\sqrt{2}a^2}{4} \right)} \quad \ldots \ldots \ldots (V)
\]

Hence, by substituting all the corresponding values, the solid angle \((\omega_o)\) subtended by each regular octagonal face (KLMNPQRS) at the centre of great rhombicuboctahedron is given as follows

\[
\omega_o = 2\pi - 2 \times 8 \sin^{-1} \left( \frac{2 \left( \sqrt{\frac{4R_o}{4} - \left( \frac{4 + 2\sqrt{2}a^2}{4} \right)} \sin \frac{\pi}{8} \right)}{4 \left( \sqrt{\frac{4R_o}{4} - \left( \frac{4 + 2\sqrt{2}a^2}{4} \right)} \right)^2 + a^2 \cot^2 \frac{\pi}{8}} \right)
\]
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\[
2\pi - 16 \sin^{-1}\left( \frac{\sqrt{4R_o^2 - (4 + 2\sqrt{2})a^2 \sqrt{\frac{\sqrt{2} - 1}{2\sqrt{2}}}}}{\sqrt{4R_o^2 - (4 + 2\sqrt{2})a^2 + a^2(\sqrt{2} + 1)^2}} \right) = 2\pi - 16 \sin^{-1}\left( \frac{\sqrt{2 - \sqrt{2}}R_o^2 - a^2}{\sqrt{4R_o^2 - a^2}} \right) = 2\pi - 16 \sin^{-1}\left( \frac{(2 - \sqrt{2})a^2 - a^2}{2R_o^2 - a^2} \right)
\]

\[\Rightarrow \omega_o = 2\pi - 16 \sin^{-1}\left( \frac{(2 - \sqrt{2})a^2 - a^2}{2R_o^2 - a^2} \right) = 2\pi - 16 \sin^{-1}\left( \frac{(2 - \sqrt{2})x^2 - 1}{4x^2 - 1} \right) \ldots \ldots \ldots (VI)\]

Since a great rhombicuboctahedron is a closed surface & we know that the total solid angle, subtended by any closed surface at any point lying inside it, is \(4\pi\) sr (Ste-radian) hence the sum of solid angles subtended by 12 congruent square faces, 8 congruent regular hexagonal faces & 6 congruent regular octagonal faces at the centre of the great rhombicuboctahedron must be \(4\pi\) sr. Thus we have

\[12[\omega_s] + 8[\omega_h] + 6[\omega_o] = 4\pi\]

Now, by substituting the values of \(\omega_s\), \(\omega_h\) & \(\omega_o\) from eq(II), (IV) & (VI) in the above expression we get

\[
12 \left[ 2\pi - 8 \sin^{-1} \left( \frac{2x^2 - 1}{4x^2 - 1} \right) \right] + 8 \left[ 2\pi - 12 \sin^{-1} \left( \frac{x^2 - 1}{4x^2 - 1} \right) \right] + 6 \left[ 2\pi - 16 \sin^{-1} \left( \frac{(2 - \sqrt{2})x^2 - 1}{4x^2 - 1} \right) \right] = 4\pi
\]

\[\Rightarrow 96 \left[ \sin^{-1} \left( \frac{2x^2 - 1}{4x^2 - 1} \right) + \sin^{-1} \left( \frac{x^2 - 1}{4x^2 - 1} \right) + \sin^{-1} \left( \frac{(2 - \sqrt{2})x^2 - 1}{4x^2 - 1} \right) \right] = 52\pi - 4\pi = 48\pi\]

\[\Rightarrow \sin^{-1} \left( \frac{2x^2 - 1}{4x^2 - 1} \right) + \sin^{-1} \left( \frac{x^2 - 1}{4x^2 - 1} \right) + \sin^{-1} \left( \frac{(2 - \sqrt{2})x^2 - 1}{4x^2 - 1} \right) = \frac{48\pi}{96} = \frac{\pi}{2}\]

\[\Rightarrow \sin^{-1} \left( \frac{2x^2 - 1}{4x^2 - 1} \right) + \sin^{-1} \left( \frac{x^2 - 1}{4x^2 - 1} \right) = \frac{\pi}{2} - \sin^{-1} \left( \frac{(2 - \sqrt{2})x^2 - 1}{4x^2 - 1} \right)\]

\[\Rightarrow \sin^{-1} \left( \frac{2x^2 - 1 - \left( \frac{x^2 - 1}{4x^2 - 1} \right)^2}{4x^2 - 1} \right) + \sin^{-1} \left( \frac{x^2 - 1 - \left( \frac{2x^2 - 1}{4x^2 - 1} \right)^2}{4x^2 - 1} \right) = \cos^{-1} \left( \frac{(2 - \sqrt{2})x^2 - 1}{4x^2 - 1} \right)\]

(since, \(\sin^{-1} X + \sin^{-1} Y = \sin^{-1} \left( X\sqrt{1 - Y^2} + Y\sqrt{1 - X^2} \right)\) & \(\frac{\pi}{2} - \sin^{-1} X = \cos^{-1} X\))

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Now, solving the above bi-quadratic equation for the values of \( x \) as follows

\[ x^2 = \frac{-(-8(7 + 3\sqrt{2})) \pm \sqrt{(-8(7 + 3\sqrt{2}))^2 - 4(16)(13 + 6\sqrt{2})}}{2 \times 16} \]

\[ = \frac{8(7 + 3\sqrt{2}) \pm 8\sqrt{67 + 42\sqrt{2} - (13 + 6\sqrt{2})}}{32} = \frac{(7 + 3\sqrt{2}) \pm \sqrt{54 + 36\sqrt{2}}}{4} = \frac{(7 + 3\sqrt{2}) \pm 3\sqrt{6 + 4\sqrt{2}}}{4} \]

\[ = \frac{(7 + 3\sqrt{2}) \pm 3(2 + \sqrt{2})}{4} = \frac{(7 + 3\sqrt{2}) \pm 3(2 + \sqrt{2})}{4} \]

1. Taking positive sign, we have

\[ x^2 = \frac{(7 + 3\sqrt{2}) + 3(2 + \sqrt{2})}{4} = \frac{13 + 6\sqrt{2}}{4} \quad \text{or} \quad x = \sqrt{\frac{13 + 6\sqrt{2}}{4}} = \frac{1}{2} \sqrt{13 + 6\sqrt{2}} \]

Since, \( x > 1 \) hence, the above value is acceptable.

2. Taking negative sign, we have
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\[ x^2 = \frac{(7 + 3\sqrt{2}) - 3(2 + \sqrt{2})}{4} = \frac{1}{4} \quad \text{or} \quad x = \frac{1}{\sqrt{4}} = \frac{1}{2} \Rightarrow x < 1 \quad \text{but} \quad x > 1 \quad \text{(required condition)} \]

Hence, the above value is discarded, now we have

\[ x = \frac{1}{2} \sqrt{13 + 6\sqrt{2}} \Rightarrow \frac{R_o}{a} = x = \frac{1}{2} \sqrt{13 + 6\sqrt{2}} \quad \text{or} \quad R_o = \frac{a}{2} \sqrt{13 + 6\sqrt{2}} \]

Hence, the outer (circumscribed) radius \((R_o)\) of a great rhombicuboctahedron with edge length \(a\) is given as

\[ R_o = \frac{a}{2} \sqrt{13 + 6\sqrt{2}} \approx 2.317610913a \quad \ldots \ldots \quad (VII) \]

Normal distance \((H_s)\) of square faces from the centre of great rhombicuboctahedron: The normal distance \((H_s)\) of each of 12 congruent square faces from the centre \(O\) of a great rhombicuboctahedron is given from eq(I) as follows

\[ H_s = OO_1 = \sqrt{\frac{2R_o^2 - a^2}{2}} = \sqrt{\frac{2 \left( \frac{a}{2} \sqrt{13 + 6\sqrt{2}} \right)^2 - a^2}{2}} = a \sqrt{\frac{13 + 6\sqrt{2} - 2}{4}} = \frac{a}{2} \sqrt{11 + 6\sqrt{2}} = \frac{a}{2} \sqrt{(3 + \sqrt{2})^2} \]

\[ \therefore \quad H_s = \frac{(3 + \sqrt{2})a}{2} \approx 2.207106781a \]

It’s clear that all 12 congruent square faces are at an equal normal distance \(H_S\) from the centre of any great rhombicuboctahedron.

Solid angle \((\omega_s)\) subtended by each of the square faces at the centre of great rhombicuboctahedron: solid angle \((\omega_s)\) subtended by each square face is given from eq(II) as follows

\[ \omega_s = 2\pi - 8\sin^{-1} \left( \frac{2x^2 - a^2}{4x^2 - a^2} \right) = 2\pi - 8\sin^{-1} \left( \frac{2R_o^2 - a^2}{4R_o^2 - a^2} \right) \quad (\text{since}, \quad x = \frac{R_o}{a}) \]

Hence, by substituting the corresponding value of \(R_o\) in the above expression, we get

\[ \omega_s = 2\pi - 8\sin^{-1} \left( \frac{2 \left( \frac{a}{2} \sqrt{13 + 6\sqrt{2}} \right)^2 - a^2}{4 \left( \frac{a}{2} \sqrt{13 + 6\sqrt{2}} \right)^2 - a^2} \right) = 2\pi - 8\sin^{-1} \left( \frac{13 + 6\sqrt{2} - 2}{2(13 + 6\sqrt{2} - 1)} \right) \]

\[ = 2\pi - 8\sin^{-1} \left( \frac{11 + 6\sqrt{2}}{12(2 + \sqrt{2})} \right) = 2\pi - 8\sin^{-1} \left( \frac{(11 + 6\sqrt{2})(2 - \sqrt{2})}{12(2 + \sqrt{2})(2 - \sqrt{2})} \right) = 2\pi - 8\sin^{-1} \left( \frac{10 + \sqrt{2}}{24} \right) \]

\[ \therefore \quad \omega_s = 2\pi - 8\sin^{-1} \left( \frac{1}{2} \sqrt{\frac{10 + \sqrt{2}}{6}} \right) = 4\sin^{-1} \left( \frac{2 - \sqrt{2}}{12} \right) \approx 0.195339779 sr \]
Normal distance ($H_h$) of regular hexagonal faces from the centre of great rhombicuboctahedron:
The normal distance ($H_h$) of each of 8 congruent regular hexagonal faces from the centre O of a great rhombicuboctahedron is given from eq(III) as follows

\[ H_h = OO_2 = \sqrt{R_o^2 - a^2} = \frac{\left(\frac{a}{2}\sqrt{13 + 6\sqrt{2}}\right)^2 - a^2}{\sqrt{\frac{13 + 6\sqrt{2} - 4}{4}}} = \frac{a}{2}\sqrt{\frac{9 + 6\sqrt{2}}{3(1 + \sqrt{2})}} \]

\[ = \frac{\sqrt{3}(1 + \sqrt{2})a}{2} \]

\[ \therefore H_h = \frac{\sqrt{3}(1 + \sqrt{2})a}{2} \approx 2.090770275a \]

It’s clear that all 8 congruent regular hexagonal faces are at an equal normal distance $H_h$ from the centre of any great rhombicuboctahedron.

Solid angle ($\omega_h$) subtended by each of the regular hexagonal faces at the centre of great rhombicuboctahedron: solid angle ($\omega_h$) subtended by each regular hexagonal face is given from eq(IV) as follows

\[ \omega_h = 2\pi - 12\sin^{-1}\left(\frac{x^2 - a^2}{4x^2 - a^2}\right) = 2\pi - 12\sin^{-1}\left(\frac{R_o^2 - a^2}{4R_o^2 - a^2}\right) \quad (\text{since, } x = \frac{R_o}{a}) \]

Hence, by substituting the corresponding value of $R_o$ in the above expression, we get

\[ \omega_h = 2\pi - 12\sin^{-1}\left(\frac{\frac{a}{2}\sqrt{13 + 6\sqrt{2}} - a^2}{4\left(\frac{a}{2}\sqrt{13 + 6\sqrt{2}} - a^2\right)}\right) = 2\pi - 12\sin^{-1}\left(\frac{13 + 6\sqrt{2} - 4}{4(13 + 6\sqrt{2} - 1)}\right) \]

\[ = 2\pi - 12\sin^{-1}\left(\frac{3 + 2\sqrt{2}}{8(2 + \sqrt{2})}\right) = 2\pi - 12\sin^{-1}\left(\frac{(3 + 2\sqrt{2})(2 - \sqrt{2})}{8(2 + \sqrt{2})(2 - \sqrt{2})}\right) = 2\pi - 12\sin^{-1}\left(\frac{2 + \sqrt{2}}{16}\right) \]

\[ \therefore \omega_h = 2\pi - 12\sin^{-1}\left(\frac{\sqrt{2} + \sqrt{2}}{4}\right) \approx 0.5210126 \text{ sr} \]

Normal distance ($H_o$) of regular octagonal faces from the centre of great rhombicuboctahedron:
The normal distance ($H_o$) of each of 6 congruent regular octagonal faces from the centre O of a great rhombicuboctahedron is given from eq(V) as follows

\[ H_o = OO_3 = \sqrt{\frac{4R_o^2 - (4 + 2\sqrt{2})a^2}{4}} = \sqrt{\frac{4\left(\frac{a}{2}\sqrt{13 + 6\sqrt{2}}\right)^2 - (4 + 2\sqrt{2})a^2}{4}} = \frac{a}{2}\sqrt{\frac{13 + 6\sqrt{2} - (4 + 2\sqrt{2})}{\frac{9 + 4\sqrt{2}}{2}}} \]

\[ = \frac{a}{2}\sqrt{(1 + 2\sqrt{2})^2} = \frac{(1 + 2\sqrt{2})a}{2} \]

\[ \therefore H_o = \frac{(1 + 2\sqrt{2})a}{2} \approx 1.914213562a \]
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It’s clear that all 6 congruent regular octagonal faces are at an equal normal distance \(H_o\) from the centre of any great rhombicuboctahedron.

Solid angle \((\omega_o)\) subtended by each of the regular octagonal faces at the centre of great rhombicuboctahedron: solid angle \((\omega_o)\) subtended by each regular octagonal face is given from eq(VI) as follows

\[
\omega_o = 2\pi - 16\sin^{-1}\left(\frac{(2 - \sqrt{2})x^2 - 1}{4x^2 - 1}\right) = 2\pi - 16\sin^{-1}\left(\frac{(2 - \sqrt{2})R_o^2 - a^2}{4R_o^2 - a^2}\right) \quad (since, \ x = \frac{R_o}{a})
\]

Hence, by substituting the corresponding value of \(R_o\) in the above expression, we get

\[
\omega_o = 2\pi - 16\sin^{-1}\left(\frac{(2 - \sqrt{2})\left(\frac{a}{2}\sqrt{13 + 6\sqrt{2}}\right)^2 - a^2}{4\left(\frac{a}{2}\sqrt{13 + 6\sqrt{2}}\right)^2 - a^2}\right) = 2\pi - 16\sin^{-1}\left(\frac{(2 - \sqrt{2})(13 + 6\sqrt{2}) - 4}{4(13 + 6\sqrt{2} - 1)}\right)
\]

\[
= 2\pi - 16\sin^{-1}\left(\frac{10 - \sqrt{2}}{24(2 + \sqrt{2})}\right) = 2\pi - 16\sin^{-1}\left(\frac{10 - \sqrt{2}}{24(2 + \sqrt{2} - 2)}\right)
\]

\[
= 2\pi - 16\sin^{-1}\left(\frac{22 - 12\sqrt{2}}{48}\right) = 2\pi - 16\sin^{-1}\left(\frac{(3 - \sqrt{2})^2}{24}\right) = 2\pi - 16\sin^{-1}\left(\frac{3 - \sqrt{2}}{2\sqrt{6}}\right)
\]

\[
\therefore \omega_o = 2\pi - 16\sin^{-1}\left(\frac{3 - \sqrt{2}}{2\sqrt{6}}\right) \approx 1.009032076 \text{ sr}
\]

It’s clear from the above results that the solid angle subtended by each of 6 regular octagonal faces is greater than the solid angle subtended by each of 12 square faces & each of 8 regular hexagonal faces at the centre of any great rhombicuboctahedron.

It’s also clear from the above results that \(H_s > H_h > H_o\) i.e. the normal distance \((H_s)\) of square faces is greater than the normal distance \((H_h)\) of the regular hexagonal faces & the normal distance \((H_o)\) of the regular octagonal faces from the centre of a great rhombicuboctahedron i.e. regular octagonal faces are closer to the centre as compared to the square & regular hexagonal faces in any great rhombicuboctahedron.

Important parameters of a great rhombicuboctahedron:

1. **Inner (inscribed) radius** \((R_i)\): It is the radius of the largest sphere inscribed (trapped inside) by a great rhombicuboctahedron. The largest inscribed sphere always touches all 6 congruent regular octagonal faces but does not touch any of 12 congruent square & any of 8 congruent regular hexagonal faces at all since all 6 octagonal faces are closest to the centre in all the faces. Thus, inner radius is always equal to the normal distance \((H_o)\) of regular octagonal faces from the centre of a great rhombicuboctahedron & is given as follows

\[
R_i = H_o = \frac{(1 + 2\sqrt{2})a}{2} = 1.914213562a
\]

Hence, the **volume of inscribed sphere** is given as
2. Outer (circumscribed) radius ($R_o$): It is the radius of the smallest sphere circumscribing a great rhombicuboctahedron or it’s the radius of a spherical surface passing through all 48 vertices of a great rhombicuboctahedron. It is from the eq(VII) as follows

$$R_o = \frac{a}{2} \sqrt{13 + 6\sqrt{2}} \approx 2.317610913a$$

Hence, the volume of circumscribed sphere is given as

$$V_{circumscribed} = \frac{4}{3} \pi (R_o)^3 = \frac{4}{3} \pi \left(\frac{a}{2} \sqrt{13 + 6\sqrt{2}}\right)^3 = 52.14470211a^3$$

3. Surface area ($A_s$): We know that a great rhombicuboctahedron has 12 congruent square faces, 8 congruent regular hexagonal faces & 6 congruent regular octagonal faces each of edge length $a$. Hence, its surface area is given as follows

$$A_s = 12(area\ of\ square) + 8(area\ of\ regular\ hexagon) + 6(area\ of\ regular\ octagon)$$

We know that area of any regular n-polygon with each side of length $a$ is given as

$$A = \frac{1}{4} na^2\cot\frac{\pi}{n}$$

Hence, by substituting all the corresponding values in the above expression, we get

$$A_s = 12 \times \left(\frac{1}{4} \times 4a^2\cot\frac{\pi}{4}\right) + 8 \times \left(\frac{1}{4} \times 6a^2\cot\frac{\pi}{6}\right) + 6 \times \left(\frac{1}{4} \times 8a^2\cot\frac{\pi}{8}\right)$$

$$= 12a^2 + 12\sqrt{3}a^2 + 12(1 + \sqrt{2})a^2 = 12(2 + \sqrt{2} + \sqrt{3})a^2$$

$$A_s = 12(2 + \sqrt{2} + \sqrt{3})a^2 \approx 61.75517244a^2$$

4. Volume ($V$): We know that a great rhombicuboctahedron with edge length $a$ has 12 congruent square faces, 8 congruent regular hexagonal faces & 6 congruent regular octagonal faces. Hence, the volume ($V$) of the great rhombicuboctahedron is the sum of volumes of all its elementary right pyramids with square base, regular hexagonal base & regular octagonal base (face) (see figure 2 above) & is given as follows

$$V = 12(volume\ of\ right\ pyramid\ with\ square\ base)$$

$$+ 8(volume\ of\ right\ pyramid\ with\ regular\ hexagonal\ base)$$

$$+ 6(volume\ of\ right\ pyramid\ with\ regular\ octagonal\ base)$$

$$= 12\left(\frac{1}{3}(area\ of\ square) \times H_s\right) + 8\left(\frac{1}{3}(area\ of\ regular\ hexagon) \times H_n\right)$$

$$+ 6\left(\frac{1}{3}(area\ of\ regular\ octagon) \times H_o\right)$$

$$= 12\left(\frac{1}{3}\left(4a^2\cot\frac{\pi}{4}\right) \times \left(3 + \sqrt{2}\right)a\right) + 8\left(\frac{1}{3}\left(6a^2\cot\frac{\pi}{6}\right) \times \sqrt{3}(1 + \sqrt{2})a\right)$$

$$+ 6\left(\frac{1}{3}\left(8a^2\cot\frac{\pi}{8}\right) \times \left(1 + 2\sqrt{2}\right)a\right)$$

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\[ V = (22 + 14\sqrt{2})a^3 \approx 41.79898987a^3 \]

5. **Mean radius \((R_m)\):** It is the radius of the sphere having a volume equal to that of a great rhombicuboctahedron. It is calculated as follows

\[
\frac{4}{3} \pi (R_m)^3 = (22 + 14\sqrt{2})a^3 \Rightarrow (R_m)^3 = \frac{3(11 + 7\sqrt{2})a^3}{2\pi} \text{ or } R_m = a \left(\frac{3(11 + 7\sqrt{2})}{2\pi}\right)^{\frac{1}{3}}
\]

\[ R_m = a \left(\frac{3(11 + 7\sqrt{2})}{2\pi}\right)^{\frac{1}{3}} \approx 2.15290926a \]

It’s clear from above results that \( R_i < R_m < R_o \)

6. **Dihedral angles between the adjacent faces:** In order to calculate dihedral angles between the different adjacent faces with a common edge in a great rhombicuboctahedron, let’s consider one-by-one all three pairs of adjacent faces with a common edge as follows

a. **Angle between square face & regular hexagonal face:** Draw the perpendiculars \( OO_1 \) & \( OO_2 \) from the centre \( O \) of great rhombicuboctahedron to the square face & the regular hexagonal face which have a common edge (See figure 3). We know that the inscribed radius \((r_i)\) of any regular \(n\)-gon with each side \(a\) is given as follows

\[
r_i = \text{inscribed radius of any regular } \ n \ - \ \text{gon} = \frac{a}{2} \cot \frac{\pi}{n}
\]

\[ \therefore \ O_1T = \text{inscribed radius of square} = \frac{a}{2} \cot \frac{\pi}{4} = \frac{a}{2} \]

\[ \therefore \ O_2T = \text{inscribed radius of regular hexagon} = \frac{a}{2} \cot \frac{\pi}{6} = \frac{a\sqrt{3}}{2} \]

In right \(\Delta OO_1T\)

\[
\tan \theta_s = \frac{OO_1}{O_1T} = \frac{H_s}{\left(\frac{a\sqrt{2}}{2}\right)} = \frac{(3 + \sqrt{2})a}{2} = (3 + \sqrt{2})
\]

\[ \therefore \ \theta_s = \tan^{-1}(3 + \sqrt{2}) \approx 77.23561032^\circ \quad \ldots \ldots \ldots \ (VIII) \]

In right \(\Delta OO_2T\)

\[
\tan \theta_h = \frac{OO_2}{O_2T} = \frac{H_h}{\left(\frac{a\sqrt{3}}{2}\right)} = \frac{\sqrt{3}(1 + \sqrt{2})a}{\frac{a\sqrt{3}}{2}} = (1 + \sqrt{2})
\]

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b. \[ \theta_h = \tan^{-1}(1 + \sqrt{2}) = 67.5^\circ \quad \ldots \ldots \ldots \quad (IX) \]

\[ \Rightarrow \theta_s + \theta_h = \tan^{-1}(3 + \sqrt{2}) + \tan^{-1}(1 + \sqrt{2}) = \tan^{-1}\left(\frac{3 + \sqrt{2}}{1 - (3 + \sqrt{2})(1 + \sqrt{2})}\right) = \tan^{-1}\left(\frac{4 + 2\sqrt{2}}{1 - (5 + 4\sqrt{2})}\right) \]

\[ = \tan^{-1}\left(-\frac{4 + 2\sqrt{2}}{4 + 4\sqrt{2}}\right) = \pi - \tan^{-1}\left(\frac{4 + 2\sqrt{2}}{4 + 4\sqrt{2}}\right) = \pi - \tan^{-1}\left(\frac{1 + \sqrt{2}}{2 + \sqrt{2}}\right) = \pi - \tan^{-1}\left(\frac{1 + \sqrt{2}}{\sqrt{2}}\right) = \pi - \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) \]

Hence, dihedral angle between the square face & the regular hexagonal face is given as

\[ \theta_s + \theta_h = \pi - \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) \approx 144.7356103^\circ \]

b. Angle between square face & regular octagonal face: Draw the perpendiculars \( OO_1 \) & \( OO_3 \) from the centre \( O \) of great rhombicuboctahedron to the square face & the regular octagonal face which have a common edge (See figure 4).

\[ O_3U = \text{inscribed radius of regular octagon} = \frac{a}{2} \cot \frac{\pi}{8} = \frac{a(1 + \sqrt{2})}{2} \]

In right \( \Delta O_3O_3U \)

\[ \tan \theta_o = \frac{OO_3}{O_3U} = \frac{H_o}{\frac{a(1 + \sqrt{2})}{2}} = \frac{(1 + 2\sqrt{2})a}{\frac{a(1 + \sqrt{2})}{2}} = \frac{(1 + 2\sqrt{2})}{(1 + \sqrt{2})} = \frac{1 + 2\sqrt{2}}{1 + \sqrt{2}} = \frac{3 - \sqrt{2}}{\sqrt{2} + 1} = 3 - \sqrt{2} \]

\[ : \quad \theta_o = \tan^{-1}(3 - \sqrt{2}) \approx 57.76438969^\circ \quad \ldots \ldots \ldots \quad (X) \]

\[ \Rightarrow \theta_s + \theta_o = \tan^{-1}(3 + \sqrt{2}) + \tan^{-1}(3 - \sqrt{2}) = \tan^{-1}\left(\frac{(3 + \sqrt{2})(3 - \sqrt{2})}{1 - (3 + \sqrt{2})(3 - \sqrt{2})}\right) = \tan^{-1}\left(\frac{6}{1 - 7}\right) \]

\[ = \tan^{-1}\left(\frac{6}{6}\right) = \tan^{-1}(1) = \pi - \tan^{-1}(1) = 180^\circ - 45^\circ = 135^\circ \]

Hence, dihedral angle between the square face & the regular octagonal face is given as

\[ \theta_s + \theta_o = 135^\circ \]

c. Angle between regular hexagonal face & regular octagonal face: Draw the perpendiculars \( OO_2 \) & \( OO_3 \) from the centre \( O \) of great rhombicuboctahedron to the regular hexagonal face & the regular octagonal face which have a common edge (See figure 5). Now from eq(IX) & (X), we get

\[ \theta_h + \theta_o = \tan^{-1}(1 + \sqrt{2}) + \tan^{-1}(3 - \sqrt{2}) = \tan^{-1}\left(\frac{(1 + \sqrt{2})(3 - \sqrt{2})}{1 - (1 + \sqrt{2})(3 - \sqrt{2})}\right) \]

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Mathematical Analysis of Great Rhombicuboctahedron/Archimedean Solid

\[ \theta_H + \theta_O = \pi - \tan^{-1}(\sqrt{2}) \approx 125.2643897^\circ \]

Hence, dihedral angle between the regular hexagonal face & the regular octagonal face is given as

Construction of a solid great rhombicuboctahedron: In order to construct a solid great rhombicuboctahedron with edge length \( a \) there are two methods

1. Construction from elementary right pyramids: In this method, first we construct all elementary right pyramids as follows

Construct 12 congruent right pyramids with square base of side length \( a \) & normal height \( (H_s) \)

\[ H_s = \frac{(3 + \sqrt{2})a}{2} \approx 2.207106781a \]

Construct 8 congruent right pyramids with regular hexagonal base of side length \( a \) & normal height \( (H_h) \)

\[ H_h = \frac{\sqrt{3}(1 + \sqrt{2})a}{2} \approx 2.090770275a \]

Construct 6 congruent right pyramids with regular octagonal base of side length \( a \) & normal height \( (H_o) \)

\[ H_o = \frac{(1 + 2\sqrt{2})a}{2} \approx 1.914213562a \]

Now, paste/bond by joining all these elementary right pyramids by overlapping their lateral surfaces & keeping their apex points coincident with each other such that 4 edges of each square base (face) coincide with the edges of 2 regular hexagonal bases & 2 regular octagonal bases (faces). Thus a solid great rhombicuboctahedron, with 12 congruent square faces, 8 congruent regular hexagonal faces & 6 congruent regular octagonal faces each of edge length \( a \), is obtained.

2. Facing a solid sphere: It is a method of facing, first we select a blank as a solid sphere of certain material (i.e. metal, alloy, composite material etc.) & with suitable diameter in order to obtain the maximum desired edge length of a great rhombicuboctahedron. Then, we perform the facing operations on the solid sphere to generate 12 congruent square faces, 8 congruent regular hexagonal faces & 6 congruent regular octagonal faces each of equal edge length.

Let there be a blank as a solid sphere with a diameter \( D \). Then the edge length \( a \), of a great rhombicuboctahedron of the maximum volume to be produced, can be co-related with the diameter \( D \) by relation of outer radius \( (R_o) \) with edge length \( (a) \) of the great rhombicuboctahedron as follows

\[ R_o = \frac{a}{2} \sqrt{13 + 6\sqrt{2}} \]
Mathematical Analysis of Great Rhombicuboctahedron/Archimedean Solid

Now, substituting $R_o = \frac{D}{2}$ in the above expression, we have

$$\frac{D}{2} = \frac{a}{2} \sqrt{13 + 6\sqrt{2}} \quad \text{or} \quad a = \frac{D}{\sqrt{13 + 6\sqrt{2}}}$$

$$a = \frac{D}{\sqrt{13 + 6\sqrt{2}}} \approx 0.215739405D$$

Above relation is very useful for determining the edge length $a$ of a great rhombicuboctahedron to be produced from a solid sphere with known diameter $D$ for manufacturing purpose.

Hence, the maximum volume of great rhombicuboctahedron produced from a solid sphere is given as follows

$$V_{\text{max}} = (22 + 14\sqrt{2})a^3 = (22 + 14\sqrt{2})\left(\frac{D}{\sqrt{13 + 6\sqrt{2}}}\right)^3 = \frac{(22 + 14\sqrt{2})D^3}{(13 + 6\sqrt{2})\sqrt{13 + 6\sqrt{2}}}$$

$$V_{\text{max}} = \frac{2(59 + 25\sqrt{2})D^3}{97\sqrt{13 + 6\sqrt{2}}} \approx 0.419714736D^3$$

Minimum volume of material removed is given as

$$(V_{\text{removed}})_{\text{min}} = \text{(volume of parent sphere with diameter D)} - \text{(volume of great rhombicuboctahedron)}$$

$$= \frac{\pi}{6} D^3 - \frac{2(59 + 25\sqrt{2})D^3}{97\sqrt{13 + 6\sqrt{2}}} = \left(\frac{\pi}{6} - \frac{2(59 + 25\sqrt{2})}{97\sqrt{13 + 6\sqrt{2}}}\right)D^3$$

$$(V_{\text{removed}})_{\text{min}} = \left(\frac{\pi}{6} - \frac{2(59 + 25\sqrt{2})}{97\sqrt{13 + 6\sqrt{2}}}\right)D^3 \approx 0.103884038D^3$$

Percentage (%) of minimum volume of material removed

$$\% \text{ of } V_{\text{removed}} = \frac{\text{minimum volume removed}}{\text{total volume of sphere}} \times 100$$

$$= \left(\frac{\pi}{6} - \frac{2(59 + 25\sqrt{2})}{97\sqrt{13 + 6\sqrt{2}}}\right)D^3 \times 100 = \frac{12(59 + 25\sqrt{2})}{97\pi\sqrt{13 + 6\sqrt{2}}} \times 100 \approx 19.84\%$$

It’s obvious that when a great rhombicuboctahedron of the maximum volume is produced from a solid sphere then about 19.84% volume of material is removed as scraps. Thus, we can select optimum diameter of blank as a solid sphere to produce a solid great rhombicuboctahedron of the maximum volume (or with maximum desired edge length)

Conclusions: Let there be any great rhombicuboctahedron having 12 congruent square faces, 8 congruent regular hexagonal faces & 6 congruent regular octagonal faces each with edge length $a$ then all its important parameters are calculated/determined as tabulated below
### Mathematical Analysis of Great Rhombicuboctahedron/Archimedean Solid

<table>
<thead>
<tr>
<th>Congruent polygonal faces</th>
<th>No. of faces</th>
<th>Normal distance of each face from the centre of the great rhombicuboctahedron</th>
<th>Solid angle subtended by each face at the centre of the great rhombicuboctahedron</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square</td>
<td>12</td>
<td>( \frac{(3 + \sqrt{2})a}{2} \approx 2.207106781a )</td>
<td>( 4 \sin^{-1} \left( \frac{2 - \sqrt{2}}{12} \right) \approx 0.195339779 \text{ sr} )</td>
</tr>
<tr>
<td>Regular hexagon</td>
<td>8</td>
<td>( \frac{\sqrt{3}(1 + \sqrt{2})a}{2} \approx 2.090770275a )</td>
<td>( 2\pi - 12 \sin^{-1} \left( \frac{\sqrt{2} + \sqrt{2}}{4} \right) \approx 0.5210126 \text{ sr} )</td>
</tr>
<tr>
<td>Regular octagon</td>
<td>6</td>
<td>( \frac{(1 + 2\sqrt{2})a}{2} \approx 1.914213562a )</td>
<td>( 2\pi - 16 \sin^{-1} \left( \frac{3 - \sqrt{2}}{2\sqrt{6}} \right) \approx 1.009032076 \text{ sr} )</td>
</tr>
</tbody>
</table>

**Inner (inscribed) radius \( (R_i) \)**

\[ R_i = \frac{(1 + 2\sqrt{2})a}{2} \approx 1.914213562a \]

**Outer (circumscribed) radius \( (R_o) \)**

\[ R_o = \frac{a}{2} \sqrt{13 + 6\sqrt{2}} \approx 2.317610913a \]

**Mean radius \( (R_m) \)**

\[ R_m = a \left( \frac{3(11 + 7\sqrt{2})}{2\pi} \right)^{\frac{1}{3}} \approx 2.15290926a \]

**Surface area \( (A_s) \)**

\[ A_s = 12(2 + \sqrt{2} + \sqrt{3})a^2 \approx 61.75517244a^2 \]

**Volume \( (V) \)**

\[ V = (22 + 14\sqrt{2})a^3 \approx 41.79898987a^3 \]

### Table for the dihedral angles between the adjacent faces of a great rhombicuboctahedron

<table>
<thead>
<tr>
<th>Pair of the adjacent faces with a common edge</th>
<th>Square &amp; regular hexagon</th>
<th>Square &amp; regular octagon</th>
<th>Regular hexagon &amp; regular octagon</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dihedral angle of the corresponding pair (of the adjacent faces)</td>
<td>( \theta_s + \theta_h = \pi - \tan^{-1} \left( \frac{1}{\sqrt{2}} \right) \approx 144.7356103^\circ )</td>
<td>( \theta_s + \theta_o = 135^\circ )</td>
<td>( \theta_h + \theta_o = \pi - \tan^{-1}(\sqrt{2}) \approx 125.2643897^\circ )</td>
</tr>
</tbody>
</table>

**Note:** Above articles had been developed & illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)

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