Winter March 20, 2015

Mathematical Analysis of Great Rhombicuboctahedron (an Archimedean solid) by H.C. Rajpoot

Harish Chandra Rajpoot Rajpoot, HCR

$Mathematical\ Analysis\ of\ Great\ Rhombicuboctahedron$

$Application\ of\ HCR's\ Theory\ of\ Polygon$

Mr Harish Chandra Rajpoot

M.M.M. University of Technology, Gorakhpur-273010 (UP), India ***March, 2015***

Introduction: A great rhombicuboctahedron is an Archimedean solid which has 12 congruent square faces, 8 congruent regular hexagonal faces & 6 congruent regular octagonal faces each having equal edge length. It has 72 edges & 48 vertices lying on a spherical surface with a certain radius. It is created/generated by expanding a truncated cube having 8 equilateral triangular faces & 6 regular octagonal faces. Thus by the expansion, each of 12 originally truncated edges changes into a square face, each of 8 triangular faces of the original solid changes into a regular hexagonal face & 6 regular octagonal faces of original solid remain unchanged i.e. octagonal faces are shifted radially. Thus a solid with 12 squares, 8 hexagonal & 6 octagonal faces, is obtained which is called **great rhombicuboctahedron** which is an **Archimedean solid**. (See figure 1), thus we have

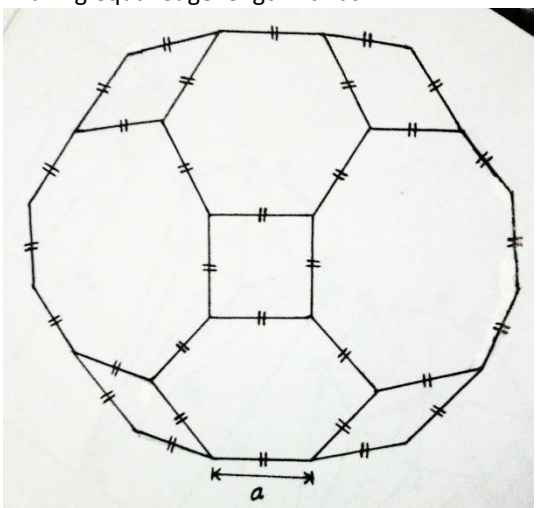

Figure 1: A great rhombicuboctahedron having 12 congruent square faces, 8 congruent regular hexagonal faces & 6 congruent regular octagonal faces each of equal edge length a

$no.of\ square\ faces = 12,\ no.of\ regular\ hexagonal\ faces = 8$

$no.of\ regular\ octagonal\ faces = 6,$

$no.of\ edges =$
$6(no.of\ edges\ in\ an\ octagonal\ face) + 2(no.of\ square\ faces) =$
$6(8) + 2(12) = 48 + 24 = 72$

$no.of\ vertices = 6(no.of\ vertices\ in\ an\ octagonal\ face) = 6(8) = 48$

We would apply HCR's Theory of Polygon to derive a mathematical relationship between radius R of the spherical surface passing through all 48 vertices & the edge length a of a great rhombicuboctahedron for calculating its important parameters such as normal distance of each face, surface area, volume etc.

Derivation of outer (circumscribed) radius (R_o) of great rhombicuboctahedron:

Let R_o be the radius of the spherical surface passing through all 48 vertices of a great rhombicuboctahedron with edge length a & the centre O. Consider a square face ABCD with centre O_1, regular hexagonal face EFGHIJ with centre O_2 & regular octagonal face KLMNPQRS with centre O_3 (see the figure 2 below)

Normal distance (H_s) of square face ABCD from the centre O of the great rhombicuboctahedron is calculated as follows

In right ΔAO_1O (figure 2)

$$OO_1 = \sqrt{(OA)^2 - (O_1A)^2} \qquad \left(\boldsymbol{O_1A = \frac{a}{\sqrt{2}}} = circumscribed\ radius\ of\ square\right)$$

$$\Rightarrow \boldsymbol{H_s} = \sqrt{(R_o)^2 - \left(\frac{a}{\sqrt{2}}\right)^2} = \sqrt{\frac{\boldsymbol{2{R_o}^2 - a^2}}{\boldsymbol{2}}} \qquad \ldots\ldots\ldots\ldots\ldots.(I)$$

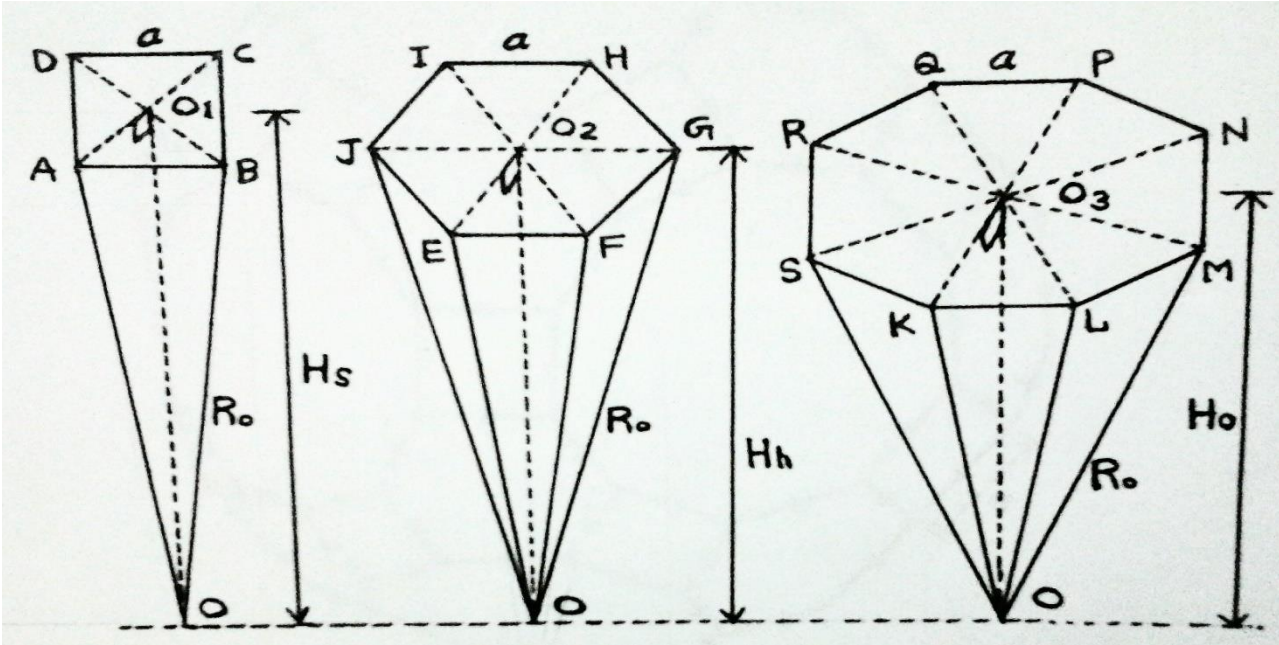

Figure 2: A square face ABCD, regular hexagonal face EFGHIJ & regular octagonal face KLMNPQRS each of edge length a

We know that the solid angle (ω) subtended by any regular polygon with each side of length a at any point lying at a distance H on the vertical axis passing through the centre of plane is given by "**HCR's Theory of Polygon**" as follows

$$\omega = 2\pi - 2n\sin^{-1}\left(\frac{2Hsin\frac{\pi}{n}}{\sqrt{4H^2 + a^2cot^2\frac{\pi}{n}}}\right)$$

Hence, by substituting the corresponding values in the above expression, we get the solid angle $(\boldsymbol{\omega_s})$ subtended by each square face (ABCD) at the centre of great rhombicuboctahedron as follows

$$\omega_s = 2\pi - 2\times 4\sin^{-1}\left(\frac{2\left(\sqrt{\frac{2{R_o}^2 - a^2}{2}}\right)sin\frac{\pi}{4}}{\sqrt{4\left(\sqrt{\frac{2{R_o}^2 - a^2}{2}}\right)^2 + a^2cot^2\frac{\pi}{4}}}\right)$$

$$= 2\pi - 8\sin^{-1}\left(\frac{\sqrt{2}\sqrt{2{R_o}^2 - a^2}\times\frac{1}{\sqrt{2}}}{\sqrt{4{R_o}^2 - 2a^2 + a^2(1)^2}}\right) = 2\pi - 8\sin^{-1}\left(\frac{\sqrt{2{R_o}^2 - a^2}}{\sqrt{4{R_o}^2 - a^2}}\right) = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{2{R_o}^2 - a^2}{4{R_o}^2 - a^2}}\right)$$

$$\boldsymbol{Let\ \frac{R_o}{a} = x > 1\ (any\ arbitrary\ variable)}$$

$$\Rightarrow \boldsymbol{\omega_s} = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{2\left(\frac{R_o}{a}\right)^2 - 1}{4\left(\frac{R_o}{a}\right)^2 - 1}}\right) = \boldsymbol{2\pi - 8\sin^{-1}\left(\sqrt{\frac{2x^2 - 1}{4x^2 - 1}}\right)} \qquad \ldots\ldots\ldots\ldots(II)$$

Similarly, **Normal distance** $(\boldsymbol{H_h})$ of **regular hexagonal face** EFGHIJ from the centre O of great rhombicuboctahedron is calculated as follows

In right $\Delta E O_2 O$ (figure 2)

$$OO_2 = \sqrt{(OE)^2 - (O_2E)^2} \qquad (\boldsymbol{O_2E = a} = circumscribed\ radius\ of\ regular\ hexagon)$$

$$\Rightarrow \boldsymbol{H_h} = \sqrt{(R_o)^2 - (a)^2} = \sqrt{\boldsymbol{R_o}^2 - \boldsymbol{a^2}} \qquad \ldots\ldots\ldots\ldots\ldots (III)$$

Hence, by substituting all the corresponding values, the solid angle $(\boldsymbol{\omega_h})$ subtended by each regular hexagonal face (EFGHIJ) at the centre of great rhombicuboctahedron is given as follows

$$\omega_h = 2\pi - 2 \times 6 \sin^{-1}\left(\frac{2\left(\sqrt{R_o{}^2 - a^2}\right) sin\frac{\pi}{6}}{\sqrt{4\left(\sqrt{R_o{}^2 - a^2}\right)^2 + a^2 cot^2\frac{\pi}{6}}}\right)$$

$$= 2\pi - 12 \sin^{-1}\left(\frac{2\sqrt{R_o{}^2 - a^2} \times \frac{1}{2}}{\sqrt{4R_o{}^2 - 4a^2 + a^2\left(\sqrt{3}\right)^2}}\right) = 2\pi - 12 \sin^{-1}\left(\frac{\sqrt{R_o{}^2 - a^2}}{\sqrt{4R_o{}^2 - a^2}}\right)$$

$$\Rightarrow \boldsymbol{\omega_h} = 2\pi - 12 \sin^{-1}\left(\sqrt{\frac{R_o{}^2 - a^2}{4R_o{}^2 - a^2}}\right) = \boldsymbol{2\pi - 12 \sin^{-1}\left(\sqrt{\frac{x^2 - 1}{4x^2 - 1}}\right)} \qquad \ldots\ldots\ldots\ldots\ldots (IV)$$

Similarly, **Normal distance** $(\boldsymbol{H_o})$ of **regular octagonal face** KLMNPQRS from the centre O of great rhombicuboctahedron is calculated as follows

In right $\Delta K O_3 O$ (figure 2)

$$\Rightarrow OO_3 = \sqrt{(OE)^2 - (O_3K)^2}$$

$$\boldsymbol{O_3K} = circumscribed\ radius\ of\ regular\ octagon = \boldsymbol{\frac{a}{2} cosec 22.5^o = \frac{a}{2}\sqrt{4 + 2\sqrt{2}}}$$

$$\Rightarrow \boldsymbol{H_o} = \sqrt{(R_o)^2 - \left(\frac{a}{2}\sqrt{4 + 2\sqrt{2}}\right)^2} = \sqrt{\frac{\boldsymbol{4R_o{}^2 - \left(4 + 2\sqrt{2}\right)a^2}}{\boldsymbol{4}}} \qquad \ldots\ldots\ldots\ldots\ldots (V)$$

Hence, by substituting all the corresponding values, the solid angle $(\boldsymbol{\omega_o})$ subtended by each regular octagonal face (KLMNPQRS) at the centre of great rhombicuboctahedron is given as follows

$$\omega_o = 2\pi - 2 \times 8 \sin^{-1}\left(\frac{2\left(\sqrt{\frac{4R_o{}^2 - \left(4 + 2\sqrt{2}\right)a^2}{4}}\right) sin\frac{\pi}{8}}{\sqrt{4\left(\sqrt{\frac{4R_o{}^2 - \left(4 + 2\sqrt{2}\right)a^2}{4}}\right)^2 + a^2 cot^2\frac{\pi}{8}}}\right)$$

$$= 2\pi - 16\sin^{-1}\left(\frac{\sqrt{4{R_o}^2 - (4+2\sqrt{2})a^2}\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}}{\sqrt{4{R_o}^2 - (4+2\sqrt{2})a^2 + a^2(\sqrt{2}+1)^2}}\right)$$

$$= 2\pi - 16\sin^{-1}\left(\frac{\sqrt{4\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right){R_o}^2 - (4+2\sqrt{2})\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)a^2}}{\sqrt{4{R_o}^2 - (4+2\sqrt{2})a^2 + (3+2\sqrt{2})a^2}}\right) = 2\pi - 16\sin^{-1}\left(\frac{\sqrt{(2-\sqrt{2}){R_o}^2 - a^2}}{\sqrt{4{R_o}^2 - a^2}}\right)$$

$$\Rightarrow \boldsymbol{\omega_o} = 2\pi - 16\sin^{-1}\left(\sqrt{\frac{(2-\sqrt{2}){R_o}^2 - a^2}{4{R_o}^2 - a^2}}\right) = \boldsymbol{2\pi - 16\sin^{-1}\left(\sqrt{\frac{(2-\sqrt{2})x^2 - 1}{4x^2 - 1}}\right)} \quad \ldots\ldots\ldots\ldots (VI)$$

Since a great rhombicuboctahedron is a **closed surface** & we know that **the total solid angle, subtended by any closed surface at any point lying inside it, is $4\pi\ sr$ (Ste-radian)** hence the sum of solid angles subtended by **12 congruent square faces, 8 congruent regular hexagonal faces & 6 congruent regular octagonal faces** at the centre of the great rhombicuboctahedron must be $4\pi\ sr$. Thus we have

$$\boldsymbol{12[\omega_s] + 8[\omega_h] + 6[\omega_o] = 4\pi}$$

Now, by substituting the values of $\boldsymbol{\omega_s}, \boldsymbol{\omega_h}$ & $\boldsymbol{\omega_o}$ from eq(II), (IV) & (VI) in the above expression we get

$$12\left[2\pi - 8\sin^{-1}\left(\sqrt{\frac{2x^2-1}{4x^2-1}}\right)\right] + 8\left[2\pi - 12\sin^{-1}\left(\sqrt{\frac{x^2-1}{4x^2-1}}\right)\right] + 6\left[2\pi - 16\sin^{-1}\left(\sqrt{\frac{(2-\sqrt{2})x^2-1}{4x^2-1}}\right)\right] = 4\pi$$

$$\Rightarrow 96\left[\sin^{-1}\left(\sqrt{\frac{2x^2-1}{4x^2-1}}\right) + \sin^{-1}\left(\sqrt{\frac{x^2-1}{4x^2-1}}\right) + \sin^{-1}\left(\sqrt{\frac{(2-\sqrt{2})x^2-1}{4x^2-1}}\right)\right] = 52\pi - 4\pi = 48\pi$$

$$\Rightarrow \sin^{-1}\left(\sqrt{\frac{2x^2-1}{4x^2-1}}\right) + \sin^{-1}\left(\sqrt{\frac{x^2-1}{4x^2-1}}\right) + \sin^{-1}\left(\sqrt{\frac{(2-\sqrt{2})x^2-1}{4x^2-1}}\right) = \frac{48\pi}{96} = \frac{\pi}{2}$$

$$\Rightarrow \sin^{-1}\left(\sqrt{\frac{2x^2-1}{4x^2-1}}\right) + \sin^{-1}\left(\sqrt{\frac{x^2-1}{4x^2-1}}\right) = \frac{\pi}{2} - \sin^{-1}\left(\sqrt{\frac{(2-\sqrt{2})x^2-1}{4x^2-1}}\right)$$

$$\Rightarrow \sin^{-1}\left(\sqrt{\frac{2x^2-1}{4x^2-1}}\sqrt{1-\left(\sqrt{\frac{x^2-1}{4x^2-1}}\right)^2} + \sqrt{\frac{x^2-1}{4x^2-1}}\sqrt{1-\left(\sqrt{\frac{2x^2-1}{4x^2-1}}\right)^2}\right) = \cos^{-1}\left(\sqrt{\frac{(2-\sqrt{2})x^2-1}{4x^2-1}}\right)$$

$$\left(\boldsymbol{since},\ \ \boldsymbol{\sin^{-1}X + \sin^{-1}Y = \sin^{-1}\left(X\sqrt{1-Y^2} + Y\sqrt{1-X^2}\right)}\ \ \&\ \ \boldsymbol{\frac{\pi}{2} - \sin^{-1}X = \cos^{-1}X}\right)$$

$$\Rightarrow \sin^{-1}\left(\sqrt{\frac{2x^2-1}{4x^2-1}}\sqrt{\frac{3x^2}{4x^2-1}}+\sqrt{\frac{x^2-1}{4x^2-1}}\sqrt{\frac{2x^2}{4x^2-1}}\right)=\sin^{-1}\left(\sqrt{1-\left(\sqrt{\frac{(2-\sqrt{2})x^2-1}{4x^2-1}}\right)^2}\right)$$

$$\Rightarrow \sin^{-1}\left(\frac{x\sqrt{3(2x^2-1)}}{4x^2-1}+\frac{x\sqrt{2(x^2-1)}}{4x^2-1}\right)=\sin^{-1}\left(\sqrt{\frac{(2+\sqrt{2})x^2}{4x^2-1}}\right)=\sin^{-1}\left(x\sqrt{\frac{(2+\sqrt{2})}{4x^2-1}}\right)$$

$$\Rightarrow \frac{x\sqrt{3(2x^2-1)}}{4x^2-1}+\frac{x\sqrt{2(x^2-1)}}{4x^2-1}=x\sqrt{\frac{(2+\sqrt{2})}{4x^2-1}}$$

$$\Rightarrow \sqrt{3(2x^2-1)}+\sqrt{2(x^2-1)}=\sqrt{(2+\sqrt{2})(4x^2-1)}$$

$$\Rightarrow \left(\sqrt{3(2x^2-1)}+\sqrt{2(x^2-1)}\right)^2=\left(\sqrt{(2+\sqrt{2})(4x^2-1)}\right)^2$$

$$\Rightarrow 3(2x^2-1)+2(x^2-1)+2\sqrt{6(x^2-1)(2x^2-1)}=(2+\sqrt{2})(4x^2-1)$$

$$\Rightarrow 4(2+\sqrt{2})x^2-(2+\sqrt{2})-8x^2+5=2\sqrt{6(x^2-1)(2x^2-1)}$$

$$\Rightarrow \left(4\sqrt{2}x^2+(3-\sqrt{2})\right)^2=\left(2\sqrt{6(x^2-1)(2x^2-1)}\right)^2$$

$$\Rightarrow 32x^4+8\sqrt{2}(3-\sqrt{2})x^2+(11-6\sqrt{2})=24(x^2-1)(2x^2-1)=48x^4-72x^2+24$$

$$\Rightarrow 16x^4-8(7+3\sqrt{2})x^2+(13+6\sqrt{2})=0$$

Now, solving the above **biquadratic equation for the values of $x>1$** as follows

$$\Rightarrow x^2=\frac{-\left(-8(7+3\sqrt{2})\right)\pm\sqrt{\left(-8(7+3\sqrt{2})\right)^2-4(16)(13+6\sqrt{2})}}{2\times 16}$$

$$=\frac{8(7+3\sqrt{2})\pm 8\sqrt{67+42\sqrt{2}-(13+6\sqrt{2})}}{32}=\frac{(7+3\sqrt{2})\pm\sqrt{54+36\sqrt{2}}}{4}=\frac{(7+3\sqrt{2})\pm 3\sqrt{6+4\sqrt{2}}}{4}$$

$$=\frac{(7+3\sqrt{2})\pm 3\sqrt{(2+\sqrt{2})^2}}{4}=\frac{(7+3\sqrt{2})\pm 3(2+\sqrt{2})}{4}$$

1. **Taking positive sign, we have**

$$x^2=\frac{(7+3\sqrt{2})+3(2+\sqrt{2})}{4}=\frac{13+6\sqrt{2}}{4}\quad or\ \ x=\sqrt{\frac{13+6\sqrt{2}}{4}}=\mathbf{\frac{1}{2}\sqrt{13+6\sqrt{2}}}$$

Since, $x>1$ hence, the above value is acceptable.

2. **Taking negative sign, we have**

$$x^2 = \frac{(7+3\sqrt{2}) - 3(2+\sqrt{2})}{4} = \frac{1}{4} \quad or \;\; x = \sqrt{\frac{1}{4}} = \frac{1}{2} \quad \Rightarrow \boldsymbol{x < 1 \;\; but \; x > 1 \; (required\ condition)}$$

Hence, the above value is discarded, now we have

$$x = \frac{1}{2}\sqrt{13+6\sqrt{2}} \;\Rightarrow\; \frac{R_o}{a} = x = \frac{1}{2}\sqrt{13+6\sqrt{2}} \quad or\; R_o = \frac{a}{2}\sqrt{13+6\sqrt{2}}$$

Hence, the **outer (circumscribed) radius $(\boldsymbol{R_o})$ of a great rhombicuboctahedron with edge length $\boldsymbol{a}$** is given as

$$\boldsymbol{R_o = \frac{a}{2}\sqrt{13+6\sqrt{2}} \approx 2.317610913a} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\; (VII)$$

Normal distance $(\boldsymbol{H_s})$ of square faces from the centre of great rhombicuboctahedron: The normal distance (H_s) of each of 12 congruent square faces from the centre O of a great rhombicuboctahedron is given from eq(I) as follows

$$H_s = OO_1 = \sqrt{\frac{2{R_o}^2 - a^2}{2}} = \sqrt{\frac{2\left(\frac{a}{2}\sqrt{13+6\sqrt{2}}\right)^2 - a^2}{2}} = a\sqrt{\frac{13+6\sqrt{2}-2}{4}} = \frac{a}{2}\sqrt{11+6\sqrt{2}} = \frac{a}{2}\sqrt{(3+\sqrt{2})^2}$$
$$= \frac{(3+\sqrt{2})a}{2}$$

$$\therefore\; \boldsymbol{H_s = \frac{(3+\sqrt{2})a}{2} \approx 2.207106781a}$$

It's clear that all 12 congruent square faces are at an equal normal distance $\boldsymbol{H_s}$ from the centre of any great rhombicuboctahedron.

Solid angle $(\boldsymbol{\omega_s})$ subtended by each of the square faces at the centre of great rhombicuboctahedron: solid angle $(\boldsymbol{\omega_s})$ subtended by each square face is given from eq(II) as follows

$$\omega_s = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{2x^2-a^2}{4x^2-a^2}}\right) = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{2{R_o}^2-a^2}{4{R_o}^2-a^2}}\right) \qquad \left(since,\;\; \boldsymbol{x = \frac{R_o}{a}}\right)$$

Hence, by substituting the corresponding value of R_o in the above expression, we get

$$\omega_s = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{2\left(\frac{a}{2}\sqrt{13+6\sqrt{2}}\right)^2 - a^2}{4\left(\frac{a}{2}\sqrt{13+6\sqrt{2}}\right)^2 - a^2}}\right) = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{13+6\sqrt{2}-2}{2(13+6\sqrt{2}-1)}}\right)$$

$$= 2\pi - 8\sin^{-1}\left(\sqrt{\frac{11+6\sqrt{2}}{12(2+\sqrt{2})}}\right) = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{(11+6\sqrt{2})(2-\sqrt{2})}{12(2+\sqrt{2})(2-\sqrt{2})}}\right) = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{10+\sqrt{2}}{24}}\right)$$

$$\therefore\; \boldsymbol{\omega_s = 2\pi - 8\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{10+\sqrt{2}}{6}}\right) = 4\sin^{-1}\left(\frac{2-\sqrt{2}}{12}\right) \approx 0.195339779\; sr}$$

Normal distance (H_h) of regular hexagonal faces from the centre of great rhombicuboctahedron:
The normal distance (H_h) of each of 8 congruent regular hexagonal faces from the centre O of a great rhombicuboctahedron is given from eq(III) as follows

$$H_h = OO_2 = \sqrt{{R_o}^2 - a^2} = \sqrt{\left(\frac{a}{2}\sqrt{13+6\sqrt{2}}\right)^2 - a^2} = a\sqrt{\frac{13+6\sqrt{2}-4}{4}} = \frac{a}{2}\sqrt{9+6\sqrt{2}} = \frac{a}{2}\sqrt{3\left(1+\sqrt{2}\right)^2}$$
$$= \frac{\sqrt{3}\left(1+\sqrt{2}\right)a}{2}$$

$$\therefore\ \boldsymbol{H_h = \frac{\sqrt{3}\left(1+\sqrt{2}\right)a}{2} \approx 2.090770275a}$$

It's clear that all 8 congruent regular hexagonal faces are at an equal normal distance H_h from the centre of any great rhombicuboctahedron.

Solid angle (ω_h) subtended by each of the regular hexagonal faces at the centre of great rhombicuboctahedron: solid angle $(\boldsymbol{\omega_h})$ subtended by each regular hexagonal face is given from eq(IV) as follows

$$\omega_h = 2\pi - 12\sin^{-1}\left(\sqrt{\frac{x^2-a^2}{4x^2-a^2}}\right) = 2\pi - 12\sin^{-1}\left(\sqrt{\frac{{R_o}^2-a^2}{4{R_o}^2-a^2}}\right) \qquad \left(since,\ \boldsymbol{x = \frac{R_o}{a}}\right)$$

Hence, by substituting the corresponding value of R_o in the above expression, we get

$$\omega_h = 2\pi - 12\sin^{-1}\left(\sqrt{\frac{\left(\frac{a}{2}\sqrt{13+6\sqrt{2}}\right)^2 - a^2}{4\left(\frac{a}{2}\sqrt{13+6\sqrt{2}}\right)^2 - a^2}}\right) = 2\pi - 12\sin^{-1}\left(\sqrt{\frac{13+6\sqrt{2}-4}{4\left(13+6\sqrt{2}-1\right)}}\right)$$

$$= 2\pi - 12\sin^{-1}\left(\sqrt{\frac{3+2\sqrt{2}}{8\left(2+\sqrt{2}\right)}}\right) = 2\pi - 12\sin^{-1}\left(\sqrt{\frac{\left(3+2\sqrt{2}\right)\left(2-\sqrt{2}\right)}{8\left(2+\sqrt{2}\right)\left(2-\sqrt{2}\right)}}\right) = 2\pi - 12\sin^{-1}\left(\sqrt{\frac{2+\sqrt{2}}{16}}\right)$$

$$\therefore\ \boldsymbol{\omega_h = 2\pi - 12\sin^{-1}\left(\frac{\sqrt{2+\sqrt{2}}}{4}\right) \approx 0.5210126\ sr}$$

Normal distance (H_o) of regular octagonal faces from the centre of great rhombicuboctahedron:
The normal distance (H_o) of each of 6 congruent regular octagonal faces from the centre O of a great rhombicuboctahedron is given from eq(V) as follows

$$H_o = OO_3 = \sqrt{\frac{4{R_o}^2 - \left(4+2\sqrt{2}\right)a^2}{4}} = \sqrt{\frac{4\left(\frac{a}{2}\sqrt{13+6\sqrt{2}}\right)^2 - \left(4+2\sqrt{2}\right)a^2}{4}} = \frac{a}{2}\sqrt{13+6\sqrt{2}-\left(4+2\sqrt{2}\right)}$$
$$= \frac{a}{2}\sqrt{9+4\sqrt{2}} = \frac{a}{2}\sqrt{\left(1+2\sqrt{2}\right)^2} = \frac{\left(1+2\sqrt{2}\right)a}{2}$$

$$\therefore\ \boldsymbol{H_o = \frac{\left(1+2\sqrt{2}\right)a}{2} \approx 1.914213562a}$$

It's clear that all 6 congruent regular octagonal faces are at an equal normal distance H_o from the centre of any great rhombicuboctahedron.

Solid angle (ω_o) subtended by each of the regular octagonal faces at the centre of great rhombicuboctahedron: solid angle (ω_o) subtended by each regular octagonal face is given from eq(VI) as follows

$$\omega_o = 2\pi - 16\sin^{-1}\left(\sqrt{\frac{(2-\sqrt{2})x^2-1}{4x^2-1}}\right) = 2\pi - 16\sin^{-1}\left(\sqrt{\frac{(2-\sqrt{2})R_o{}^2-a^2}{4R_o{}^2-a^2}}\right) \quad \left(since,\ \boldsymbol{x = \frac{R_o}{a}}\right)$$

Hence, by substituting the corresponding value of R_o in the above expression, we get

$$\omega_o = 2\pi - 16\sin^{-1}\left(\sqrt{\frac{(2-\sqrt{2})\left(\frac{a}{2}\sqrt{13+6\sqrt{2}}\right)^2-a^2}{4\left(\frac{a}{2}\sqrt{13+6\sqrt{2}}\right)^2-a^2}}\right) = 2\pi - 16\sin^{-1}\left(\sqrt{\frac{(2-\sqrt{2})(13+6\sqrt{2})-4}{4(13+6\sqrt{2}-1)}}\right)$$

$$= 2\pi - 16\sin^{-1}\left(\sqrt{\frac{10-\sqrt{2}}{24(2+\sqrt{2})}}\right) = 2\pi - 16\sin^{-1}\left(\sqrt{\frac{(10-\sqrt{2})(2-\sqrt{2})}{24(2+\sqrt{2})(2-\sqrt{2})}}\right)$$

$$= 2\pi - 16\sin^{-1}\left(\sqrt{\frac{22-12\sqrt{2}}{48}}\right) = 2\pi - 16\sin^{-1}\left(\sqrt{\frac{(3-\sqrt{2})^2}{24}}\right) = 2\pi - 16\sin^{-1}\left(\frac{3-\sqrt{2}}{2\sqrt{6}}\right)$$

$$\therefore\ \boldsymbol{\omega_o = 2\pi - 16\sin^{-1}\left(\frac{3-\sqrt{2}}{2\sqrt{6}}\right) \approx 1.009032076\ sr}$$

It's clear from the above results that the solid angle subtended by each of 6 regular octagonal faces is greater than the solid angle subtended by each of 12 square faces & each of 8 regular hexagonal faces at the centre of any great rhombicuboctahedron.

It's also clear from the above results that $\boldsymbol{H_s > H_h > H_o}$ i.e. the normal distance (H_s) of square faces is greater than the normal distance H_h of the regular hexagonal faces & the normal distance H_o of the regular octagonal faces from the centre of a great rhombicuboctahedron i.e. **regular octagonal faces are closer to the centre as compared to the square & regular hexagonal faces in any great rhombicuboctahedron**.

Important parameters of a great rhombicuboctahedron:

1. **Inner (inscribed) radius (R_i):** It is the radius of the largest sphere inscribed (trapped inside) by a great rhombicuboctahedron. The largest inscribed sphere always touches all 6 congruent regular octagonal faces but does not touch any of 12 congruent square & any of 8 congruent regular hexagonal faces at all since all 6 octagonal faces are closest to the centre in all the faces. Thus, inner radius is always equal to the normal distance (H_o) of regular octagonal faces from the centre of a great rhombicuboctahedron & is given as follows

$$\boldsymbol{R_i = H_o = \frac{(1+2\sqrt{2})a}{2} \approx 1.914213562a}$$

Hence, the **volume of inscribed sphere** is given as

$$\boldsymbol{V_{inscribed}} = \frac{4}{3}\pi(R_i)^3 = \boldsymbol{\frac{4}{3}\pi\left(\frac{(1+2\sqrt{2})a}{2}\right)^3 \approx 29.38054016a^3}$$

2. **Outer (circumscribed) radius (R_o):** It is the radius of the smallest sphere circumscribing a great rhombicuboctahedron or it's the radius of a spherical surface passing through all 48 vertices of a great rhombicuboctahedron. It is from the eq(VII) as follows

$$\boldsymbol{R_o = \frac{a}{2}\sqrt{13+6\sqrt{2}} \approx 2.317610913a}$$

Hence, the **volume of circumscribed sphere** is given as

$$\boldsymbol{V_{circumscribed}} = \frac{4}{3}\pi(R_o)^3 = \boldsymbol{\frac{4}{3}\pi\left(\frac{a}{2}\sqrt{13+6\sqrt{2}}\right)^3 = 52.14470211a^3}$$

3. **Surface area (A_s):** We know that a great rhombicuboctahedron has 12 congruent square faces, 8 congruent regular hexagonal faces & 6 congruent regular octagonal faces each of edge length a. Hence, its surface area is given as follows

$$A_s = 12(area\ of\ square) + 8(area\ of\ regular\ hexagon) + 6(area\ of\ regular\ octagon)$$

We know that **area of any regular n-polygon** with each side of length a is given as

$$\boldsymbol{A = \frac{1}{4}na^2 cot\frac{\pi}{n}}$$

Hence, by substituting all the corresponding values in the above expression, we get

$$A_s = 12 \times \left(\frac{1}{4} \times 4a^2 cot\frac{\pi}{4}\right) + 8 \times \left(\frac{1}{4} \times 6a^2 cot\frac{\pi}{6}\right) + 6 \times \left(\frac{1}{4} \times 8a^2 cot\frac{\pi}{8}\right)$$

$$= 12a^2 + 12\sqrt{3}a^2 + 12(1+\sqrt{2})a^2 = 12(2+\sqrt{2}+\sqrt{3})a^2$$

$$\boldsymbol{A_s = 12(2+\sqrt{2}+\sqrt{3})a^2 \approx 61.75517244a^2}$$

4. **Volume (V):** We know that a great rhombicuboctahedron with edge length a has 12 congruent square faces, 8 congruent regular hexagonal faces & 6 congruent regular octagonal faces. Hence, the volume (V) of the great rhombicuboctahedron is the sum of volumes of all its elementary right pyramids with square base, regular hexagonal base & regular octagonal base (face) (see figure 2 above) & is given as follows

$$\begin{aligned} V = {} & 12(volume\ of\ right\ pyramid\ with\ square\ base) \\ & + 8(volume\ of\ right\ pyramid\ with\ regular\ hexagonal\ base) \\ & + 6(volume\ of\ right\ pyramid\ with\ regular\ octagonal\ base) \end{aligned}$$

$$\begin{aligned} = {} & 12\left(\frac{1}{3}(area\ of\ square) \times H_s\right) + 8\left(\frac{1}{3}(area\ of\ regular\ hexagon) \times H_h\right) \\ & + 6\left(\frac{1}{3}(area\ of\ regular\ octagon) \times H_o\right) \end{aligned}$$

$$\begin{aligned} = {} & 12\left(\frac{1}{3}\left(\frac{1}{4} \times 4a^2 cot\frac{\pi}{4}\right) \times \frac{(3+\sqrt{2})a}{2}\right) + 8\left(\frac{1}{3}\left(\frac{1}{4} \times 6a^2 cot\frac{\pi}{6}\right) \times \frac{\sqrt{3}(1+\sqrt{2})a}{2}\right) \\ & + 6\left(\frac{1}{3}\left(\frac{1}{4} \times 8a^2 cot\frac{\pi}{8}\right) \times \frac{(1+2\sqrt{2})a}{2}\right) \end{aligned}$$

$$= 2(3+\sqrt{2})a^3 + 6(1+\sqrt{2})a^3 + 2(1+\sqrt{2})(1+2\sqrt{2})a^3 = (22+14\sqrt{2})a^3$$

$$\boldsymbol{V = (22+14\sqrt{2})a^3 \approx 41.79898987a^3}$$

5. **Mean radius (R_m):** It is the radius of the sphere having a volume equal to that of a great rhombicuboctahedron. It is calculated as follows

$$volume\ of\ sphere\ with\ mean\ radius\ R_m = volume\ of\ the\ great\ rhombicuboctahedron$$

$$\frac{4}{3}\pi(R_m)^3 = (22+14\sqrt{2})a^3 \quad \Rightarrow (R_m)^3 = \frac{3(11+7\sqrt{2})a^3}{2\pi} \quad or\ R_m = a\left(\frac{3(11+7\sqrt{2})}{2\pi}\right)^{\frac{1}{3}}$$

$$\boldsymbol{R_m = a\left(\frac{3(11+7\sqrt{2})}{2\pi}\right)^{\frac{1}{3}} \approx 2.15290926a}$$

It's clear from above results that $\boldsymbol{R_i < R_m < R_o}$

6. **Dihedral angles between the adjacent faces:** In order to calculate dihedral angles between the different adjacent faces with a common edge in a great rhombicuboctahedron, let's consider one-by-one all three pairs of adjacent faces with a common edge as follows

a. **Angle between square face & regular hexagonal face:** Draw the perpendiculars OO_1 & OO_2 from the centre O of great rhombicuboctahedron to the square face & the regular hexagonal face which have a common edge (See figure 3). We know that the inscribed radius (r_i) of any regular n-gon with each side a is given as follows

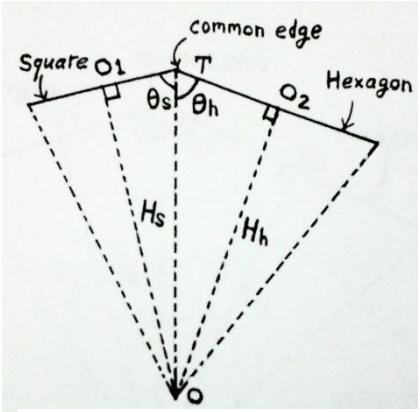

Figure 3: Square face with centre O_1 & regular hexagonal face with centre O_2 with a common edge (denoted by point T) normal to the plane of paper

$$\boldsymbol{r_i = inscribed\ radius\ of\ any\ regular\ n-gon\ = \frac{a}{2}cot\frac{\pi}{n}}$$

$$\therefore\ O_1T = inscribed\ radius\ of\ square = \frac{a}{2}cot\frac{\pi}{4} = \frac{a}{2}\ \ \&$$

$$\therefore\ O_2T = inscribed\ radius\ of\ regular\ hexagon = \frac{a}{2}cot\frac{\pi}{6} = \frac{a\sqrt{3}}{2}$$

In right ΔOO_1T

$$tan\theta_s = \frac{OO_1}{O_1T} = \frac{H_s}{\left(\frac{a}{2}\right)} = \frac{\left(\frac{(3+\sqrt{2})a}{2}\right)}{\left(\frac{a}{2}\right)} = (3+\sqrt{2})$$

$$\boldsymbol{\therefore\ \theta_s = \tan^{-1}(3+\sqrt{2}) \approx 77.23561032^o} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (VIII)$$

In right ΔOO_2T

$$tan\theta_h = \frac{OO_2}{O_2T} = \frac{H_h}{\left(\frac{a\sqrt{3}}{2}\right)} = \frac{\left(\frac{\sqrt{3}(1+\sqrt{2})a}{2}\right)}{\left(\frac{a\sqrt{3}}{2}\right)} = (1+\sqrt{2})$$

$$\therefore\ \boldsymbol{\theta_h = \tan^{-1}(1+\sqrt{2}) = 67.5^o} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (IX)$$

$$\Rightarrow \theta_s + \theta_h = \tan^{-1}(3+\sqrt{2}) + \tan^{-1}(1+\sqrt{2}) = \tan^{-1}\left(\frac{(3+\sqrt{2})+(1+\sqrt{2})}{1-(3+\sqrt{2})(1+\sqrt{2})}\right) = \tan^{-1}\left(\frac{4+2\sqrt{2}}{1-(5+4\sqrt{2})}\right)$$

$$= \tan^{-1}\left(\frac{-(4+2\sqrt{2})}{4+4\sqrt{2}}\right) = \pi - \tan^{-1}\left(\frac{4+2\sqrt{2}}{4+4\sqrt{2}}\right) = \pi - \tan^{-1}\left(\frac{1+\sqrt{2}}{2+\sqrt{2}}\right) = \pi - \tan^{-1}\left(\frac{(1+\sqrt{2})(2-\sqrt{2})}{(2+\sqrt{2})(2-\sqrt{2})}\right)$$

$$= \pi - \tan^{-1}\left(\frac{\sqrt{2}}{2}\right) = \pi - \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

Hence**, dihedral angle between the square face & the regular hexagonal face is given as**

$$\boldsymbol{\theta_s + \theta_h = \pi - \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) \approx 144.7356103^o}$$

b. Angle between square face & regular octagonal face: Draw the perpendiculars OO_1 & OO_3 from the centre O of great rhombicuboctahedron to the square face & the regular octagonal face which have a common edge (See figure 4).

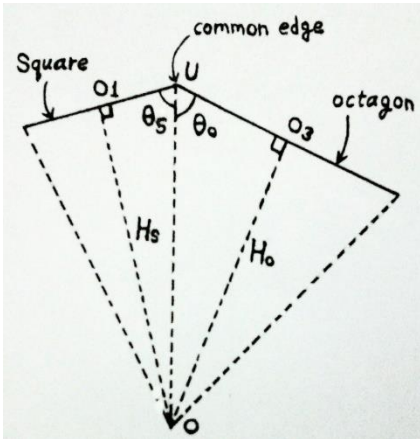

Figure 4: Square face with centre O_1 & regular octagonal face with centre O_3 with a common edge (denoted by point U) normal to the plane of paper

$$O_3U = inscribed\ radius\ of\ regular\ octagon = \frac{a}{2}cot\frac{\pi}{8} = \frac{a(1+\sqrt{2})}{2}$$

In right ΔOO_3U

$$tan\theta_o = \frac{OO_3}{O_3U} = \frac{H_o}{\left(\frac{a(1+\sqrt{2})}{2}\right)} = \frac{\left(\frac{(1+2\sqrt{2})a}{2}\right)}{\left(\frac{a(1+\sqrt{2})}{2}\right)} = \frac{(1+2\sqrt{2})}{(1+\sqrt{2})}$$

$$= \frac{(1+2\sqrt{2})(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)} = \frac{3-\sqrt{2}}{(2-1)} = 3-\sqrt{2}$$

$$\therefore\ \boldsymbol{\theta_o = \tan^{-1}(3-\sqrt{2}) \approx 57.76438969^o} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (X)$$

$$\Rightarrow \theta_s + \theta_o = \tan^{-1}(3+\sqrt{2}) + \tan^{-1}(3-\sqrt{2}) = \tan^{-1}\left(\frac{(3+\sqrt{2})+(3-\sqrt{2})}{1-(3+\sqrt{2})(3-\sqrt{2})}\right) = \tan^{-1}\left(\frac{6}{1-7}\right)$$

$$= \tan^{-1}\left(\frac{-6}{6}\right) = \tan^{-1}(-1) = \pi - \tan^{-1}(1) = 180^o - 45^o = 135^o$$

Hence**, dihedral angle between the square face & the regular octagonal face is given as**

$$\boldsymbol{\theta_s + \theta_o = 135^o}$$

c. Angle between regular hexagonal face & regular octagonal face: Draw the perpendiculars OO_2 & OO_3 from the centre O of great rhombicuboctahedron to the regular hexagonal face & the regular octagonal face which have a common edge (See figure 5). Now from eq(IX) & (X), we get

$$\theta_h + \theta_o = \tan^{-1}(1+\sqrt{2}) + \tan^{-1}(3-\sqrt{2}) = \tan^{-1}\left(\frac{(1+\sqrt{2})+(3-\sqrt{2})}{1-(1+\sqrt{2})(3-\sqrt{2})}\right)$$

$$= \tan^{-1}\left(\frac{4}{1-(1+2\sqrt{2})}\right) = \tan^{-1}\left(\frac{-4}{2\sqrt{2}}\right) = \tan^{-1}(-\sqrt{2}) = \pi - \tan^{-1}(\sqrt{2})$$

Hence, **dihedral angle between the regular hexagonal face & the regular octagonal face is given as**

$$\boldsymbol{\theta_h + \theta_o = \pi - \tan^{-1}(\sqrt{2}) \approx 125.2643897^o}$$

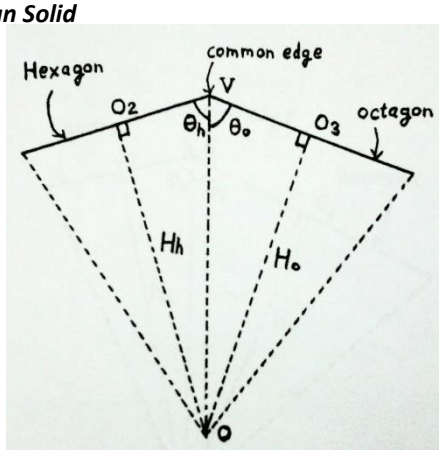

Figure 5: Regular hexagonal face with centre O_2 & regular octagonal face with centre O_3 with a common edge (denoted by point V) normal to the plane of paper

Construction of a solid great rhombicuboctahedron: In order to construct a solid great rhombicuboctahedron with edge length a there are two methods

1. Construction from elementary right pyramids: In this method, first we construct all elementary right pyramids as follows

Construct 12 congruent right pyramids with square base of side length a & normal height (H_s)

$$H_s = \frac{(3+\sqrt{2})a}{2} \approx 2.207106781a$$

Construct 8 congruent right pyramids with regular hexagonal base of side length a & normal height (H_h)

$$H_h = \frac{\sqrt{3}(1+\sqrt{2})a}{2} \approx 2.090770275a$$

Construct 6 congruent right pyramids with regular octagonal base of side length a & normal height (H_o)

$$H_o = \frac{(1+2\sqrt{2})a}{2} \approx 1.914213562a$$

Now, paste/bond by joining all these elementary right pyramids by overlapping their lateral surfaces & keeping their apex points coincident with each other such that 4 edges of each square base (face) coincide with the edges of 2 regular hexagonal bases & 2 regular octagonal bases (faces). Thus a solid great rhombicuboctahedron, with 12 congruent square faces, 8 congruent regular hexagonal faces & 6 congruent regular octagonal faces each of edge length a, is obtained.

2. Facing a solid sphere: It is a method of facing, first we select a **blank as a solid sphere** of certain material (i.e. metal, alloy, composite material etc.) & with suitable diameter in order to obtain the maximum desired edge length of a great rhombicuboctahedron. Then, we perform the facing operations on the solid sphere to generate 12 congruent square faces, 8 congruent regular hexagonal faces & 6 congruent regular octagonal faces each of equal edge length.

Let there be a blank as a solid sphere with a diameter D. Then the edge length a, of a great rhombicuboctahedron of the maximum volume to be produced, can be co-related with the diameter D by **relation of outer radius (R_o) with edge length (a) of the great rhombicuboctahedron** as follows

$$R_o = \frac{a}{2}\sqrt{13+6\sqrt{2}}$$

Now, substituting $R_o = {}^{D}/_{2}$ in the above expression, we have

$$\frac{D}{2} = \frac{a}{2}\sqrt{13+6\sqrt{2}} \quad or \quad a = \frac{D}{\sqrt{13+6\sqrt{2}}}$$

$$\boldsymbol{a = \frac{D}{\sqrt{13+6\sqrt{2}}} \approx 0.215739405D}$$

Above relation is very useful for determining the edge length a of a great rhombicuboctahedron to be produced from a solid sphere with known diameter D for manufacturing purpose.

Hence, the **maximum volume of great rhombicuboctahedron** produced from a solid sphere is given as follows

$$V_{max} = (22+14\sqrt{2})a^3 = (22+14\sqrt{2})\left(\frac{D}{\sqrt{13+6\sqrt{2}}}\right)^3 = \frac{(22+14\sqrt{2})D^3}{(13+6\sqrt{2})\sqrt{13+6\sqrt{2}}}$$

$$= \frac{(22+14\sqrt{2})(13-6\sqrt{2})D^3}{97\sqrt{13+6\sqrt{2}}} = \frac{(118+50\sqrt{2})D^3}{97\sqrt{13+6\sqrt{2}}}$$

$$\boldsymbol{V_{max} = \frac{2(59+25\sqrt{2})D^3}{97\sqrt{13+6\sqrt{2}}} \approx 0.419714736D^3}$$

M**inimum volume of material removed** is given as

$$(V_{removed})_{min} = (volume\ of\ parent\ sphere\ with\ diameter\ D) - (volume\ of\ great\ rhombicuboctahedron)$$

$$= \frac{\pi}{6}D^3 - \frac{2(59+25\sqrt{2})D^3}{97\sqrt{13+6\sqrt{2}}} = \left(\frac{\pi}{6} - \frac{2(59+25\sqrt{2})D^3}{97\sqrt{13+6\sqrt{2}}}\right)D^3$$

$$\boldsymbol{(V_{removed})_{min} = \left(\frac{\pi}{6} - \frac{2(59+25\sqrt{2})}{97\sqrt{13+6\sqrt{2}}}\right)D^3 \approx 0.103884038D^3}$$

Percentage $(\%)$ **of minimum volume of material removed**

$$\%\ \boldsymbol{of\ V_{removed}} = \frac{minimum\ volume\ removed}{total\ volume\ of\ sphere} \times 100$$

$$= \frac{\left(\frac{\pi}{6} - \frac{2(59+25\sqrt{2})}{97\sqrt{13+6\sqrt{2}}}\right)D^3}{\frac{\pi}{6}D^3} \times 100 = \boldsymbol{\left(1 - \frac{12(59+25\sqrt{2})}{97\pi\sqrt{13+6\sqrt{2}}}\right) \times 100 \approx 19.84\%}$$

It's obvious that when a great rhombicuboctahedron of the maximum volume is produced from a solid sphere then about 19.84% volume of material is removed as scraps. Thus, we can select optimum diameter of blank as a solid sphere to produce a solid great rhombicuboctahedron of the maximum volume (or with maximum desired edge length)

Conclusions: Let there be any great rhombicuboctahedron having 12 congruent square faces, 8 congruent regular hexagonal faces & 6 congruent regular octagonal faces each with edge length a then all its important parameters are calculated/determined as tabulated below

Congruent polygonal faces	No. of faces	Normal distance of each face from the centre of the great rhombicuboctahedron	Solid angle subtended by each face at the centre of the great rhombicuboctahedron
Square	12	$\frac{(3+\sqrt{2})a}{2} \approx 2.207106781a$	$4\sin^{-1}\left(\frac{2-\sqrt{2}}{12}\right) \approx 0.195339779\ sr$
Regular hexagon	8	$\frac{\sqrt{3}(1+\sqrt{2})a}{2} \approx 2.090770275a$	$2\pi - 12\sin^{-1}\left(\frac{\sqrt{2+\sqrt{2}}}{4}\right) \approx 0.5210126\ sr$
Regular octagon	6	$\frac{(1+2\sqrt{2})a}{2} \approx 1.914213562a$	$2\pi - 16\sin^{-1}\left(\frac{3-\sqrt{2}}{2\sqrt{6}}\right) \approx 1.009032076\ sr$

Inner (inscribed) radius (R_i)	$R_i = \frac{(1+2\sqrt{2})a}{2} \approx 1.914213562a$
Outer (circumscribed) radius (R_o)	$R_o = \frac{a}{2}\sqrt{13+6\sqrt{2}} \approx 2.317610913a$
Mean radius (R_m)	$R_m = a\left(\frac{3(11+7\sqrt{2})}{2\pi}\right)^{\frac{1}{3}} \approx 2.15290926a$
Surface area (A_s)	$A_s = 12(2+\sqrt{2}+\sqrt{3})a^2 \approx 61.75517244a^2$
Volume (V)	$V = (22+14\sqrt{2})a^3 \approx 41.79898987a^3$

Table for the dihedral angles between the adjacent faces of a great rhombicuboctahedron

Pair of the adjacent faces with a common edge	Square & regular hexagon	Square & regular octagon	Regular hexagon & regular octagon
Dihedral angle of the corresponding pair (of the adjacent faces)	$\theta_s + \theta_h = \pi - \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) \approx 144.7356103^o$	$\theta_s + \theta_o = 135^o$	$\theta_h + \theta_o = \pi - \tan^{-1}(\sqrt{2}) \approx 125.2643897^o$

Note: Above articles had been developed & *illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)*

M.M.M. University of Technology, Gorakhpur-273010 (UP) India *March, 2015*

Email: rajpootharishchandra@gmail.com

Author's Home Page: https://notionpress.com/author/HarishChandraRajpoot