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Mathematical Analysis of Elliptical Path in the Annular Region Between Two Circles, Smaller Inside the Bigger One (Ellipse Between Two Circles by H.C. Rajpoot)

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Mathematical Analysis of Elliptical Path in the Annulus of Two Circles

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1. Introduction:

When a smaller circle exactly lies inside the bigger one (either touching or not touching internally) with their centres separated by a certain distance, then the **locus of the centre of a third tangent (parametric) circle**, **in the annular region between them, touching the smaller circle externally & the bigger one internally is always an ellipse having the centres of former circles as its foci & its centre at the midpoint of the centres of former circles.** This path becomes a circle if both the smaller & bigger circles are concentric i.e. centres of both the circles, smaller inside bigger, are coincident. (See figure 1 below, showing an elliptical path (dotted curve A'BAB'A') in the annular region between two circles, smaller inside bigger, with their centres separated by a certain distance)

2. Derivation of major axis, minor axis, eccentricity & equation of elliptical path in the annular region between two circles, one inside other, with their centres separated by a certain distance:

Consider a smaller circle, with a radius r, centred at the origin (point O) completely inside the bigger one, with a radius R, centred at the point O' such that distance between their centres O & O' is d. Now consider an **elliptical path** (dotted curve A'BAB'A' with the centre C) traced by the centre of a third (parametric) circle, in the annular region between them, which touches the smaller circle externally & the bigger one internally.

For convenience, let's assume

$$major\ axis,\ \ AA' = 2a \quad \& \quad minor\ axis,\ \ BB' = 2b$$

Now, we have

$$OE = OF = r,\ O'D = O'G = R\ \ \&\ OO' = d$$

$$DE = O'D - O'E = O'D - (O'O + OE)$$
$$= R - (d + r)$$

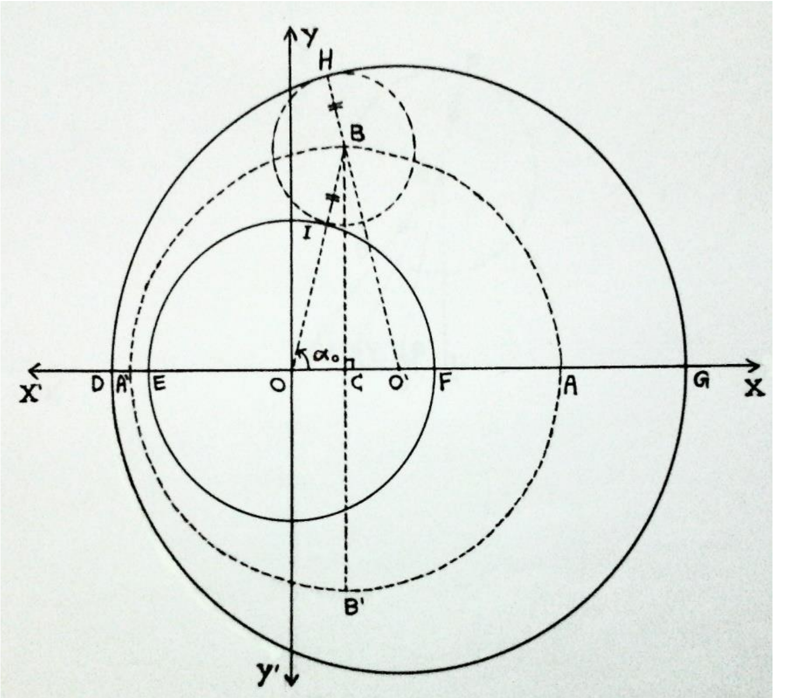

Figure 1: Elliptical path (dotted curve A'BAB'A' with the centre C) is the locus of the centre of a third tangent (parametric) circle, in the annular region between two circles (smaller inside bigger) with their centres separated by a certain distance d, touching the smaller circle (with the centre O) externally & the bigger one (with the centre O') externally.

$$\therefore \boldsymbol{A'E} = \frac{DE}{2} = \frac{R-(d+r)}{2} = \frac{\boldsymbol{R-d-r}}{\boldsymbol{2}}$$

$$FG = OG - OF = OO' + O'G - OF = d + R - r$$

$$\therefore \boldsymbol{FA} = \frac{FG}{2} = \frac{\boldsymbol{R+d-r}}{\boldsymbol{2}}$$

$$\Rightarrow\ A'A = A'E + OE + OF + FA = \frac{R-d-r}{2} + r + r + \frac{R+d-r}{2} = \frac{R-d-r+4r+R+d-r}{2} = R + r$$

$$\therefore \boldsymbol{Major\ axis},\quad \boldsymbol{2a = AA' = R + r} \quad \ldots\ldots\ldots\ldots\ldots..(I)$$

Now, join the vertex B of the transverse axis BCB’ to the origin O so as the line OB intersects the smaller circle at the point I & then join B to the centre O’ & extend the line O’B to intersect the bigger circle at the point H. Now, draw a third tangent circle, with a radius a & centre at the vertex B, which touches the smaller circle externally & the bigger one internally. Thus we have

$$BC = semi\ minor\ axis = b,\ \ BI = BH = radius\ of\ tangent\ circle = a \quad \&$$

$$\Rightarrow\ OC = OG - CG = (OF + FG) - (CA + AG)$$

$$= (OF + FG) - \left(\frac{AA'}{2} + FA\right) \qquad \left(CA = CA' = \frac{AA'}{2}\ \ \&\ AG = FA\right)$$

$$OC = (r + R + d - r) - \left(\frac{R + r}{2} + \frac{R + d - r}{2}\right) = \frac{2R + 2d - 2R - d}{2} = \frac{d}{2}$$

$$\therefore\ CO' = OO' - OC = d - \frac{d}{2} = \frac{d}{2} \qquad \Rightarrow\ \boldsymbol{CO = CO' = \frac{d}{2}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (II)$$

Thus, the centre C is the midpoint of line segment OO’. Now in the right triangles BCO & BCO’ we have

$$CO = CO' = \frac{d}{2}\ ,\ \ BO = BO = b\ \ \&\ \angle BCO = \angle BCO' = 90^o$$

Hence, the right triangles BCO & BCO’ are congruent.

In right ΔBCO

$$OB = \sqrt{(BC)^2 + (OC)^2} = \sqrt{b^2 + \left(\frac{d}{2}\right)^2} = \sqrt{\frac{4b^2 + d^2}{4}} = \frac{\sqrt{4b^2 + d^2}}{2}$$

$$\Rightarrow\ OB = OI + IB = r + a \quad \therefore\ \boldsymbol{r + a = \frac{\sqrt{4b^2 + d^2}}{2}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (III)$$

$$but,\ \ OB = O'B = O'H - BH = R - a \qquad (since,\ \ right\ triangles\ BCO\ \&\ BCO'\ are\ congruent)$$

Now, equating the value of OB from the eq(III), we get

$$R - a = r + a\ \ \Rightarrow\ 2a = R - r \quad or\ \ a = \frac{R - r}{2}$$

Now, substituting the value of a in the above eq(III), we get

$$r + \frac{R - r}{2} = \frac{\sqrt{4b^2 + d^2}}{2} \quad or \quad \frac{\sqrt{4b^2 + d^2}}{2} = \frac{R + r}{2}$$

$$\Rightarrow\ \ 4b^2 + d^2 = (R + r)^2\ \ or\ \ \ 2b = \sqrt{(R + r)^2 - d^2}$$

$$\therefore\ \ \boldsymbol{Minor\ axis,} \qquad \boldsymbol{2b = BB' = \sqrt{(R + r)^2 - d^2}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (IV)$$

$$\Rightarrow\ Eccentricity,\ \ e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \left(\frac{2b}{2a}\right)^2} = \sqrt{1 - \left(\frac{\sqrt{(R + r)^2 - d^2}}{(R + r)}\right)^2}$$

$$= \sqrt{1 - \frac{(R+r)^2 - d^2}{(R+r)^2}} = \sqrt{1 - 1 + \frac{d^2}{(R+r)^2}} = \sqrt{\frac{d^2}{(R+r)^2}} = \frac{d}{R+r}$$

$$\therefore \ \boldsymbol{Eccentricity,\ e = \frac{d}{R+r} \quad (0 \le e < 1 \quad \forall\ R \ge d+r)} \qquad \dots\dots\dots\dots\dots\dots \ (V)$$

Equation of elliptical path: Since, the elliptical path has its centre at the point $C\left({}^{d}/_{2}, 0\right)$, major axis $2a$ & minor axis $2b$, hence the equation of the elliptical path, assuming centre of the smaller circle as the origin, can be obtained by using **standard formula of ellipse centred at the origin** as follows

$$\frac{\boldsymbol{X^2}}{\boldsymbol{A^2}} + \frac{\boldsymbol{Y^2}}{\boldsymbol{B^2}} = \boldsymbol{1} \quad \Rightarrow \frac{\left(x - \frac{d}{2}\right)^2}{a^2} + \frac{(y-0)^2}{b^2} = 1 \quad or \quad \frac{\left(x - \frac{d}{2}\right)^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\therefore \boldsymbol{Equation\ of\ elliptical\ path,\ \frac{\left(x - \frac{d}{2}\right)^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \forall\ a = \frac{R+r}{2} \ \ \&\ \ b = \sqrt{(R+r)^2 - d^2}}$$

Above equation is very useful for determining the co-ordinates of any point on the elliptical path & the radius of the parametric circle with centre at any arbitrary point on the elliptical path.

Now, the distance of foci from the centre C of elliptical path, can be determined as follows

$$X = \pm ae \quad \Rightarrow \quad x - \frac{d}{2} = \pm\left(\frac{R+r}{2}\right)\left(\frac{d}{R+r}\right) \quad or \quad x = \frac{d}{2} \pm \frac{d}{2} \ \Rightarrow \ \boldsymbol{x = 0 \ \&\ x = d}$$

$$but\ \ x = 0 \ \ \&\ \ x = d \quad represents\ the\ coordinates\ of\ the\ centres\ O\ \&\ O'\ \ respectively$$

Hence, it's obvious from the above values that **the centres of two circles are always the foci of the elliptical path in the annulus of these circles, smaller inside bigger, with their centres separated by a certain distance.**

3. Radius of the tangent circle with centre at the vertex B: The radius (α) of tangent circle, having centre at the vertex B of the transverse axis BCB' of the elliptical path, touching the smaller circle externally & the bigger one internally (See the above figure 1), is given as

$$\boldsymbol{a = \frac{R-r}{2} \qquad \forall\ R \ge d + r}$$

And **angle $\boldsymbol{\alpha_o}$** of the line BO joining the centre B of the tangent circle to the origin O with the x-axis (line OO')

In right ΔBCO

$$tan\angle BOC = \frac{BC}{OC} \quad \Rightarrow \quad tan\alpha_o = \frac{b}{\left(\frac{d}{2}\right)} = \frac{2\left(\frac{\sqrt{(R+r)^2 - d^2}}{2}\right)}{d} = \frac{\sqrt{(R+r)^2 - d^2}}{d} = \sqrt{\left(\frac{R+r}{d}\right)^2 - 1}$$

$$or\ \ tan\alpha_o = \sqrt{\left(\frac{1}{e}\right)^2 - 1} = \sqrt{\frac{1-e^2}{e^2}} = \frac{\sqrt{1-e^2}}{e} \quad or \quad \alpha_o = \tan^{-1}\left(\frac{\sqrt{1-e^2}}{e}\right)$$

$$\therefore \ \boldsymbol{\alpha_o = \tan^{-1}\left(\sqrt{\left(\frac{R+r}{d}\right)^2 - 1}\right) = \tan^{-1}\left(\frac{\sqrt{1-e^2}}{e}\right) \quad \forall\ e = \frac{d}{R+r} \ \ \&\ \ R \ge d + r}$$

Above, equation can be used to calculate angle $\boldsymbol{\alpha_o}$ for the tangent circle with centre at the vertex B.

4. Derivation of the radius of tangent (parametric) circle touching the smaller circle externally & the bigger one internally: Consider a **tangent (parametric) circle** with the centre at **any arbitrary point say E on the elliptical path** such that it touches the smaller circle at the point D externally & the bigger one at the point F internally. Now join the centre E of the tangent circle to the origin O, centre C & the centre O' by the dotted lines & also draw a perpendicular EM on the x-axis (see the figure 2) such that

$$\angle EOM = \alpha, \ \angle ECM = \gamma \ \& \ \angle EO'M = \beta$$

Now, the **equation of line OE** passing through the origin O & inclined at an angle α with the positive direction of x-axis is given as follows

$$\boldsymbol{y = mx = x\tan\alpha \quad \forall \ \ 0 \le \alpha \le \pi}$$

Now, solving the equation of the line OE & the equation of elliptical path to calculate the co-ordinates of the point of intersection E of the ellipse & the line by substituting $x = y\cot\alpha$ in the equation of elliptical path as follows

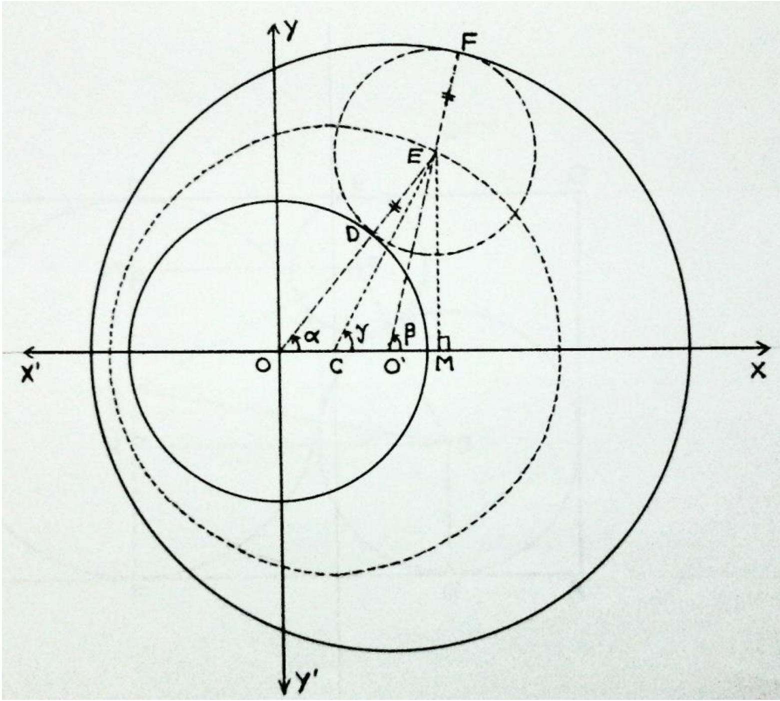

Figure 2: A tangent (parametric) circle, with the centre at any arbitrary point E on the elliptical path, is considered such that the angle of the line OE with the positive direction of x-axis is α in order to calculate its radius

$$\frac{\left(x-\frac{d}{2}\right)^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{\left(y\cot\alpha-\frac{d}{2}\right)^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\Rightarrow \frac{\left(y\cot\alpha-\frac{d}{2}\right)^2}{\left(\frac{R+r}{2}\right)^2}+\frac{y^2}{\left(\frac{\sqrt{(R+r)^2-d^2}}{2}\right)^2}=1 \quad (on\ substituting\ the\ values\ of\ a\ \&\ b)$$

$$or \left(\frac{\sqrt{(R+r)^2-d^2}}{2}\right)^2\left(y\cot\alpha-\frac{d}{2}\right)^2+\left(\frac{R+r}{2}\right)^2 y^2=\left(\frac{R+r}{2}\right)^2\left(\frac{\sqrt{(R+r)^2-d^2}}{2}\right)^2$$

$$\Rightarrow \left(\frac{(R+r)^2-d^2}{4}\right)\left(y^2\cot^2\alpha+\frac{d^2}{4}-yd\cot\alpha\right)+\frac{(R+r)^2}{4}y^2=\frac{(R+r)^2}{4}\left(\frac{(R+r)^2-d^2}{4}\right)$$

$$\Rightarrow \left\{\left(\frac{(R+r)^2-d^2}{4}\right)\cot^2\alpha+\frac{(R+r)^2}{4}\right\}y^2-\left\{\left(\frac{(R+r)^2-d^2}{4}\right)d\cot\alpha\right\}y+\left(\frac{(R+r)^2-d^2}{4}\right)\frac{d^2}{4}-\frac{(R+r)^2}{4}\left(\frac{(R+r)^2-d^2}{4}\right)=0$$

$$\left\{\frac{(R+r)^2 cosec^2\alpha}{4}-\frac{d^2}{4}\cot^2\alpha\right\}y^2-\left\{\left(\frac{(R+r)^2-d^2}{4}\right)d\cot\alpha\right\}y-\left(\frac{(R+r)^2-d^2}{4}\right)\left(\frac{(R+r)^2}{4}-\frac{d^2}{4}\right)=0$$

$$\Rightarrow \left\{\frac{(R+r)^2 cosec^2\alpha}{4}-\frac{d^2}{4}\cot^2\alpha\right\}y^2-\left\{\left(\frac{(R+r)^2-d^2}{4}\right)d\cot\alpha\right\}y-\left(\frac{(R+r)^2-d^2}{4}\right)^2=0$$

Now, solving above quadratic for the values of y as follows

$$y = \frac{\left(\frac{(R+r)^2-d^2}{4}\right)dcot\alpha \pm \sqrt{\left\{\left(\frac{(R+r)^2-d^2}{4}\right)dcot\alpha\right\}^2 + 4\left\{\frac{(R+r)^2cosec^2\alpha}{4} - \frac{d^2}{4}cot^2\alpha\right\}\left(\frac{(R+r)^2-d^2}{4}\right)^2}}{2\left\{\frac{(R+r)^2cosec^2\alpha}{4} - \frac{d^2}{4}cot^2\alpha\right\}}$$

$$= \frac{\left(\frac{(R+r)^2-d^2}{4}\right)dcot\alpha \pm \left(\frac{(R+r)^2-d^2}{4}\right)\sqrt{d^2cot^2\alpha + (R+r)^2cosec^2\alpha - d^2cot^2\alpha}}{\frac{1}{2}\{(R+r)^2cosec^2\alpha - d^2cot^2\alpha\}}$$

$$= 2\left(\frac{(R+r)^2-d^2}{4}\right)\left(\frac{dcot\alpha \pm (R+r)cosec\alpha}{(R+r)^2cosec^2\alpha - d^2cot^2\alpha}\right) = \left(\frac{(R+r)^2-d^2}{2}\right)\left(\frac{dcos\alpha \pm (R+r)}{(R+r)^2 - d^2cos^2\alpha}\right)sin\alpha$$

Taking positive sign, we get

$$y = \left(\frac{(R+r)^2-d^2}{2}\right)\left(\frac{dcos\alpha + (R+r)}{(R+r)^2 - d^2cos^2\alpha}\right)sin\alpha$$

$$= \left(\frac{(R+r)^2-d^2}{2}\right)\left(\frac{dcos\alpha + (R+r)}{\big((R+r)+dcos\alpha\big)\big((R+r)-dcos\alpha\big)}\right)sin\alpha = \frac{sin\alpha}{2}\left(\frac{(R+r)^2-d^2}{(R+r)-dcos\alpha}\right)$$

$$\Rightarrow\ y = \frac{sin\alpha}{2}\left(\frac{(R+r)^2-d^2}{(R+r)-dcos\alpha}\right) \geq 0\quad \forall\ 0 \leq \alpha \leq \pi\ \ \boldsymbol{hence\ this\ value\ of\ y\ is\ accepted}$$

$$\therefore\ x = ycot\alpha = \frac{sin\alpha cot\alpha}{2}\left(\frac{(R+r)^2-d^2}{(R+r)-dcos\alpha}\right) = \frac{cos\alpha}{2}\left(\frac{(R+r)^2-d^2}{(R+r)-dcos\alpha}\right)$$

Taking negative sign, we get

$$y = \left(\frac{(R+r)^2-d^2}{2}\right)\left(\frac{dcos\alpha - (R+r)}{(R+r)^2 - d^2cos^2\alpha}\right)sin\alpha$$

$$= \left(\frac{(R+r)^2-d^2}{2}\right)\left(\frac{dcos\alpha - (R+r)}{\big((R+r)+dcos\alpha\big)\big((R+r)-dcos\alpha\big)}\right)sin\alpha = \frac{-sin\alpha}{2}\left(\frac{(R+r)^2-d^2}{(R+r)+dcos\alpha}\right)$$

$$\Rightarrow\ y = \frac{-sin\alpha}{2}\left(\frac{(R+r)^2-d^2}{(R+r)+dcos\alpha}\right) \leq 0\quad \forall\ 0 \leq \alpha \leq \pi\ \ \boldsymbol{hence\ this\ value\ of\ y\ is\ discarded}$$

Hence, the co-ordinates of the point of intersection E (i.e. the centre of tangent circle) is given as

$$\boldsymbol{E \equiv (x,y) \equiv \left(\frac{cos\alpha}{2}\left(\frac{(R+r)^2-d^2}{(R+r)-dcos\alpha}\right), \frac{sin\alpha}{2}\left(\frac{(R+r)^2-d^2}{(R+r)-dcos\alpha}\right)\right)\quad \forall\ 0 \leq \alpha \leq \pi}$$

Now, in the right ΔEMO, we have

$$\boldsymbol{OM = x = \frac{cos\alpha}{2}\left(\frac{(R+r)^2-d^2}{(R+r)-dcos\alpha}\right)\ \ \&\ \ EM = y = \frac{sin\alpha}{2}\left(\frac{(R+r)^2-d^2}{(R+r)-dcos\alpha}\right)}$$

$$\therefore\ OE = \sqrt{(OM)^2+(EM)^2} = \sqrt{\left(\frac{cos\alpha}{2}\left(\frac{(R+r)^2-d^2}{(R+r)-dcos\alpha}\right)\right)^2 + \left(\frac{sin\alpha}{2}\left(\frac{(R+r)^2-d^2}{(R+r)-dcos\alpha}\right)\right)^2}$$

$$\Rightarrow \boldsymbol{OE} = \frac{1}{2}\left(\frac{(R+r)^2 - d^2}{(R+r) - dcos\alpha}\right)\sqrt{sin^2\alpha + cos^2\alpha} = \frac{1}{2}\left(\frac{(R+r)^2 - d^2}{(R+r) - dcos\alpha}\right)$$

$$but, OE = OD + DE = r + ED \;\;\Rightarrow\;\; ED = OE - r$$

$$\therefore ED = \frac{1}{2}\left(\frac{(R+r)^2 - d^2}{(R+r) - dcos\alpha}\right) - r = \frac{1}{2}\left(\frac{(R+r)^2 - d^2 - 2r(R+r) + 2rdcos\alpha}{(R+r) - dcos\alpha}\right)$$

$$= \frac{1}{2}\left(\frac{R^2 + r^2 + 2rR - d^2 - 2rR - 2r^2 + 2rdcos\alpha}{(R+r) - dcos\alpha}\right) = \frac{1}{2}\left(\frac{R^2 - d^2 - r^2 + 2rdcos\alpha}{(R+r) - dcos\alpha}\right)$$

Hence, the **radius** $(\boldsymbol{R_\alpha})$ **of the tangent (parametric) circle** for the given value of the angle α is given as follows

$$\boldsymbol{R_\alpha} = \frac{\mathbf{1}}{\mathbf{2}}\left(\frac{\boldsymbol{R^2 - d^2 - r^2 + 2rdcos\alpha}}{\boldsymbol{(R+r) - dcos\alpha}}\right) \quad \forall\ \mathbf{0} \le \boldsymbol{\alpha} \le \boldsymbol{\pi}\ \&\ \boldsymbol{R} \ge \boldsymbol{d} + \boldsymbol{r}$$

The above expression is very useful to determine the radius $(\boldsymbol{R_\alpha})$ of the tangent (parametric) circle for the given value of parametric angle $\boldsymbol{\alpha}$.

5. Parametric angles $\boldsymbol{\beta}$ & $\boldsymbol{\gamma}$: The parametric angles β & γ are determined for the given value of parametric angle α as follows

In the right ΔEMC, we have

$$tan\angle ECM = \frac{EM}{CM} = \frac{EM}{OM - OC} \;\Rightarrow\; tan\gamma = \frac{\frac{sin\alpha}{2}\left(\frac{(R+r)^2 - d^2}{(R+r) - dcos\alpha}\right)}{\frac{cos\alpha}{2}\left(\frac{(R+r)^2 - d^2}{(R+r) - dcos\alpha}\right) - \frac{d}{2}}$$

$$= \frac{((R+r)^2 - d^2)sin\alpha}{(R+r)^2cos\alpha - d^2cos\alpha - (R+r)d + d^2cos\alpha} = \frac{((R+r)^2 - d^2)sin\alpha}{(R+r)^2cos\alpha - (R+r)d}$$

$$\therefore\; \boldsymbol{tan\gamma} = \frac{\boldsymbol{((R+r)^2 - d^2)sin\alpha}}{\boldsymbol{(R+r)^2cos\alpha - (R+r)d}} \quad \mathbf{0} \le \boldsymbol{\gamma} \le \boldsymbol{\pi} \quad \forall\ \mathbf{0} \le \boldsymbol{\alpha} \le \boldsymbol{\pi}$$

In the right $\Delta EMO'$, we have

$$tan\angle EO'M = \frac{EM}{O'M} = \frac{EM}{OM - OO'} \;\Rightarrow\; tan\beta = \frac{\frac{sin\alpha}{2}\left(\frac{(R+r)^2 - d^2}{(R+r) - dcos\alpha}\right)}{\frac{cos\alpha}{2}\left(\frac{(R+r)^2 - d^2}{(R+r) - dcos\alpha}\right) - d}$$

$$= \frac{((R+r)^2 - d^2)sin\alpha}{(R+r)^2cos\alpha - d^2cos\alpha - 2(R+r)d + 2d^2cos\alpha} = \frac{((R+r)^2 - d^2)sin\alpha}{(R+r)^2cos\alpha + d^2cos\alpha - 2(R+r)d}$$

$$\therefore\; \boldsymbol{tan\beta} = \frac{\boldsymbol{((R+r)^2 - d^2)sin\alpha}}{\boldsymbol{((R+r)^2 + d^2)cos\alpha - 2(R+r)d}} \quad \mathbf{0} \le \boldsymbol{\beta} \le \boldsymbol{\pi} \quad \forall\ \mathbf{0} \le \boldsymbol{\alpha} \le \boldsymbol{\pi}$$

Thus, the parametric angles $\boldsymbol{\beta}$ & $\boldsymbol{\gamma}$ can be determined for the given value of angle $\boldsymbol{\alpha}$. Also these angles can be co-related with each other by using above expressions.

Conclusion: For given two circles, smaller inside the bigger one, with their radii r & R respectively and their centres separated by a certain distance d $(\forall\ d \geq 0)$, the **locus of the centre of a third tangent (parametric) circle** (in the annulus of two circles) touching the smaller circle externally & the bigger one internally, is always an **ellipse** (except circle in case of concentric circles) then all the parameters of this elliptical path are determined as follows

Major axis $(2a)$:

$$\boldsymbol{2a = R + r}$$

Minor axis $(2b)$:

$$\boldsymbol{2b = \sqrt{(R+r)^2 - d^2}}$$

Eccentricity (e):

$$\boldsymbol{e = \frac{d}{R+r} \quad (0 \leq e < 1 \quad \forall\ R \geq d + r)}$$

Radius (R_α) of the tangent (parametric) circle: For the given value of the parametric angle α

$$\boldsymbol{R_\alpha = \frac{1}{2}\left(\frac{R^2 - d^2 - r^2 + 2rd\cos\alpha}{(R+r) - d\cos\alpha}\right) \qquad \forall\ 0 \leq \alpha \leq \pi\ \&\ R \geq d + r}$$

Parametric angles β & γ :

$$\boldsymbol{\tan\beta = \frac{((R+r)^2 - d^2)\sin\alpha}{((R+r)^2 + d^2)\cos\alpha - 2(R+r)d} \qquad 0 \leq \beta \leq \pi \quad \forall\ 0 \leq \alpha \leq \pi}$$

$$\boldsymbol{\tan\gamma = \frac{((R+r)^2 - d^2)\sin\alpha}{(R+r)^2\cos\alpha - (R+r)d} \qquad 0 \leq \gamma \leq \pi \quad \forall\ 0 \leq \alpha \leq \pi}$$

*All the articles above have been derived by the author by using **simple geometry & trigonometry**. All above articles (formula) are very practical & simple to apply in case studies & practical applications of 2-D Geometry. These articles are the most generalised expressions which can be used for the analysis of the elliptical path in the annular region between two circles, smaller inside the bigger one & their centres separated by a certain distance in order to calculate major axis, minor axis, eccentricity & the radius of the tangent (parametric) circle.*

Note: Above articles had been derived *& illustrated by* ***Mr H.C. Rajpoot** (**B Tech, Mechanical Engineering**)*

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