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Mathematical Analysis of Truncated Dodecahedron by H.C. Rajpoot

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$Mathematical\ Analysis\ of\ Truncated\ Dodecahedron$

$Application\ of\ HCR's\ Theory\ of\ Polygon\ \&\ HCR's\ Formula$

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Introduction: A truncated dodecahedron is a solid which has 20 congruent equilateral triangular & 12 congruent regular decagonal faces each having equal edge length. It is obtained by truncating a regular dodecahedron (having 20 vertices & 12 congruent faces each as a regular pentagon) at all the vertices to generate 20 equilateral triangular & 12 regular decagonal faces of equal edge length. For calculating all the parameters of a truncated dodecahedron, we would use the equations of right pyramid & regular dodecahedron. When a regular dodecahedron is truncated at the vertex, a right pyramid, with base as an equilateral triangle & certain normal height, is obtained. Since, a regular dodecahedron has 20 vertices hence we obtain 20 truncated off congruent right pyramids each with an equilateral triangular base.

Truncation of a regular dodecahedron: For ease of calculations, let there be a regular dodecahedron with edge length PQ $(unknown)$ & its centre at the point C. Now it is truncated at all 20 vertices to obtain a truncated dodecahedron. Thus each of the congruent regular pentagonal faces with edge length PQ is changed into a regular decagonal face with edge length a $(known)$ (see figure 1) & we obtain 20 truncated off congruent right pyramids with base as an equilateral triangle corresponding to 20 vertices of the parent solid. (See figure 1 which shows one regular pentagonal face & a right pyramid with equilateral triangular base & normal height h being truncated from the regular dodecahedron).

$$No.of\ congruent\ equilateral\ triangular\ faces\ in\ the\ truncated\ dodecahedron$$
$$= no.of\ vertices\ in\ parent\ dodecahedron = 20$$

$$No.of\ congruent\ regular\ decagonal\ faces\ in\ the\ truncated\ dodecahedron$$
$$= no.of\ regular\ pentagonal\ faces\ in\ parent\ dodecahedron = 12$$

$$No.of\ vertices\ in\ the\ truncated\ dodecahedron = 20 \times 3 = 60$$

Calculation of edge length of parent regular dodecahedron:

Let a be the edge length of each face of a truncated dodecahedron to be generated by truncating a parent regular dodecahedron with edge length PQ (unknown).

$$\angle PAM = 90^o - \frac{\angle APB}{2}\quad (from\ figure\ 1)$$

$$= 90^o - \frac{108^o}{2} = 90^o - 54^o = 36^o$$

In right ΔPMA

$$cos\angle PAM = \frac{AM}{AP}\quad or\quad cos36^o = \frac{\frac{a}{2}}{AP}$$

$$or\ AP = \frac{a}{2cos36^o}$$

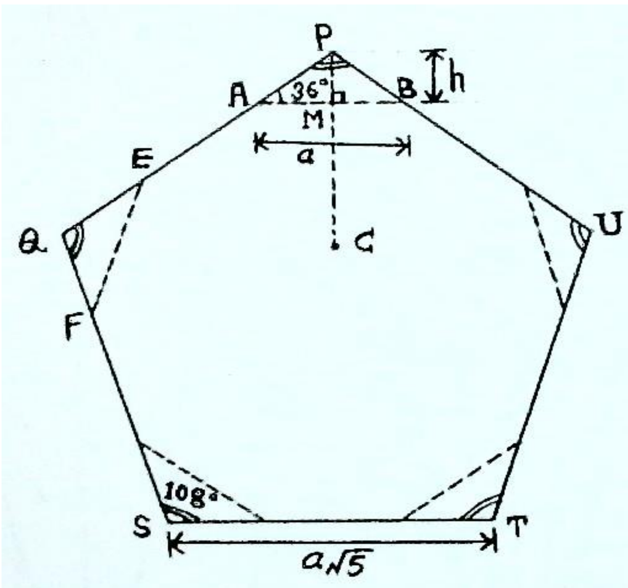

Figure 1: Each of 12 congruent pentagonal faces with edge length $a\sqrt{5}$ of a regular dodecahedron is changed into a regular decagonal face with edge length a by the truncation of vertices. A right pyramid with base as an equilateral triangle with side length a & normal height h is being truncated off from a regular dodecahedron with edge length $a\sqrt{5}$

$$\therefore\ edge\ length\ of\ parent\ dodecahedron, PQ = PA + AE + EQ$$

$$= 2PA + AB\ (since,\ \ AB = AE = EF = \ \dots\dots\ = edge\ length\ of\ truncated\ dodecahedron = \ a)$$

$$= 2 \times \frac{a}{2cos36^o} + a = a\left(\frac{4}{\sqrt{5}+1} + 1\right) = a\left(\frac{5+\sqrt{5}}{\sqrt{5}+1}\right) = a\left(\frac{\sqrt{5}(\sqrt{5}+1)}{\sqrt{5}+1}\right) = a\sqrt{5}$$

$$\therefore\ \boldsymbol{edge\ length\ of\ parent\ regular\ dodecahedron,\ \ PQ = a\sqrt{5}}$$

Above result shows that if we are to generate a truncated dodecahedron with edge length a then we have to truncate all 20 vertices of a regular dodecahedron of edge length $a\sqrt{5}$

Analysis of truncated dodecahedron by using equations of right pyramid & regular dodecahedron

Now consider any of 20 truncated off congruent right pyramids having base as an equilateral triangle ABD with side length a, normal height h & an angle 108^obetween any two consecutive lateral edges (see figure 2 below)

Normal height (h) of truncated off right pyramid: We know that the normal height of any **right pyramid with regular polygonal base,** having n no. of sides each of length a & an angle α between any two consecutive lateral edges, is given as

$$H = \frac{a}{2}\sqrt{cot^2\frac{\alpha}{2} - cot^2\frac{\pi}{n}}$$

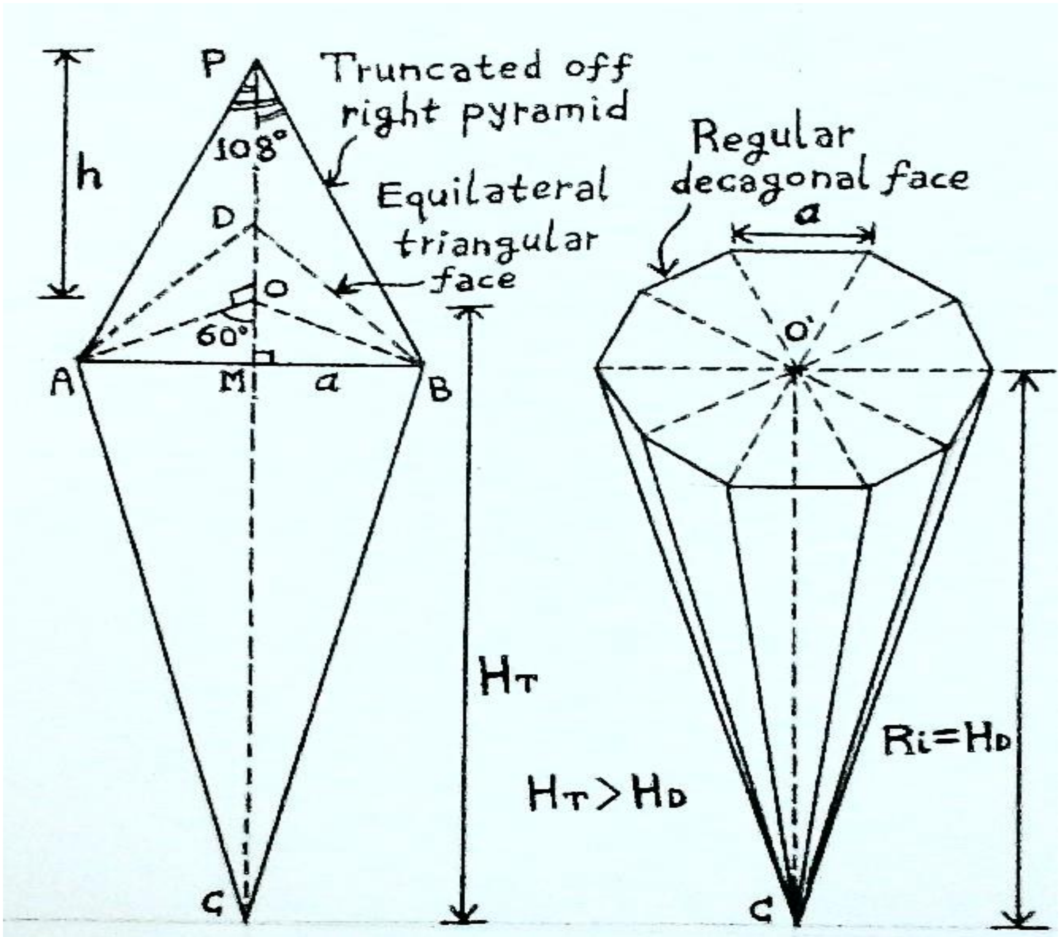

Figure 2: Normal distance (H_T) of equilateral triangular faces is always greater than the normal distance (H_D) of regular decagonal faces measured from the centre C of any truncated dodecahedron.

$$\therefore h = \frac{a}{2}\sqrt{cot^2\frac{108^o}{2} - cot^2\frac{\pi}{3}} = \frac{a}{2}\sqrt{tan^2 36^o - \left(\frac{1}{\sqrt{3}}\right)^2} \quad (\,for\ equilateral\ triangular\ base,\ \ n = 3)$$

$$\Rightarrow h = \frac{a}{2}\sqrt{\left(\frac{\sqrt{10-2\sqrt{5}}}{\sqrt{5}+1}\right)^2 - \frac{1}{3}} = \frac{a}{2}\sqrt{\frac{3(10-2\sqrt{5}) - (\sqrt{5}+1)^2}{3(\sqrt{5}+1)^2}} = \frac{a}{2(\sqrt{5}+1)}\sqrt{\frac{30-6\sqrt{5}-6-2\sqrt{5}}{3}}$$

$$= \frac{a}{2(\sqrt{5}+1)}\sqrt{\frac{4(6-2\sqrt{5})}{3}} = \frac{2a}{2(\sqrt{5}+1)}\sqrt{\frac{(\sqrt{5}-1)^2}{3}} = \frac{a(\sqrt{5}-1)}{\sqrt{3}(\sqrt{5}+1)} = \frac{a(\sqrt{5}-1)(\sqrt{5}-1)}{\sqrt{3}(\sqrt{5}+1)(\sqrt{5}-1)}$$

$$= \frac{a(6-2\sqrt{5})}{\sqrt{3}(5-1)} = \frac{2a(3-\sqrt{5})}{4\sqrt{3}} = \frac{a(3-\sqrt{5})}{2\sqrt{3}}$$

$$\therefore \boldsymbol{truncated\ normal\ height,\ \ h = \frac{(3-\sqrt{5})a}{2\sqrt{3}}} \quad \dots\dots\dots\dots\dots\dots\dots\dots.(I)$$

Volume (V') of truncated off right pyramid: We know that the volume of a right pyramid is given as

$$= \frac{1}{3}(area\ of\ base\ (equilateral\ triangle)) \times (normal\ height)$$

$$\therefore V' = \frac{1}{3}\left(\frac{1}{4}na^2 cot\frac{\pi}{n}\right) \times h = \frac{1}{3}\left(\frac{1}{4} \times 3 \times a^2 cot\frac{\pi}{3}\right) \times \frac{a(3-\sqrt{5})}{2\sqrt{3}}$$

$$= \frac{a^3}{4} \times \frac{1}{\sqrt{3}} \times \frac{(3-\sqrt{5})}{2\sqrt{3}} = \frac{(3-\sqrt{5})a^3}{24}$$

$$\boldsymbol{V' = \frac{(3-\sqrt{5})a^3}{24}} \quad \dots\dots\dots\dots\dots\dots(II)$$

Normal distance (H_T) of equilateral triangular faces from the centre of truncated dodecahedron:
The normal distance (H_T) of each of the equilateral triangular faces from the centre C of truncated dodecahedron is given as

$$H_T = OC = CP - OP \quad (from\ the\ figure\ 2\ above)$$

$$\Rightarrow\ H_T = (outer\ (circumscribed)\ radius\ of\ parent\ regular\ dodecahedron) - h$$

$$\Rightarrow\ H_T = \frac{\sqrt{3}(\sqrt{5}+1)(a\sqrt{5})}{4} - \frac{(3-\sqrt{5})a}{2\sqrt{3}} \quad (\boldsymbol{outer\ radius\ is\ given\ from\ HCR's\ \ Formula})$$

$$= \frac{a(15+3\sqrt{5}-6+2\sqrt{5})}{4\sqrt{3}} = \frac{a(9+5\sqrt{5})}{4\sqrt{3}}$$

$$\Rightarrow \boldsymbol{H_T = \frac{(9+5\sqrt{5})a}{4\sqrt{3}} \approx 2.912781167a} \quad \dots\dots\dots\dots\dots.(III)$$

It's clear that all 20 congruent equilateral triangular faces are at an equal normal distance H_T from the centre of any truncated dodecahedron.

Solid angle (ω_T) subtended by each of the equilateral triangular faces at the centre of truncated dodecahedron: we know that the solid angle (ω) subtended by any regular polygon with each side of length a at any point lying at a distance H on the vertical axis passing through the centre of plane is given by "**HCR's Theory of Polygon**" as follows

$$\omega = 2\pi - 2n\sin^{-1}\left(\frac{2H\sin\frac{\pi}{n}}{\sqrt{4H^2 + a^2\cot^2\frac{\pi}{n}}}\right)$$

Hence, by substituting the corresponding values in the above expression, we get the solid angle subtended by each equilateral triangular face at the centre of truncated dodecahedron as follows

$$\omega_T = 2\pi - 2\times 3\sin^{-1}\left(\frac{2\left(\frac{(9+5\sqrt{5})a}{4\sqrt{3}}\right)\sin\frac{\pi}{3}}{\sqrt{4\left(\frac{(9+5\sqrt{5})a}{4\sqrt{3}}\right)^2 + a^2\cot^2\frac{\pi}{3}}}\right)$$

$$= 2\pi - 6\sin^{-1}\left(\frac{\frac{(9+5\sqrt{5})}{2\sqrt{3}}\times\frac{\sqrt{3}}{2}}{\sqrt{\frac{81+125+90\sqrt{5}}{12}+\frac{1}{3}}}\right) = 2\pi - 6\sin^{-1}\left(\frac{9+5\sqrt{5}}{4\sqrt{\frac{206+90\sqrt{5}+4}{12}}}\right)$$

$$= 2\pi - 6\sin^{-1}\left(\frac{9+5\sqrt{5}}{4\sqrt{\frac{70+30\sqrt{5}}{4}}}\right) = 2\pi - 6\sin^{-1}\left(\frac{9+5\sqrt{5}}{2\sqrt{70+30\sqrt{5}}}\right)$$

$$\boldsymbol{\omega_T = 2\pi - 6\sin^{-1}\left(\frac{9+5\sqrt{5}}{2\sqrt{70+30\sqrt{5}}}\right) \approx 0.05029947\,sr} \qquad \dots\dots\dots\dots\dots\dots\dots..(IV)$$

Normal distance (H_D) of regular decagonal faces from the centre of truncated dodecahedron: The normal distance (H_h) of each of the regular decagonal faces from the centre C of truncated dodecahedron is given as

$$H_D = O'C \quad (from\ the\ figure\ 2\ above)$$

$$\Rightarrow\ H_D = (inner\ (inscribed)\ radius\ of\ parent\ regular\ dodecahedron)$$

$$\Rightarrow\ H_D = \frac{(3+\sqrt{5})(a\sqrt{5})}{2\sqrt{10-2\sqrt{5}}} = \frac{(5+3\sqrt{5})a}{2\sqrt{10-2\sqrt{5}}} \qquad (\boldsymbol{inner\ radius\ is\ given\ from\ HCR's\ \ Formula})$$

$$\Rightarrow\ \boldsymbol{H_D = \frac{(5+3\sqrt{5})a}{2\sqrt{10-2\sqrt{5}}} = \frac{(5+3\sqrt{5})a}{8\sin 36^o} \approx 2.489898285a} \qquad \dots\dots\dots\dots\dots\dots\dots.(V)$$

It's clear that all 12 congruent regular decagonal faces are at an equal normal distance $\boldsymbol{H_D}$ from the centre of any truncated dodecahedron.

It's also clear from eq(III) & (V) $\boldsymbol{H_T > H_D}$ i.e. the normal distance (H_T) of equilateral triangular faces is greater than the normal distance (H_D) of regular decagonal faces from the centre of truncated dodecahedron i.e.

regular decagonal faces are much closer to the centre as compared to the equilateral triangular faces in any truncated dodecahedron.

Solid angle (ω_D) subtended by each of the regular decagonal faces at the centre of truncated dodecahedron: we know that the solid angle (ω) subtended by any regular polygon is given by "**HCR's Theory of Polygon**" as follows

$$\omega = 2\pi - 2n\sin^{-1}\left(\frac{2H\sin\frac{\pi}{n}}{\sqrt{4H^2 + a^2\cot^2\frac{\pi}{n}}}\right)$$

Hence, by substituting the corresponding value of normal distance $H = H_D$ in the above expression, we get the solid angle subtended by each regular decagonal face at the centre of truncated dodecahedron as follows

$$\omega_D = 2\pi - 2\times 10\sin^{-1}\left(\frac{2\left(\frac{(5+3\sqrt{5})a}{2\sqrt{10-2\sqrt{5}}}\right)\sin\frac{\pi}{10}}{\sqrt{4\left(\frac{(5+3\sqrt{5})a}{2\sqrt{10-2\sqrt{5}}}\right)^2 + a^2\cot^2\frac{\pi}{10}}}\right) \quad (for\ decagon,\ \ n = 10)$$

$$= 2\pi - 20\sin^{-1}\left(\frac{\frac{(5+3\sqrt{5})}{\sqrt{10-2\sqrt{5}}}\times\frac{(\sqrt{5}-1)}{4}}{\sqrt{\frac{(25+45+30\sqrt{5})}{10-2\sqrt{5}} + \left(\frac{\sqrt{10+2\sqrt{5}}}{\sqrt{5}-1}\right)^2}}\right)$$

$$= 2\pi - 20\sin^{-1}\left(\frac{\left(\frac{5\sqrt{5}+15-5-3\sqrt{5}}{4\sqrt{10-2\sqrt{5}}}\right)}{\sqrt{\frac{(70+30\sqrt{5})(6-2\sqrt{5}) + (10+2\sqrt{5})(10-2\sqrt{5})}{(10-2\sqrt{5})(\sqrt{5}-1)^2}}}\right)$$

$$= 2\pi - 20\sin^{-1}\left(\frac{\frac{(5+\sqrt{5})}{2\sqrt{10-2\sqrt{5}}}\sqrt{10-2\sqrt{5}}\,(\sqrt{5}-1)}{\sqrt{420+180\sqrt{5}-140\sqrt{5}-300+100-20}}\right)$$

$$= 2\pi - 20\sin^{-1}\left(\frac{5\sqrt{5}+5-5-\sqrt{5}}{2\sqrt{420+180\sqrt{5}-140\sqrt{5}-300+100-20}}\right) = 2\pi - 20\sin^{-1}\left(\frac{2\sqrt{5}}{\sqrt{200+40\sqrt{5}}}\right)$$

$$= 2\pi - 20\sin^{-1}\left(\frac{2\sqrt{5}}{\sqrt{20(10+2\sqrt{5})}}\right) = 2\pi - 20\sin^{-1}\left(\frac{2\sqrt{5}}{2\sqrt{5}\sqrt{10+2\sqrt{5}}}\right) = 2\pi - 20\sin^{-1}\left(\frac{1}{\sqrt{10+2\sqrt{5}}}\right)$$

$$\boldsymbol{\omega_D = 2\pi - 20\sin^{-1}\left(\frac{1}{\sqrt{10+2\sqrt{5}}}\right) \approx 0.963365099\,sr} \qquad \ldots\ldots\ldots\ldots\ldots..(VI)$$

It's clear that the solid angle subtended by each of the regular decagonal faces is greater than the solid angle subtended by each of the equilateral triangular faces at the centre of any truncated dodecahedron.

Important parameters of a truncated dodecahedron:

1. **Inner (inscribed) radius$(\boldsymbol{R_i})$:** It is the radius of the largest sphere inscribed (trapped inside) by a truncated dodecahedron. The largest inscribed sphere always touches all 12 congruent regular decagonal faces but does not touch any of 20 equilateral triangular faces at all since all 12 decagonal faces are closer to the centre as compared to all 20 triangular faces. Thus, inner radius is always equal to the normal distance (H_D) of decagonal faces from the centre & is given from the eq(V) as follows

$$\boldsymbol{R_i = H_D = \frac{(5+3\sqrt{5})a}{2\sqrt{10-2\sqrt{5}}} \approx 2.489898285a}$$

Hence, the **volume of inscribed sphere** is given as

$$V_{inscribed} = \frac{4}{3}\pi(R_i)^3 = \boldsymbol{\frac{4}{3}\pi\left(\frac{(5+3\sqrt{5})a}{2\sqrt{10-2\sqrt{5}}}\right)^3 \approx 64.65966161a^3}$$

2. **Outer (circumscribed) radius$(\boldsymbol{R_o})$:** It is the radius of the smallest sphere circumscribing a given truncated dodecahedron or it's the radius of a spherical surface passing through all 60 vertices of a given truncated dodecahedron. It is calculated as follows (See figure 2 above).

$$R_o = distance\ of\ any\ of\ the\ vertices\ from\ the\ centre\ C = CA = CB = CD$$

In right ΔOMA

$$sin\angle AOM = \frac{AM}{OA} \Rightarrow sin60^o = \frac{\left(\frac{a}{2}\right)}{OA} \quad or\ OA = \frac{a}{2\left(\frac{\sqrt{3}}{2}\right)} = \frac{a}{\sqrt{3}} \quad \left(since,\ \ \angle AOB = \frac{2\pi}{3} = 120^o\right)$$

In right ΔAOC

$$\Rightarrow CA = \sqrt{(OA)^2+(OC)^2} = \sqrt{\left(\frac{a}{\sqrt{3}}\right)^2 + \left(\frac{(9+5\sqrt{5})a}{4\sqrt{3}}\right)^2} \qquad (since,\ \ OC = H_T)$$

$$= a\sqrt{\frac{1}{3}+\frac{81+125+90\sqrt{5}}{48}} = a\sqrt{\frac{16+206+90\sqrt{5}}{48}} = a\sqrt{\frac{222+90\sqrt{5}}{48}} = a\sqrt{\frac{74+30\sqrt{5}}{16}}$$

$$= \frac{a}{4}\sqrt{74+30\sqrt{5}} = outer\ (circumscribed)\ radius$$

Hence, the outer radius of truncated dodecahedron is given as

$$\boldsymbol{R_o = \frac{a}{4}\sqrt{74+30\sqrt{5}} \approx 2.969449016a}$$

Hence, the **volume of circumscribed sphere** is given as

$$V_{circumscribed} = \frac{4}{3}\pi(R_o)^3 = \boldsymbol{\frac{4}{3}\pi\left(\frac{a}{4}\sqrt{74+30\sqrt{5}}\right)^3 \approx 109.6771681a^3}$$

3. **Surface area(A_s):** We know that a truncated dodecahedron has 20 congruent equilateral triangular & 12 regular decagonal faces each of edge length a. Hence, its surface area is given as follows

$$A_s = 20 \times (area\ of\ equilateral\ triangle) + 12 \times (area\ of\ regular\ decagon)$$

We know that **area of any regular n-polygon** with each side of length a is given as

$$\boldsymbol{A = \frac{1}{4}na^2 cot\frac{\pi}{n}}$$

Hence, by substituting all the corresponding values in the above expression, we get

$$A_s = 20 \times \left(\frac{1}{4} \times 3a^2 cot\frac{\pi}{3}\right) + 12 \times \left(\frac{1}{4} \times 10a^2 cot\frac{\pi}{10}\right) = 15a^2 \times \frac{1}{\sqrt{3}} + 30a^2 cot18^o$$

$$= \left(5\sqrt{3} + 30cot18^o\right)a^2 = 5\left(\sqrt{3} + 6cot18^o\right)a^2$$

$$\boldsymbol{A_s = 5\left(\sqrt{3} + 6cot18^o\right)a^2 \approx 100.9907602a^2}$$

4. **Volume(V):** We know that a truncated dodecahedron with edge length a is obtained by truncating a regular tetrahedron with edge length $a\sqrt{5}$ at all its 20 vertices. Thus, 20 congruent right pyramids with equilateral triangular base are truncated off from the parent regular dodecahedron. Hence, the volume (V) of the truncated dodecahedron is given as follows

$$V = (volume\ of\ parent\ regular\ dodecahedron) - 20(volume\ of\ truncated\ off\ right\ pyramid)$$

$$= \frac{\left(15 + 7\sqrt{5}\right)\left(a\sqrt{5}\right)^3}{4} - 20 \times (V') \quad (volume\ of\ dodecahedron\ is\ given\ from\ HCR's\ formula)$$

$$= \frac{\left(15 + 7\sqrt{5}\right)5\sqrt{5}a^3}{4} - 20 \times \frac{\left(3 - \sqrt{5}\right)a^3}{24} \quad (substituting\ the\ value\ of\ V'\ from\ eq\ (II))$$

$$= \frac{\left(450\sqrt{5} + 1050 - 60 + 20\sqrt{5}\right)a^3}{24} = \frac{\left(990 + 470\sqrt{5}\right)a^3}{24} = \frac{\left(495 + 235\sqrt{5}\right)a^3}{12} = \frac{5}{12}\left(99 + 47\sqrt{5}\right)a^3$$

$$\boldsymbol{V = \frac{5}{12}\left(99 + 47\sqrt{5}\right)a^3 \approx 85.03966456a^3}$$

5. **Mean radius(R_m):** It is the radius of the sphere having a volume equal to that of a given truncated dodecahedron. It is calculated as follows

$$volume\ of\ sphere\ with\ mean\ radius\ R_m = volume\ of\ given\ truncated\ dodecahedron$$

$$\frac{4}{3}\pi(R_m)^3 = \frac{\left(495 + 235\sqrt{5}\right)a^3}{12} \quad \Rightarrow (R_m)^3 = \frac{\left(495 + 235\sqrt{5}\right)a^3}{16\pi}\ or\ R_m = a\left(\frac{495 + 235\sqrt{5}}{16\pi}\right)^{\frac{1}{3}}$$

$$\boldsymbol{R_m = a\left(\frac{495 + 235\sqrt{5}}{16\pi}\right)^{\frac{1}{3}} \approx 2.727999647a}$$

It's clear from above results that $\boldsymbol{R_i < R_m < R_o}$

Construction of a solid truncated dodecahedron: In order to construct a solid truncated dodecahedron with edge length a there are two methods

1. Construction from elementary right pyramids: In this method, first we construct all elementary right pyramids as follows

Construct 20 congruent right pyramids with equilateral triangular base of side length a & normal height (H_T)

$$H_T = \frac{(9+5\sqrt{5})a}{4\sqrt{3}} \approx 2.912781167a$$

Construct 12 congruent right pyramids with regular decagonal base of side length a & normal height (H_D)

$$H_D = \frac{(5+3\sqrt{5})a}{8sin36^o} \approx 2.489898285a$$

Now, paste/bond by joining all these right pyramids by overlapping their lateral surfaces & keeping their apex points coincident with each other such that all the edges of each equilateral triangular base (face) coincide with the edges of three decagonal bases (faces). Thus, a solid truncated dodecahedron, with 20 congruent equilateral triangular & 12 congruent regular decagonal faces each of edge length a, is obtained.

2. Facing a solid sphere: It is a method of facing, first we select a **blank as a solid sphere** of certain material (i.e. metal, alloy, composite material etc.) & with suitable diameter in order to obtain the maximum desired edge length of truncated dodecahedron. Then, we perform facing operations on the solid sphere to generate 20 congruent equilateral triangular & 12 congruent regular decagonal faces each of equal edge length.

Let there be a blank as a solid sphere with a diameter D. Then the edge length a, of a truncated dodecahedron of maximum volume to be produced, can be co-related with the diameter D by **relation of outer radius (R_o) with edge length $(a)$of a truncated dodecahedron** as follows

$$R_o = \frac{a}{4}\sqrt{74+30\sqrt{5}}$$

Now, substituting $R_o = {}^{D}/_{2}$ in the above expression, we have

$$\frac{D}{2} = \frac{a}{4}\sqrt{74+30\sqrt{5}} \quad or \quad D = \frac{a}{2}\sqrt{74+30\sqrt{5}}$$

$$\boldsymbol{a = \frac{2D}{\sqrt{74+30\sqrt{5}}} \approx 0.168381405D}$$

Above relation is very useful for determining the edge length a of a truncated dodecahedron to be produced from a solid sphere with known diameter D for manufacturing purpose.

Hence, the **maximum volume of truncated dodecahedron** produced from the solid sphere is given as follows

$$V_{max} = \frac{5}{12}(99+47\sqrt{5})a^3 = \frac{(495+235\sqrt{5})}{12}\left(\frac{2D}{\sqrt{74+30\sqrt{5}}}\right)^3$$

$$= \frac{8(495+235\sqrt{5})D^3}{12(74+30\sqrt{5})\sqrt{74+30\sqrt{5}}} = \frac{(495+235\sqrt{5})D^3}{3(37+15\sqrt{5})\sqrt{74+30\sqrt{5}}}$$

$$= \frac{(495+235\sqrt{5})(37-15\sqrt{5})D^3}{3(37+15\sqrt{5})(37-15\sqrt{5})\sqrt{74+30\sqrt{5}}} = \frac{(18315+8695\sqrt{5}-7425\sqrt{5}-17625)D^3}{732\sqrt{74+30\sqrt{5}}}$$

$$=\frac{(690+1270\sqrt{5})D^3}{732\sqrt{74+30\sqrt{5}}}=\frac{(345+635\sqrt{5})D^3}{366\sqrt{74+30\sqrt{5}}}=\frac{5(69+127\sqrt{5})D^3}{366\sqrt{74+30\sqrt{5}}}=\frac{5}{366}\left(\frac{69+127\sqrt{5}}{\sqrt{74+30\sqrt{5}}}\right)D^3$$

$$\boldsymbol{V_{max}=\frac{5}{366}\left(\frac{69+127\sqrt{5}}{\sqrt{74+30\sqrt{5}}}\right)D^3\approx 0.405979339D^3}$$

M**inimum volume of material removed** is given as

$$(V_{removed})_{min}=(volume\ of\ parent\ sphere\ with\ diameter\ D)-(volume\ of\ truncated\ dodecahedron)$$

$$=\frac{\pi}{6}D^3-\frac{5}{366}\left(\frac{69+127\sqrt{5}}{\sqrt{74+30\sqrt{5}}}\right)D^3=\left(\frac{\pi}{6}-\frac{345+635\sqrt{5}}{366\sqrt{74+30\sqrt{5}}}\right)D^3$$

$$\boldsymbol{(V_{removed})_{min}=\left(\frac{\pi}{6}-\frac{345+635\sqrt{5}}{366\sqrt{74+30\sqrt{5}}}\right)D^3\approx 0.117619436D^3}$$

Percentage $(\%)$ **of minimum volume of material removed**

$$\%\ \boldsymbol{of\ V_{removed}}=\frac{minimum\ volume\ removed}{total\ volume\ of\ sphere}\times 100$$

$$=\frac{\left(\frac{\pi}{6}-\frac{345+635\sqrt{5}}{366\sqrt{74+30\sqrt{5}}}\right)D^3}{\frac{\pi}{6}D^3}\times 100=\boldsymbol{\left(1-\frac{345+635\sqrt{5}}{61\pi\sqrt{74+30\sqrt{5}}}\right)\times 100\approx 22.46\%}$$

It's obvious that when a truncated dodecahedron of maximum volume is produced from a solid sphere then about 22.46% of material is removed as scraps. Thus, we can select optimum diameter of blank as a solid sphere to produce a solid truncated dodecahedron of the maximum volume (or with maximum desired edge length)

Conclusions: let there be any truncated dodecahedron having 20 congruent equilateral triangular & 12 congruent regular decagonal faces each with edge length a then all its important parameters are calculated/determined as tabulated below

Congruent polygonal faces	**No. of faces**	**Normal distance of each face from the centre of the given truncated dodecahedron**	**Solid angle subtended by each face at the centre of the given truncated dodecahedron**
Equilateral triangle	20	$\frac{(9+5\sqrt{5})a}{4\sqrt{3}}\approx 2.912781167a$	$2\pi-6\sin^{-1}\left(\frac{9+5\sqrt{5}}{2\sqrt{70+30\sqrt{5}}}\right)\approx 0.05029947sr$
Regular decagon	12	$\frac{(5+3\sqrt{5})a}{2\sqrt{10-2\sqrt{5}}}\approx 2.489898285a$	$2\pi-20\sin^{-1}\left(\frac{1}{\sqrt{10+2\sqrt{5}}}\right)\approx 0.963365099sr$

Inner (inscribed) radius $(\boldsymbol{R_i})$	$R_i = \frac{(5+3\sqrt{5})a}{2\sqrt{10-2\sqrt{5}}} \approx 2.4898982 85a$
Outer (circumscribed) radius $(\boldsymbol{R_o})$	$R_o = \frac{a}{4}\sqrt{74+30\sqrt{5}} \approx 2.969449016a$
Mean radius $(\boldsymbol{R_m})$	$R_m = a\left(\frac{495+235\sqrt{5}}{16\pi}\right)^{\frac{1}{3}} \approx 2.727999647a$
Surface area $(\boldsymbol{A_s})$	$A_s = 5(\sqrt{3}+6cot18^o)a^2 \approx 100.9907602a^2$
Volume $(\boldsymbol{V})$	$V = \frac{5}{12}(99+47\sqrt{5})a^3 \approx 85.03966456a^3$

Note: Above articles had been developed *& illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)*

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