2008

Credible Ratings

ettore damiano, *University of Toronto*
hao li, *University of Toronto*
wing suen

Credible Ratings

Ettore Damiano
University of Toronto

Li, Hao
University of Toronto

Wing Suen
The University of Hong Kong

October 20, 2006

Abstract: This paper considers a model of a rating agency with multiple clients. Each client has a separate market (end-user of the rating); the only exogenous connection among them is that the underlying qualities of the clients are correlated. In the benchmark case of individual rating, the market for each client does not know the ratings for other clients. In centralized rating, the agency rates all clients together and shares the rating information among all markets. In decentralized rating, the ratings are again shared among all markets, but each client is rated by a self-interested rater of the agency with no access to the quality information of other clients. Both centralized rating and decentralized rating weakly dominate individual rating for the agency. When the underlying qualities are weakly correlated, centralized rating can dominate decentralized rating, but the reverse holds when the qualities are strongly correlated.

Acknowledgments: Part of the research was conducted when Damiano and Li visited Suen at the University of Hong Kong. We thank the university for its hospitality. We also benefited from seminar audiences at University of British Columbia, Shanghai University of Finance and Economics and Duke University.

1. Introduction

Consider a rating agency that issues a report on each of its clients. The rating agency is informed of the quality of each client and its report on the client is received as a signal by the market that the client faces. The agency cares about the payoff to each client. Examples of a rating agency with multiple clients include an economics department that ranks its PhD graduates, a stock brokerage firm that deals with multiple stocks, and a consumer electronics magazine that issues ratings on multiple products. We are interested in an environment in which the payoff to each client depends only on the perceived quality of that client, and not on the perceived qualities of other clients, so that there is no direct payoff link among the clients. The only possible link is indirect, and informational: when the markets are given access to all client ratings, the perceived quality of each client can depend on the ratings of other clients, either exogenously through some statistical correlation among client qualities, or endogenously through the reporting strategy of the agency, or both. In the economics department example, the payoff link is likely to be absent if the PhD graduates are in different fields so that their markets are separate, or if the markets are sufficiently thick that each graduate receives a competitive wage, while the informational link will be present if there are strong cohort effects in the graduate program or if the department ranks the students by comparing them. Similarly, for the stock brokerage firm example and the consumer magazine example, there may be little demand substitutability or complementarity in the aggregate so that the price of a rated stock or an electronic product depends only on the valuation of that stock or product,[1] but a positive correlation among the client qualities can still arise, for example, if the future returns of all the stocks are affected by an economy-wide shock or the electronic products share significant common parts or designs. In our model, because the agency cares about the perceived qualities of its clients, credibility of the ratings is at issue. The objective of this paper is to compare the credibility of ratings under three schemes that differ in

[1] The literature on asset pricing focuses on the case where the price of a stock depends on the probability distribution of the future cash flow and some "pricing kernel." In a large market, the cash flow on any single stock does not affect the pricing kernel, and so the payoffs for different stocks are separable. For the electronics example, payoff separability is a more appropriate assumption if the products belong to different categories, or if consumers have strong brand loyalty.

whether the markets have access to all the reports and in whether the raters in the agency share the knowledge about client qualities.

In "individual rating," the market for each client does not observe the ratings for other clients. This is a natural benchmark due to the absence of any direct payoff linkage. The rating scheme can be analyzed as a simple signaling model with one sender (the rating agency with a single client) and a receiver (the market for the client), with the market only interested in making the right inference about the client's underlying quality. We make assumptions on the payoff function of the agency regarding its reputational concerns and how these concerns interact with the derived benefits from an improved perception of the client quality. These assumptions imply that the incentive to exaggerate the quality always outweighs the reverse incentive to downplay it regardless of the resulting belief of the market regarding the quality. This "single crossing" property allows us to focus on the "inflationary equilibrium," which is a semi-pooling equilibrium where the client's quality is truthfully revealed whenever it is good and sometimes exaggerated when it is bad. The benchmark model of individual rating can be interpreted as a model of credibility, with the equilibrium perception of a good rating as the measure of credibility and a one-to-one correspondence between credibility and the equilibrium ex ante payoff of the agency. The inability of the rating agency to commit to an honest rating policy dilutes the meaning of a good rating without changing the meaning of the bad rating, and therefore reduces the rating agency's ex ante payoff. We ask the following question in the rest of the paper: can the rating agency obtain a higher ex ante payoff than in the inflationary equilibrium in individual rating by improving credibility of good ratings?

In "centralized rating," the agency rates all clients together and shares the reports among all markets. Each market can use the ratings of other clients as well as its own client to make inference about the quality of the latter. Sharing the rating information among all markets allows the agency to coordinate the ratings with a correlated randomization between good and bad ratings across clients of bad quality, even when client qualities are statistically independent. It turns out that all inflationary equilibria that are not payoff equivalent to the equilibrium under individual ratings have the property that there is a minimal number, larger than zero, of good ratings issued regardless of the number of clients of good quality. If such coordination is credible or incentive compatible, it results in a higher payoff to the agency than under individual rating. We show that there exists an equilibrium that weakly dominates the benchmark inflationary equilibrium under individual rating for the agency. This coordinated ratings equilibrium is unique when it strictly dominates the benchmark inflationary equilibrium.

In "decentralized rating," the ratings are shared among all markets, as in centralized rating, but each client is rated by a self-interested rater of the agency with no access to the quality information of other clients. This means that only independent randomization across clients of bad quality is possible, as in individual rating. However, unlike individual rating, ratings information is shared among all markets. In an inflationary equilibrium the perception of a good rating depends on the total number of good ratings in all markets: the perception improves with more good ratings when the client qualities are positively correlated, and it deteriorates when the qualities are negatively correlated. This endogenous payoff link among the clients makes it more difficult for each rater to fool the market with an exaggerated rating. As a result, the equilibrium probability of an inflationary rating can be lower and the average credibility of a good rating can be higher than in the benchmark inflationary equilibrium under individual rating, leading to a greater equilibrium payoff for the agency than the benchmark.

Comparison between centralized rating and decentralized rating in terms of equilibrium credibility of good ratings and ex ante payoff to the agency depends on the degree of correlation. When the underlying qualities are independently distributed, any inflationary equilibrium under decentralized rating is payoff-equivalent to the benchmark inflationary equilibrium under individual rating, as the ratings of other clients cannot discipline each individual rater and thus there is no gain in credibility. In contrast, under centralized rating the necessary and sufficient condition for an inflationary equilibrium that strictly dominates the benchmark equilibrium is typically satisfied under independence. Thus, centralized rating dominates decentralized rating for the agency under independence. With correlation across the underlying qualities, there is less room to manipulate ratings under both centralized rating and decentralized rating. When the underlying qualities are almost perfectly correlated, under centralized rating there is no inflationary equilibrium that strictly dominates the benchmark equilibrium under individual rating, as the strong correlation across client qualities severely reduces the credibility of coordinated rating. In contrast, under decentralized rating the discipline on credibility imposed by strong correlation allows the construction of an inflationary equilibrium that is arbitrarily close to truth-telling. Thus, centralized rating is dominated by decentralized rating for the agency with strong correlation.

Our comparison results regarding individual rating, centralized rating and decentralized rating have strong implications for how a rating agency can gain credibility of its ratings and improve its welfare. Since there exist inflationary equilibria that weakly dominate the benchmark under either centralized or decentralized rating schemes, it is always to the advantage of the agency to share ratings information among all markets it serves. Whether the agency should share information about client qualities among its raters or commit to a policy that restricts information access and preserves the raters' independent concerns for career reputation, depends on the underlying correlation structure across client qualities. Our results suggest that the agency should group together clients with weakly correlated qualities and centralize their rating, but for clients with strongly correlated qualities the agency should decentralize their rating among the raters.

It is interesting to interpret our comparison results between centralized rating and decentralized rating in terms of different market structures for rating agencies as opposed to different information structures for a single rating agency. The centralized rating scheme naturally corresponds to the monopoly market structure, while the decentralized scheme can be equivalently viewed as the competitive market structure. Although under the decentralized scheme there is no direct competition among the agencies because the clients have separate markets, the agencies indirectly compete for credibility as the ratings are observed by all markets. Our results then suggest that the comparison between the two market structures depends on the degree of correlation across the underlying states of nature. The monopoly structure performs better due to an economy of scale when the states are weakly correlated. When the states are strongly correlated, the competitive structure does better because competing ratings constrain the incentive to inflate and improve the credibility of good ratings.

The paper is organized as follows. Section 2 presents the basic ingredients of our model of rating agencies. We introduce the out-of-equilibrium belief refinement used throughout of the paper, and characterize an inflationary equilibrium under individual rating that serves as the benchmark of comparison. In Section 3 we deal with centralized rating. This turns out to be a signaling model with one-dimensional private information and multi-dimensional signals. We establish the existence of an inflationary equilibrium that weakly dominates the benchmark inflationary equilibrium of individual rating for the agency in terms of expected payoff. We provide a necessary and sufficient condition for the existence of an equilibrium that strictly dominates the benchmark inflationary equilibrium, and show that the equilibrium is unique when it exists. Section 4 presents the model of decentralized rating. We introduce a correlation structure that accommodates possibilities of both positive and negative correlation across client qualities in a multi-dimensional setting. Using the structure we show that there exists an inflationary equilibrium that weakly dominates the benchmark inflationary equilibrium of individual rating for the agency in terms of expected payoff under further assumptions on the payoff functions of the agency. In Section 5 we study how the comparison between centralized rating and decentralized rating depends on the correlation across client qualities. Section 6 provides some remarks on related literature. Proofs of all lemmas can be found in the Appendix.

2. A Model of Rating Agencies

A rating agency deals with N clients. In our model the N sets of relationship between each client i, $i = 1, \ldots, N$, and the corresponding market (end-user of the rating for the client) are identical. The underlying quality S_i of each client i is either good (G) or bad (B); the rating s_i for the client is either good (g) or bad (b). The objective function of the market is to minimize the expectation of the squared difference between a real-valued decision variable δ_i and a random variable which is equal to δ_G if the the quality of the client is G and δ_B if the quality is B. Let q_i denote the market's belief that the quality of the client is good. The solution to the maximization problem is then to set δ_i to $q_i\delta_G + (1 - q_i)\delta_B$, which depends only on the endogenous variable q_i. We write the rating agency's utility function from client i as $U(S_i, s_i, q_i)$ for $S_i = G, B$ and $s_i = g, b$. The total utility to the agency is the sum $\sum_{i=1}^{N} U(S_i, s_i, q_i)$.

For the statistical distribution of client qualities, at this point we assume only that the client qualities are exchangeable random variables: the probability of any realization of the random vector $(S_1, \ldots, S_N)$ depends only on the number of clients of good quality. The joint probability distribution of $(S_1, \ldots, S_N)$ can then be represented by a vector $(\pi_0, \ldots, \pi_N)$, where π_n is the probability that there are exactly n clients of good quality. We assume that $\pi_n > 0$ for each $n = 0, 1, \ldots, N$. Define π as the probability that any given client is of good quality, which satisfies

$$\pi = \frac{1}{N} \sum_{n=1}^{N} n\pi_n. \tag{2.1}$$

The assumption of exchangeability introduces symmetry across clients that simplifies our analysis without imposing statistical independence. In the applications of the model that we have in mind, correlated client qualities might be an important feature. For example, student qualities might be correlated through peer effects, stock valuations through some underlying common fundamental, and electronic products through common design features. It turns out that the specific correlation structure does not play any role in our analysis of individual and centralized rating schemes. We will need to make further assumptions on the correlation structure when we analyze decentralized rating.

A few remarks about the setup are in order. First, the specific preference function adopted here for the markets is meant to capture the idea that each client faces competitive bids after the market updates its belief about the quality of the client based on the reports of the agency. This reduces the role of the receiver in our signaling model to forming rational expectations of the client quality, and allows us to focus on the signaling incentives of the agency. Second, the utility of the agency in the relationship with client i is assumed to depend on the market's belief q_i about client i's quality, which summarizes the payoff to the client. This models the idea that the agency is not an impartial provider of information, in that it cares about the payoff to the client. Third, both the underlying quality S_i and the signal s_i enter the utility function of the agency. This form allows for any two-state, two-signal setup. The general idea is that the utility of the agency is affected both by the payoff to the client and by its own reputational concerns, and we are using the function U as a reduced-form representation of the agency's utility. Later we will make further assumptions on how the concern for the client's payoff and the reputational concerns interact with each other. Finally, the utility of the agency is assumed to be additively separable in the utilities from the N sets of client relationships. This separability assumption is justified if the payoff to each client i only depends on the belief q_i about the client's quality. As mentioned in the introduction, there are environments in the labor market, the financial market and the goods market in which this assumption is reasonably appropriate. We do not claim that it holds in all relevant situations for rating agencies. Rather, the separability assumption is made to allow us to focus on informational issues of ratings.

We need to make further assumptions on the common utility function U. We drop the subscript i for now as there is no risk of confusion. First, we assume that the derivative of $U(S, s, q)$ with respect to q, $U_q(S, s, q)$, exists and is strictly positive for each $q \in (0, 1)$.[2]

Assumption 1. *$U_q(S, s, q)$ exists and is strictly positive for each $S = G, B$, $s = g, b$ and $q \in (0, 1)$.*

Signaling games often have a multiplicity of equilibria. One way to minimize the equilibrium selection issue is to ensure that if the agency weakly prefers g to b when the quality is B, then it strictly prefers g to b in state G, and conversely, if the agency weakly prefers b to g in state G, then it strictly prefers b to g when the quality is B. This condition may be referred to as "single-crossing." It will be used to limit equilibrium signaling to one form of misrepresentation, referred to as "inflationary rating" (issuing a good rating when the quality is bad), and to rule out "deflation" (issuing a bad rating when the quality is good). For the single-crossing result to be effective in eliminating unwanted equilibria, we will need it to hold regardless of how different ratings induce different beliefs:

$$U(G, g, q) - U(G, b, q') > U(B, g, q) - U(B, b, q') \tag{2.2}$$

for all $q, q' \in [0, 1]$. Condition (2.2) can be thought of as payoff complementarity between the underlying quality S and the rating g, modified to suit the signaling model so that it

[2] This rules out situations where the market's response to the agency's rating is discrete, for example, where the only choice of the market is whether or not to acquire the client's service at some fixed wage.

holds whenever a switch of the underlying quality for the same rating does not affect the belief q while a switch of the rating for the same quality generally will affect q.[3]

The following assumption on utility functions $U(S, s, q)$, together with Assumption 1, immediately leads to condition (2.2).[4]

ASSUMPTION 2. *$U_q(G, g, q) > U_q(B, g, q)$, $U_q(G, b, q) < U_q(B, b, q)$ for any $q \in (0, 1)$, and*

$$U(G, g, 0) - U(G, b, 0) > U(B, g, 0) - U(B, b, 0). \tag{2.3}$$

Inequality (2.3) in the assumption is simply inequality (2.2) evaluated at $q = q' = 0$. The two conditions on the derivatives of U require that with each rating s the agency benefit more from an improvement in the belief q when the agency is telling the truth about the quality of the client.[5] One may interpret the difference $U(G, g, \cdot) - U(B, g, \cdot)$ as a measure of the agency's reputational concern for honesty. Given the same rating g and any belief q, $U(B, g, q)$ differs from $U(G, g, q)$ because the agency is concerned that the true quality of the client may be discovered, thus revealing a dishonest rating. Similarly, the difference $U(B, b, \cdot) - U(G, b, \cdot)$ is a measure of the agency's reputational concern for competence: for the same rating b and any q, $U(G, b, q)$ differs from $U(B, b, q)$ because when the true quality of the client is discovered, it reveals an inaccurate rating. Assumption 2 requires both differences to be increasing in the client's perceived quality q. This assumption is motivated by the idea that it is more likely (or faster) that the market learns the true quality of the client when the perceived quality is higher. For the consumer magazine example mentioned in the introduction, if an electronic product is new to the market and is of an experience good variety, a higher perceived quality will lead to greater sales and faster consumer learning about its true quality. Similarly, a higher market belief about

[3] Condition (2.2) is stronger than we need for the purpose of the analysis; single-crossing requires it to hold only when the right-hand-side is non-negative.

[4] To see this, note that since $U_q(G, g, q) > U_q(B, g, q)$, we have $U(G, g, q) - U(B, g, q) \geq U(G, g, 0) - U(B, g, 0)$ for any q. Similarly, since $U_q(G, b, q) < U_q(B, b, q)$, we have $U(G, b, q') - U(B, b, q') \leq U(G, b, 0) - U(B, b, 0)$ for any q'. Condition (2.2) then follows from inequality (2.2) in Assumption 2.

[5] The inequalities are sufficient but not necessary for the single-crossing condition (2.2). Our analysis of individual rating and decentralized rating goes through so long as (2.2) holds, but the two inequality conditions on U_q are used for equilibrium construction in the case of centralized rating.

the quality of a job candidate is more likely to result in a better and more challenging job placement, which can quickly reveal the true quality of the candidate, and a higher valuation about a rated stock may lead to a greater transaction volume, which motivates more subsequent research.

The next set of assumptions is made to rule out uninteresting equilibria in order to bring out our main results more effectively. It implies that there exist favorable beliefs that will induce the agency to issue an inflationary rating when the quality is B, but there is no incentive to inflate if beliefs cannot be favorably manipulated.

Assumption 3. $U(B, g, 1) > U(B, b, 0) > U(B, g, 0)$.

Assumptions 1 and 3 imply that there is a unique $q^* \in (0, 1)$ that satisfies

$$U(B, g, q^*) = U(B, b, 0). \tag{2.4}$$

The above equation is the critical indifference condition under quality B that defines a unique inflationary equilibrium in the benchmark scheme of individual rating. Under individual rating, the market for each client has no access to ratings for other clients. Since the clients are exchangeable, the model reduces to N identical signaling games involving the agency and the market. In each such game, an inflationary rating strategy is such that the agency gives g under quality G and randomizes between g and b under quality B. Suppose that there exists $p \in (0, 1)$ such that

$$\frac{\pi}{\pi + (1 - \pi)p} = q^*, \tag{2.5}$$

where π is given in equation (2.1). Then, we have a semi-separating equilibrium in which the agency gives b under B with probability p: by equation (2.4) the agency is indifferent between g and b under quality B, which by the single-crossing condition (2.2) implies that the agency strictly prefers g to b under quality G. We refer to this type of inflationary equilibrium as "full support inflationary equilibrium," as the support of the equilibrium strategy is the same as the space of the signals. Since equation (2.5) can be satisfied by some $p \in (0, 1)$ only if $\pi < q^*$, a full support equilibrium does not exist if $\pi \geq q^*$. Instead, we can construct a "non-full support equilibrium" in which the agency gives g

with probability 1 under B. This is accomplished by specifying the out-of-equilibrium belief that the quality of the client is B with probability 1 when b is observed: since the equilibrium belief that the quality is G when g is observed is equal to the prior probability π, the agency weakly prefers g to b under quality B, which implies that it strictly prefers g to b under G by (2.2). Further, due to the same single-crossing condition (2.2), the above specification of the out-of-equilibrium belief is the only one consistent with the refinement concept of "Divinity" (Banks and Sobel, 1987).[6] We use this refinement throughout the paper, and we refer to a sequential equilibrium that passes the refinement test simply as equilibrium. It follows that there is a unique inflationary equilibrium under individual rating, which is full support if $q^* > \pi$ and non-full support if $q^* \leq \pi$.[7]

The model of individual rating can be interpreted as a model of credibility. The market's perception of the quality of the client given a good rating is q^* in a full support equilibrium, and is π in a non-full support equilibrium. This market belief quantifies equilibrium credibility in our model. From the equilibrium indifference condition (2.4), we see that the value of q^* depends only on the function $U(B, g, \cdot)$ and the value of $U(B, b, 0)$. When the prior probability of good quality is higher than q^*, an increase in the prior translates into an increase in the equilibrium credibility of good ratings by the same amount, which allows the agency to simply pass any client of bad quality as one of good quality. In contrast, when the prior probability is lower than q^*, an increase in the prior has no effect on the equilibrium credibility. The increase in the prior probability means that a good rating is too attractive if the agency keeps the probability of reporting g in state B unchanged, and so the probability of inflated good ratings must increase to restore the equilibrium indifference condition. As a result, the equilibrium credibility, and hence the utility to the agency, is pinned down by the indifference condition so long as the

[6] More precisely, for any out-of-equilibrium belief $\hat{q}$ that the quality is G after b is observed, $U(G, b, \hat{q}) \geq U(G, g, \pi)$ implies that $U(B, b, \hat{q}) > U(B, g, \pi)$. Thus, $\hat{q} = 0$ under the refinement of Banks and Sobel.

[7] With additional assumptions, we can show that no other equilibrium exists under individual rating. In particular, if $U(G, g, 1) > U(G, b, 1)$, then we can rule out all "deflationary" equilibria in which the agency gives b with a positive probability under quality G. However, since the focus of this paper is on the credibility of good ratings, we are only interested in constructing inflationary equilibria under different rating schemes.

agency reports b with a positive probability in equilibrium.[8]

The last assumption is a strengthening of the single-crossing condition (2.2):

ASSUMPTION 4. *For any* $q \in (q^*, 1)$,

$$\frac{U(G,g,q) - U(G,b,0)}{U(B,g,q) - U(B,b,0)} > \frac{U_q(G,g,q)}{U_q(B,g,q)}.$$

By Assumption 2, both the left-hand-side and the right-hand-side of the above inequality are greater than 1. Assumption 4 strengthens condition (2.2) for a particular range of market beliefs. Alternatively, the assumption can be viewed as imposing an upper bound on $U(G,b,0)$, which is the payoff to the agency from a client of quality G when it gives the rating b. Assumption 4 thus requires the payoff to be sufficiently low, or the reputational concerns for competence to be sufficiently great. This assumption is used in the construction of inflationary equilibria under centralized rating to regulate the incentives to issue deflationary ratings.[9]

3. Centralized Rating: A Model of Multi-dimensional Signals

This section considers centralized rating, in which a single rater of the agency rates all N clients and shares the rating information among all markets. Although the payoff to each client depends only on the market's perception of the quality of this client, under centralized rating all the reports are used to make inference about the quality of each client. This means that the agency can potentially coordinate the N ratings in an attempt to influence market perception.

It may not be intuitive that centralized rating creates opportunities for the agency to increase the credibility of good ratings relative to individual rating, especially if the

[8] In equilibrium the agency gets its complete information payoff $U(B,b,0)$ under quality B, but its equilibrium payoff under quality G is $U(G,g,q)$, which is strictly lower than the complete information payoff $U(G,g,1)$. Thus, the ex ante payoff to the agency (before the client's quality is revealed) is lower than what it would obtain if it could commit to truthful revelation of the quality.

[9] With a sufficiently tighter upper bound on the value of $U(G,b,0)$, it is possible to rule out all deflationary equilibria. For example, a sufficient condition is that $U(G,g,0) - U(G,b,0) > (N-1)(U(G,g,1) - U(G,g,0))$, which implies that the smallest loss due to a deflationary rating of a single client exceeds the largest possible gain from all other clients. This assumption would also significantly simplify our analysis for the case of centralized rating. However, it does not hold with an arbitrarily large N.

client qualities are statistically independent. Indeed, it is easy to see that in the case of independent qualities, the equilibrium outcome of individual rating can be supported under centralized rating if the agency independently randomizes between g and b for each client of bad quality with the same probability of choosing b as in individual rating. In this case, the market belief about the quality of any client i with a good rating remains q^*, regardless of the other ratings, as they provide no information about client i's quality under independent qualities and independent randomization. Moreover, this is the only equilibrium outcome under independent randomization. Indeed, a more general result is established below: even when the qualities are correlated and randomizations are coordinated among the clients, any inflationary equilibrium is payoff-equivalent to the benchmark inflationary equilibrium with belief q^* as long as N bad ratings are issued with a positive probability in equilibrium. The key to improved credibility under centralized rating relative to individual rating is to construct an inflationary equilibrium in which the agency never reports N bad ratings, and we provide a characterization of the structure of any such equilibrium. The main result of this section establishes a necessary and sufficient condition for the existence of an equilibrium with improved credibility. This condition requires the prior probability of having N bad qualities to be sufficiently low, so that it is credible for the agency never to issue N bad ratings.

Formally, for the rating agency, the state is now an N-dimensional vector $(S_1, \ldots, S_N)$ where $S_i \in \{G, B\}$ for $i = 1, \ldots, N$. The signal is similarly an N-dimensional vector $(s_1, \ldots, s_N)$ where $s_i \in \{g, b\}$ for $i = 1, \ldots, N$. Given that $S_1, \ldots, S_N$ are exchangeable, we impose a symmetry requirement that the market belief about any client i's quality depend only on the rating s_i of the client and the total number good ratings issued by the agency. For any $i = 1, \ldots, N$, let $q(m)$ be the market belief that $S_i = G$ when $s_i = g$ and $\#\{j : s_j = g\} = m$. Similarly, define $\hat{q}(m)$ to be the market belief that $S_i = G$ when $s_i = b$ and $\#\{j : s_j = g\} = m$. Given the state, the agency chooses the signal vector $(s_1, \ldots, s_N)$ to maximize the sum of utilities $\sum_{i=1}^{N} U(S_i, s_i, q_i)$ where $q_i = q(m)$ if $s_i = g$ and $q_i = \hat{q}(m)$ if $s_i = b$ for all $m = \#\{j : s_j = g\}$. It directly follows from the single-crossing condition (2.2) that while the agency may have an incentive to mislead the markets about the total number of clients of good quality, it has no incentive to mislead the markets about the identity of clients of good quality. That is, for any $i = 1, \ldots, N$, when $\#\{j : S_j = G\} \leq \#\{j : s_j = g\}$, then $S_i = G$ implies $s_i = g$.[10] The same is true about the identity of clients of bad quality when the agency deflates the number of clients of good quality. As a result, we can reduce the state space to a one-dimensional variable representing the number of clients of good quality. Denote the signaling strategy of the agency as $p(m; n)$, the probability of giving m good ratings when the n clients are of good quality. Note that the strategy is multi-dimensional because for each number n we need specify a vector of probability numbers $p(m; n)$ for $m = 0, \ldots, N$. Obviously, we require $\sum_{m=0}^{N} p(m; n) = 1$ for all $n = 0, \ldots, N$.

Let $W(m; n)$ be the expected payoff to the agency when it chooses m good ratings when the number of good quality clients is n. For $m \geq n$, we have

$$W(m; n) = nU(G, g, q(m)) + (m - n)U(B, g, q(m)) + (N - m)U(B, b, \hat{q}(m)).$$

For $m \leq n$, we have

$$W(m; n) = mU(G, g, q(m)) + (n - m)U(G, b, \hat{q}(m)) + (N - n)U(B, b, \hat{q}(m)).$$

The follow lemma imposes some restrictions on equilibrium strategies.

Lemma 1. *(i) For any $m \leq n < n' \leq m'$, if $W(m'; n) \geq W(m; n)$ then $W(m'; n') > W(m; n')$; (ii) for any $n < n' \leq m, m'$ and $q(m') > q(m)$, if $W(m'; n) \geq W(m; n)$, then $W(m'; n') > W(m; n')$; and (iii) for any $m, m' \leq n' < n$ and $\hat{q}(m') > \hat{q}(m)$, if $W(m'; n) \geq W(m; n)$, then $W(m'; n') > W(m; n')$.*

The first part of the lemma means that relative incentive to inflate rather than deflate is stronger when the number of clients of good quality is greater. It implies that if in any equilibrium $p(m'; n) > 0$ for $m' > n$, then $p(m; n') = 0$ for all $n' \in \{n + 1, \ldots, m'\}$ and

[10] To see this, let $\#\{j : S_j = G\} = n$ and $\#\{j : s_j = g\} = m$. If $\#\{j : S_j = G$ and $s_j = g\} = n$, the expected utility to the agency is $nU(G, g, q(m)) + (m - n)U(B, g, q(m)) + (N - m)U(B, b, \hat{q}(m))$. If instead $\#\{j : S_j = G$ and $\#\{s_j = g\} = n' < n$, the expected utility to the agency is reduced by $(n - n')$ times $[U(G, g, q(m)) - U(G, b, \hat{q}(m))] - [U(B, g, q(m)) - U(B, b, \hat{q}(m))]$, which is positive by condition (2.2).

$m \leq n$.[11] The second part of the lemma states that the incentive to inflate to a signal with a more favorable belief about good ratings is stronger when there are more clients of good quality, while the third part states that the incentive to deflate to a signal with a more favorable belief about bad ratings is stronger when the agency has more clients of bad quality.

An inflationary strategy satisfies $p(m;n) = 0$ for all n and all $m < n$. The assumptions made in Section 2, and Lemma 1, are in general insufficient to rule out deflationary equilibrium strategies. Nevertheless, it is natural to focus on inflationary equilibria. Given an inflationary equilibrium let $T = \{m : \sum_{n=0}^{N} p(m;n) > 0\}$ be the set of all signals which are issued with positive probability, and let $l = \min T$ be the smallest signal (with the lowest number of good ratings). Define $T_n = \{m : p(m;n) > 0\}$ as the set of signals sent with positive probabilities when there are n clients of good quality. In an inflationary equilibrium, for each $m \in T$, the market beliefs upon observing m good ratings are

$$q(m) = \frac{\sum_{n=0}^{N} \pi_n p(m;n) n}{m \sum_{n=0}^{N} \pi_n p(m;n)}, \tag{3.1}$$

and $\hat{q}(m) = 0$. The following lemma distinguishes two types of inflationary equilibria.

Lemma 2. *In any inflationary equilibrium, (i) if $l = 0$, then $q(m) = q^*$ for all $m > 0$ and $m \in T$; and (ii) if $l > 0$, then either $q(m) = q^*$ for all $m \in T$ or $q(m) > q(m') > q^*$ for $m, m' \in T$ and $m < m'$.*

An inflationary equilibrium with $l = 0$ does not have full support if $T \neq \{0, 1, \ldots, N\}$. However, part (i) of Lemma 2 establishes that any inflationary equilibrium with $l = 0$ is payoff-equivalent to the full support inflationary equilibrium under individual rating. Although each market can use the ratings of other clients as well as its own client to make inference about the quality of the latter, the rating agency gains no credibility relative to

[11] Conversely, if in any equilibrium $p(m;n') > 0$ for $m < n'$, then $p(m';n) = 0$ for all $n \in \{m, \ldots, n'-1\}$ and $m' \geq n$. Although we restrict to inflationary equilibria in the following analysis, this and the other two results in Lemma 1 are needed for restricting out-of-equilibrium beliefs. Note that part (i) of Lemma 1 does not imply that if $W(m';n) \geq W(m;n)$ for some $m' > m$ then $W(m';n') > W(m;n')$ for all $n' > n$. In other words, the incentive to increase the number of good ratings is not necessarily single-crossing in the number of good quality clients. Indeed, what satisfies single-crossing is the incentive to inflate as opposed to deflate. Similarly (ii) and (iii) of Lemma 1 are not single-crossing conditions either, because they require restrictions on the endogenous variables q.

individual rating. In any such equilibrium, when all clients have bad quality, the agency is indifferent between issuing zero good rating and issuing any number of good ratings in T. These indifference conditions reduce centralized rating to individual rating in terms of payoff to the agency.[12] Part (ii) of the above lemma establishes that in an equilibrium with $l > 0$, either the same indifference conditions are again at work and the market belief corresponding to a good rating is the same regardless of the number of good ratings issued and equal to q^*, or the market beliefs are all strictly greater than q^*. In the second case, the beliefs decrease in the number of good ratings issued, for otherwise the agency would inflate as much as possible. The second type of inflationary equilibria are more interesting, because the agency's ex ante payoff is higher than in the benchmark full support individual rating case. From now on, we distinguish equilibria according to whether they are payoff-equivalent to the full support equilibrium under individual rating: equilibria with $l > 0$ and $q(l) > q^*$ are referred to as non-full support equilibria, and those with $q(m) = q^*$ for all $m \in T$ are referred to as full support equilibria regardless of whether $l = 0$ or $l > 0$. The next lemma provides a partial characterization of the structure of the equilibrium signaling strategy in a non-full support equilibrium.

LEMMA 3. *In any non-full support equilibrium, (i)* $T_l \ni l$*; (ii)* $T_m = \{m\}$ *if* $m \in T$ *and* $m > l$*; (iii)* $\min T_m \geq \max T_{m+1}$ *for all* $m < l$*; (iv)* $T = \{l, ..., N\}$*; and (v)* $q(m) = 1$ *and* $\hat{q}(m) = 0$ *for all* $m < l$.

The structure of the equilibrium strategy described by Lemma 3 is illustrated in Figure 1. In the figure, an arrow from node n to m indicates that $p(m; n) > 0$ in a non-full support equilibrium. When the number of clients of good quality is greater than the minimum number l of good ratings issued, the agency issues a truthful report with probability 1. When the number of clients of good quality is less than l, the agency exaggerates the number of good quality clients; indeed it issues more good ratings when there are fewer

[12] The proof of this result (in the appendix) is more complicated than indicated by this reasoning, because we have to allow for non-full support strategies. This requires the use of the refinement. Later, we will show that all inflationary equilibria have the threshold property in that $T = \{l, ..., N\}$. However, if we restrict to strategies that satisfy this property, then Lemma 2 and part (i) through part (iii) of Lemma 3 below can be established using the equilibrium conditions, without resorting to the refinement.

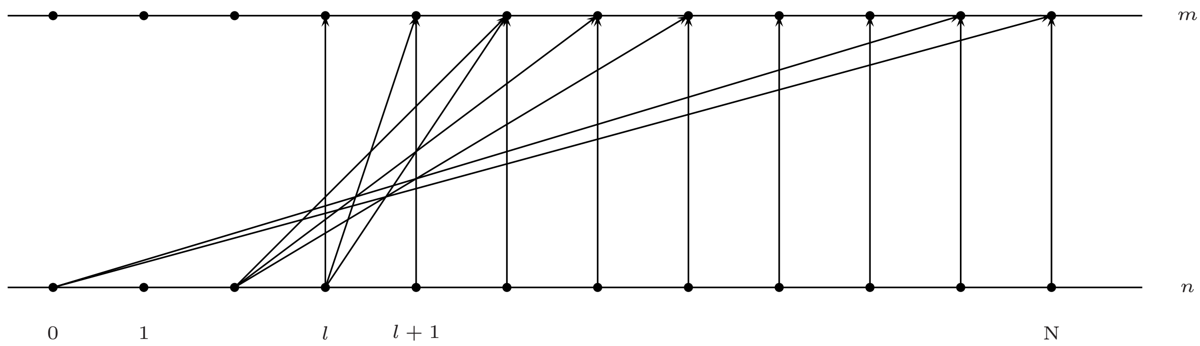

Figure 1

clients of good quality.[13] This characterization follows from the result in Lemma 2 that the credibility of a good rating decreases with the total number of good ratings, and the result in Lemma 1 that the agency has a stronger incentive to inflate to a more credible signal when there are more clients of good quality. Part (iv) of the above lemma establishes that in any non-full support equilibrium the aggregate support of the equilibrium strategy, T, satisfies the threshold property that all signals $m \geq l$ are sent with positive probability. Finally, part (v) of the lemma specifies a unique set of out-of-equilibriums beliefs $q(m)$ and $\hat{q}(m)$ for $m \notin T$ that satisfy the refinement. It is established by showing that if the agency finds it weakly optimal to send an out-of-equilibrium signal $m < l$ when there are $n \neq m$ good quality clients, then the signal is strictly optimal when there are exactly m good quality clients.

To prove the main result of this section, we first use the above characterization of non-full support equilibria to derive necessary and sufficient equilibrium conditions. We then construct an N-step, iterative algorithm. For each $l > 0$, the l-th step of the algorithm covers all possible non-full support equilibria. For each such equilibrium, the values of $p(m; n)$ for $m \geq l$ and $n \leq l$ are recursively assigned, starting from $p(m; l)$ for $m = l, \ldots, N$, such that all the equilibrium conditions are satisfied except for that the sum of probabilities $p(n; 0)$ for $n = 0, \ldots, N$ equals 1. We cover all non-full support equilibria with threshold l by continuously adjusting a "path variable" P^l, which records the recursive assignments

[13] When the number fo clients of good quality is equal to l, the agency may tell the truth in equilibrium (as depicted in Figure 1), or it may randomize between issuing l or issuing more than l good ratings.

of $p(m;n)$ for $m \geq l$ and $n \leq l$. Any point along the algorithm results in a unique value for the sum $\sum_{n=0}^{N} p(n;0)$. At the start of the algorithm, the sum $\sum_{n=0}^{N} p(n;0)$ has a value strictly greater than 1 by construction. If the value of σ exceeds 1 at the end of the last step of the algorithm, we have a non-full support equilibrium with $l = N$. Along the algorithm in each step l, $l = 0, \ldots, N-1$, the sum $\sum_{n=0}^{N} p(n;0)$ continuously decreases, as we continuously adjust the path variable. If the sum $\sum_{n=0}^{N} p(n;0)$ is equal to 1 within the step l, we can construct a non-full support equilibrium with threshold l. Since the sum $\sum_{n=0}^{N} p(n;0)$ is a continuous function, the algorithm establishes the existence of a non-full support equilibrium under centralized rating. Moreover, every non-full support equilibrium corresponds to a distinct value of the path variable in the algorithm where the sum $\sum_{n=0}^{N} p(n;0)$ equals 1. We show the sum $\sum_{n=0}^{N} p(n;0)$ is monotonically decreasing along the algorithm, and thus we have at most one equilibrium with $l > 0$ and $q(l) > q^*$.

PROPOSITION 1. *An inflationary equilibrium exists under centralized rating. Further, a unique non-full support equilibrium exists if and only if* $q^* < 1 - \pi_0$.

PROOF. See the Appendix. *Q.E.D.*

The necessary and sufficient condition for the existence of a unique non-full support equilibrium is rather weak.[14] Unless client qualities are strongly positively correlated, the probability π_0 that no client is of good quality will be small, and the condition in the proposition will be satisfied. Also, if client qualities are independently distributed, the condition is satisfied as long as N is not too small.

4. Decentralized Rating: A Model of Competing Signals

In decentralized rating, rating information is shared among all markets, as in centralized rating, but each client is rated by a self-interested rater of the agency with no access to the

[14] The algorithm in the proof of Proposition 1 considers just one particular kind of full support equilibrium, with $l = 0$ and $p(n;n) = 1$ for all $n > 1$. However, there are typically multiple equilibria with $l = 0$ and $q(m) = q^*$, and they can also coexist with a non-full support equilibrium. For example, consider the model with $N = 2$, $\pi_0 = \pi_2 = \rho/2$ and $\pi_1 = 1 - \rho$ for some $\rho \in (0,1)$, and $q^* > 1/2$. We can show that for ρ in the interval between $2(1-q^*)(2q^*-1)/q^*$ and $2(1-q^*)$, there are three equilibria: one full support equilibrium with $p(1;1) = 1$, another full support equilibrium with $p(2;1) > 0$, and one non-full support equilibrium with $l = 1$.

quality information of other clients. The implicit assumption is that it is possible for the agency to limit the information about client quality available to each rater to the single client that the rater is assigned to.[15] In terms of strategy space, decentralized rating is the same as individual rating, as only independent randomization across clients is feasible. If the underlying client qualities are independently distributed, decentralized rating produces identical equilibrium outcome as in individual rating. However, since ratings information is shared among all markets, when the underlying qualities are correlated, each market can use the other ratings to make inference about the quality of its own client.

In this section we construct an inflationary equilibrium under decentralized rating. Unlike the case of centralized rating, the analysis of decentralized rating requires a model of quality correlation across the clients. In Definition 1 below, we give precise formulations for positive and negative correlations among client qualities. These formulations allow us to give sharp characterizations of inflationary equilibria: under positive (negative) correlation each rater expects a greater number of good ratings conditional on G than conditional on B, in the sense of first order stochastic dominance, and credibility of a good rating is increasing (decreasing) in the total number ratings issued. The main result of this section establishes the existence of a symmetric inflationary equilibrium under decentralized rating, and the necessary and sufficient condition for a full support equilibrium. It turns out that this condition is identical to the condition under individual rating. We postpone to the next section a discussion of how in a decentralized scheme the rating agency can gain in credibility and ex ante payoff under correlated qualities relative to individual rating.

Define a random variable X_i, $i = 1, \ldots, N$, such that $X_i = 1$ if $S_i = G$ and $X_i = 0$ if $S_i = B$. Let $f(X_1, \ldots, X_N)$ represent the joint probability mass function of the random vector $X = (X_1, \ldots, X_N)$.

Definition 1. *We say that X is multivariate totally positive of order 2 (MTP$_2$) if, for all $x, y \in \{0, 1\}^N$,*

$$f(x \vee y) f(x \wedge y) \geq f(x) f(y),$$

[15] How to structure the incentives within the agency to motivate the raters and to restrict their information access is beyond the scope of this paper. We are instead interested in an analysis of credibility from a signaling perspective assuming that the agency has full control over information sharing within the organziation.

where $x \vee y = (\max\{x_1, y_1\}, \dots, \max\{x_N, y_N\})$*;* $x \wedge y = (\min\{x_1, y_1\}, \dots, \min\{x_N, y_N\})$*. We say that* X *is multivariate reverse rule of order 2 (MRR*$_2$*) if the above inequality is reversed.*

The definition of MTP$_2$ is the same as log-supermodularity, also referred to as affiliation. It is a commonly used concept of positive dependence among random variables in the statistics literature (see, for example, Joe, 1997) and in the auction literature (see, for example, Milgrom and Weber, 1982). Similarly, MRR$_2$ can be used to capture the idea of negative dependence among random variables. These dependence concepts are stronger than the notion of "regression dependence" used by Lehmann (1966). We will use the following result (for proof, see Karlin and Rinott, 1980). For each $i = 1, \dots, N$, let $X_{-i} = (X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_N)$.

FACT 1. *If* X *is MTP*$_2$*, then for any* $x_{-i} \in \{0,1\}^{N-1}$, $\Pr[X_{-i} \geq x_{-i} \mid X_i = x_i]$ *is increasing in* x_i*. If* X *is MRR*$_2$*, then* $\Pr[X_{-i} \geq x_{-i} \mid X_i = x_i]$ *is decreasing in* x_i*.*

Since in our model $(S_1, \dots, S_N)$ are exchangeable, for any two realizations x and x' of X such that $\sum_{i=1}^{N}(x_i - x'_i) = 0$, we have $f(x) = f(x')$. Let f_n be the probability mass of x such that $\sum_{i=1}^{N} x_i = n$. Note that the prior probability π_n, $n = 0, \dots, N$, of having n clients of good quality satisfies

$$\pi_n = \binom{N}{n} f_n.$$

The following result is also useful. (The proof follows easily from the definition of MTP$_2$ and MRR$_2$.)

FACT 2. *If* X *is MTP*$_2$*, then for any* $n, n' = 1, \dots, N$ *such that* $n < n'$*, we have* $f_n f_{n'} \geq f_{n+m} f_{n'-m}$ *for all* $m = 1, \dots, n' - n - 1$*. The inequality is reversed if* X *is MRR*$_2$*.*

We focus on symmetric inflationary equilibria in which for each $i = 1, \dots, N$, the common signaling strategy satisfies $\Pr[s_i = g \mid S_i = G] = 1$ and $\Pr[s_i = g \mid S_i = B] = p$ for some $p \in [0, 1]$. Fix some $i = 1, \dots, N$. For each $m = 1, \dots, N$, let $r^G(m)$ be the probability of a total number m of good ratings conditional on $S_i = G$ and $s_i = g$:

$$r^G(m) = \Pr[\#\{j : s_j = g\} = m \mid S_i = G, s_i = g],$$

with $r^G(0) = 0$. In any inflationary equilibrium, we have

$$r^G(m) = \sum_{n=1}^{m} \pi_n^G \beta(N-n, m-n, p), \tag{4.1}$$

where π_n^G is the probability of a total number n of clients of good quality conditional $S_i = G$, defined as

$$\pi_n^G = \Pr[\#\{j : S_j = G\} = n \mid S_i = G],$$

with $\pi_0^G = 0$, and $\beta(t, k, p)$ is the probability of having k successes out of t Bernoulli trials with probability of success p, given by

$$\beta(t, k, p) = \binom{t}{k} p^k (1-p)^{t-k}.$$

Similarly, let

$$r^B(m) = \Pr[\#\{j : s_j = g\} = m \mid S_i = B, s_i = g],$$

with $r^B(0) = 0$. We have

$$r^B(m) = \sum_{n=0}^{m-1} \pi_n^B \beta(N-n-1, m-n-1, p), \tag{4.2}$$

where π_n^B is defined by

$$\pi_n^B = \Pr[\#\{j : S_j = G\} = n \mid S_i = B]$$

with $\pi_N^B = 0$.

Intuitively, for any fixed p, under MTP_2 each individual rater expects to find more good ratings when the quality of his own client is good than when it is bad, while the reverse is true under MRR_2. This idea is formalized in the following lemma.

LEMMA 4. *In any inflationary equilibrium, $\{r^G(m)\}$ first order stochastic dominates $\{r^B(m)\}$ under MTP_2; the reverse is true under MRR_2.*

Given any inflationary equilibrium, the beliefs $q(m)$, $m = 1, \ldots, N$, are given by

$$q(m) = \frac{1}{m} \frac{\sum_{n=1}^{m} \pi_n \beta(N-n, m-n, p) n}{\sum_{n=0}^{m} \pi_n \beta(N-n, m-n, p)}. \tag{4.3}$$

The above formula is valid so long as the denominator is strictly positive, which happens if $p < 1$. We refer to an inflationary equilibrium with $p < 1$ as a full support equilibrium.

LEMMA 5 *In any full support inflationary equilibrium, $q(m)$ is increasing in m under MTP$_2$ and decreasing under MRR$_2$.*

The above result is quite intuitive. In an inflationary equilibrium the perception of a good rating depends on the total number of good ratings in all markets: the perception improves with more good ratings when the client qualities are positively correlated, and it deteriorates when the qualities are negatively correlated. We are now ready to use Lemma 4 and Lemma 5 to establish existence of an inflationary equilibrium. Note that in any inflationary equilibrium, $\hat{q}(m) = 0$ for all $m = 0, \ldots, N-1$.

PROPOSITION 2. *There exists an inflationary equilibrium under decentralized rating. Further, if $\pi < q^*$, there is a full support inflationary equilibrium.*

PROOF. A necessary and sufficient condition for the existence of a full support inflationary equilibrium is that there exists $p \in (0,1)$ such that (i) $s_i = g$ is weakly preferred to $s_i = b$ if $S_i = G$,

$$\sum_{m=1}^{N} r^G(m) U(G, g, q(m)) \geq U(G, b, 0);$$

and (ii) $s_i = g$ and $s_i = b$ yield the same expected payoff if $S_i = B$,

$$\sum_{m=1}^{N} r^B(m) U(B, g, q(m)) = U(B, b, 0). \tag{4.4}$$

Under MTP$_2$, Lemma 4 states that $\{r^G(m)\}$ first order stochastic dominates $\{r^B(m)\}$, while Lemma 5 states that $q(m)$ is increasing m. Therefore,

$$\sum_{m=1}^{N} r^G(m) U(G, g, q(m)) \geq \sum_{m=1}^{N} r^B(m) U(G, g, q(m)).$$

It follows from Assumption 2 that condition (ii) implies condition (i). Under MRR$_2$, $\{r^B(m)\}$ first order stochastic dominates $\{r^G(m)\}$ while $q(m)$ is decreasing m, so again condition (ii) implies condition (i) by Assumption 2. Now, consider the indifference condition ii). If $p = 0$, we have $q(m) = 1$ for all $m = 1, \ldots, N$. By Assumption 3,

$$\sum_{m=1}^{N} r^B(m) U(B, g, q(m)) > U(B, b, 0)$$

when $p = 0$. When $p = 1$, we have $q(N) = \sum_n \pi_n n/N = \pi$ and the left-hand-side of condition ii) becomes $U(B, g, \pi)$. Under Assumption 2, the refinement implies that the out-of-equilibrium belief $\hat{q}(N-1)$ is equal to 0. Thus, if $U(B, g, \pi) < U(B, b, 0)$, or equivalently $\pi < q^*$, then by the intermediate value theorem there exists $p \in (0, 1)$ such that the equilibrium condition ii) is satisfied, and hence there is a full support inflationary equilibrium. If instead $\pi \geq q^*$, with the out-of-equilibrium belief $\hat{q}(N-1)$ set to 0, g is weakly preferred to b under quality B, implying that g is strictly preferred to b under G. We thus have a non-full support equilibrium with $p = 1$. *Q.E.D.*

The condition for the existence of a full support inflationary equilibrium is identical to the condition for the existence of the unique full support inflationary equilibrium under individual rating.[16] When $S_1, \ldots, S_N$ are independently distributed, we have $f_n = \pi^n(1-\pi)^{N-n}$ for each $n = 0, \ldots, N$. Then, for each $m = 1, \ldots, N$, direct calculations reveal that

$$q(m) = \frac{1}{m} \frac{\sum_{n=1}^{m} (\pi/(1-\pi)p)^n (1/((m-n)!(n-1)!))}{\sum_{n=0}^{m} (\pi/(1-\pi)p)^n (1/((m-n)!n!)} = \frac{\pi}{\pi + (1-\pi)p}.$$

Thus, under independence, decentralized rating reduces to individual rating.

5. Comparing Rating Schemes: Credibility and Welfare

Comparison between centralized rating and decentralized rating in terms of equilibrium credibility of good ratings and ex ante payoffs to the agency generally depends on the underlying correlation structure. In Proposition 1, we have established that there always exists an inflationary equilibrium under centralized rating that does at least as well as the full support inflationary equilibrium under individual rating. Moreover, when $q^* < 1 - \pi_0$, there is a unique non-full support equilibrium that does strictly better. This condition is

[16] In general, multiple full support inflationary equilibria may occur. For example, in the model of $N = 2$, $\pi_0 = \pi_2 = \rho/2$ and $\pi_1 = 1 - \rho$ for some $\rho \in (0, 1)$, and $q^* > 1/2$, the indifference condition is given by $r^B(2)U(B, g, q(2)) + (1 - r^B(2))U(B, g, q(1)) = U(B, b, 0)$. Both $q(2)$ and $q(1)$ are decreasing functions of p, while $r^B(2)$ increases with p. With positive correlation ($\rho > 1/2$), we have $q(2) > q(1)$, and thus the left-hand-side of the indifference condition is not necessarily monotone in p. leading to possibly multiple full support inflationary equilibria. We note that the comparison result in the next section are not affected by the possibility of multiple equlibria.

rather weak, and is easily satisfied when the qualities are independently distributed, as long as N is not too small. In contrast, with independently distributed qualities, the unique inflationary equilibrium under decentralized rating is payoff-equivalent to the full support inflationary equilibrium under individual rating. Thus, we expect centralized rating to dominate decentralized rating for the agency when there is weak correlation among the qualities.

The next set of results shows that both equilibrium credibility of good ratings and ex ante payoff to the agency under decentralized rating improve relative to the benchmark of individual rating when the qualities are correlated. Unlike the above discussion about centralized rating, comparison of credibility between decentralized rating and individual rating requires a precise definition of equilibrium credibility of good ratings. For any p, the probability of inflating, consider the following expression:

$$\sum_{n=1}^{N} \frac{n\pi_n}{N\pi} \sum_{m=1}^{N} \beta(N-n, m-n, p) q(m). \tag{5.1}$$

The above may be thought of as an average measure of credibility of good ratings under decentralized rating, as the credibility of a single given good rating depends on the total number of good ratings. It is an average across states, with each state weighted both by the prior probability of the state and by the number of good quality clients in the state. Under individual rating, the same expression (5.1) applies, but $q(m)$ is constant and equal to q because the markets are separate. Since $\sum_{m=1}^{N} \beta(N-n, m-n, p) = 1$, the above definition of credibility is consistent with the definition given under individual rating. Further, if we replace $\beta(N-n, m-n, p)$ with $p(m; n)$ in (5.1), we have a measure of credibility under centralized rating.[17] It follows that for any non-full support equilibrium, the equilibrium credibility is greater than q^*, the equilibrium credibility in the benchmark full support inflationary equilibrium under individual rating. To make comparison of equilibrium credibility between decentralized rating and individual rating, we first note

[17] It turns out that the definition (5.1) of credibility corresponds one-to-one with the average expected loss of the N markets. With the quadratic loss function given in Section 2, the average expected loss under individual, centralized, decentralized rating schemes is $(\delta_G - \delta_B)^2 \pi(1-Q)$, where Q is given by (5.1) for the corresponding scheme.

that for each $n = 1, \ldots, N$,

$$\pi_n^G = \frac{\pi_n}{\pi} \frac{\binom{N-1}{n-1}}{\binom{N}{n}} = \frac{\pi_n}{\pi} \frac{n}{N}.$$

It then follows from equation (4.1) that under decentralized rating, our measure of credibility satisfies

$$\sum_{n=1}^{N} \frac{n\pi_n}{N\pi} \sum_{m=1}^{N} \beta(N-n, m-n, p)q(m) = \sum_{n=1}^{N} r^G(m)q(m).$$

Thus, the average measure of credibility of good ratings under decentralized rating is equal to the market belief expected by a rater with a good quality client. Next, we present a preliminary result.

LEMMA 6. *Under decentralized rating, for any $p < 1$,*

$$\sum_{m=1}^{N} r^B(m)q(m) \leq \frac{\pi}{\pi + (1-\pi)p}.$$

Under individual rating, the market's belief upon observing g is given by $\pi/(\pi + (1 - \pi)p)$ if p is the probability that rating g is issued under quality B. Thus, under decentralized rating the weighted average of the market belief conditional on a bad quality client is lower than the market belief under individual rating for the same probability of rating inflation. That is, a rater that issues an inflated rating on a bad quality client expects on average a less favorable market belief under either positive or negative correlation than when qualities are statistically independent. The intuition is that under either positive correlation (MTP_2) or negative correlation (MRR_2), for the rater with a bad quality client, the weights are smaller for higher market beliefs q, so that the weighted average is lower than the average when the qualities are independently distributed for independent randomizations with the same probability of inflation p. For example, under positive correlation, a higher market belief q is associated with a greater number of good ratings, but since a rater with a bad quality client expects statistically fewer good quality clients and thus fewer good ratings, a higher market belief receives a smaller weight. Conversely, under negative correlation, a higher market belief q is associated with a smaller number of good ratings, and again receives a smaller weight in the belief of the rater with a bad quality client.

Lemma 6 illustrates the idea that in our model of credibility correlation across client qualities imposes a discipline on incentives to inflate by making it harder for each individual rater to fool its own market. In fact, if $U(B, g, q)$ is weakly concave in q, then at any full support inflationary equilibrium in decentralized rating, the equilibrium probability of inflation is lower than the full support equilibrium probability of inflation under individual rating.[18] Furthermore, under the same condition on $U(B, g, \cdot)$, the equilibrium credibility is higher under decentralized rating than under individual rating.

Proposition 3. *Suppose $U(B, g, q)$ is concave in q. Then, in any full support inflationary equilibrium under decentralized rating, the probability of inflation is lower and the credibility is higher than in the full support inflationary equilibrium under individual rating.*

Proof. In a full support inflationary equilibrium, for each $i = 1, \ldots, N$, we must have the indifference condition between $s_i = g$ and $s_i = b$. This condition gives

$$\sum_{m=1}^{N} r^B(m) U(B, g, q(m)) = U(B, g, q^*). \tag{5.2}$$

Since $U(B, g, q)$ is concave in q, we have

$$U(B, g, \sum_{m=1}^{N} r^B(m) q(m)) \geq U(B, g, q^*). \tag{5.3}$$

It then follows from Lemma 6 that

$$U(B, g, \pi/(\pi + (1-\pi)p)) \geq U(B, g, q^*),$$

where p is the equilibrium probability of inflation. Comparing the above inequality to equation (2.5) in a full support inflationary equilibrium under individual rating, we immediately have that the equilibrium probability of inflation is lower under decentralized rating than under individual rating.

[18] This comparison between equilibrium probabilities of inflation under decentralized and individual rating holds so long as the function is not too convex. An even stronger result can be obtained if one imposes more structure on the form of quality correlation than MTP_2 or MRR_2. Consider for example the model of $N = 2$, $\pi_0 = \pi_2 = \rho/2$ and $\pi_1 = 1 - \rho$ for some $\rho \in (0, 1)$, and $q^* > 1/2$. We can show that if $U(B, g, q)$ is concave in q, then an increase in ρ for $\rho > 1/2$ or a decrease in ρ for $\rho < 1/2$ reduces the equilibrium probability of ratings inflation.

From Lemma 4 and Lemma 5, we have that $\{r^G(m)\}$ first order stochastically dominates $\{r^B(m)\}$ and $q(m)$ is increasing under MTP_2, while $\{r^B(m)\}$ stochastically dominates $\{r^B(m)\}$ and $q(m)$ is decreasing under MRR_2. In either case, we have

$$\sum_{m=1}^{N} r^G(m)q(m) \geq \sum_{m=1}^{N} r^B(m)q(m).$$

From inequality (5.3) we then have

$$\sum_{m=1}^{N} r^G(m)q(m) \geq q^*.$$

Thus, the equilibrium credibility is higher under decentralized rating than under individual rating. *Q.E.D.*

For welfare comparison between decentralized rating and individual rating under either MTP_2 or MRR_2, we say that $U(B, g, \cdot)$ is "more concave" than $U(G, g, \cdot)$ if there is a weakly concave function H such that $U(B, g, q) = H(U(G, g, q))$. We have the following result.

Proposition 4. *Suppose $U(B, g, q)$ is more concave in q than $U(G, g, q)$. Then, the agency's payoff in a full support inflationary equilibrium under decentralized rating is higher than the full support inflationary equilibrium under individual rating.*

Proof. If $U(B, g, q)$ is more concave in q than $U(G, g, q)$, the indifference condition (5.2) implies

$$\sum_{m=1}^{N} r^B(m)U(G, g, q(m)) \geq U(G, g, q^*).$$

Under MTP_2, $\{r^G(m)\}$ first order stochastically dominates $\{r^B(m)\}$ and $q(m)$ is increasing, and so

$$\sum_{m=1}^{N} r^G(m)U(G, g, q(m)) \geq U(G, g, q^*).$$

Under MRR_2, $q(m)$ is decreasing but $\{r^B(m)\}$ stochastically dominates $\{r^B(m)\}$, so again the inequality is true. *Q.E.D.*

Compared to individual rating, in decentralized rating each client i is exposed to a greater risk when $S_i = G$ because of the uncertainty regarding the ratings of other clients.

However, the beliefs are more favorable under G than under B in the sense of first order stochastic dominance. Thus, welfare improves so long as the agency is not too much more risk-averse when $S_i = G$ than when $S_i = B$.

Since the strategy space in decentralized rating is the same as in individual rating, the above results show that the gains in credibility and welfare in decentralized rating come from sharing ratings information among the markets. We expect that the gains are larger when the correlation is stronger. Indeed, the next proposition establishes that when the correlation across client qualities is almost perfect, there is a limit inflationary equilibrium with "truth-telling," i.e., the equilibrium probability of inflation converges to 0. Let $\{\pi_0^k, \ldots, \pi_N^k\}$ be a sequence of probability distributions that satisfy MTP_2, such that (i) $\lim_{k\to\infty} \sum_{n=1}^{N-1} \pi_n^k = 0$; and (ii) $\lim_{k\to\infty} \pi_N^k/(\pi_N^k + \pi_0^k) < q^*$. The first condition means that the states become almost perfectly positively correlated as k becomes arbitrarily large. The second condition guarantees that there exists no pooling equilibrium with $p(N; n) = 1$ for all n when k is large.

Proposition 5. *Under decentralized rating, truth-telling is a limit inflationary equilibrium when k goes to infinity.*

Proof. Equation (4.4) is necessary and sufficient for an inflationary equilibrium under decentralized rating. As in the proof of Proposition 2, for $p = 0$, the left-hand-side of (4.4) is strictly larger than the right-hand-side for any k. Next, for all $p > 0$, the limit of left-hand-side as k goes to infinity is strictly less than

$$p^{N-1}U(B, g, 1) + (1 - p^{N-1})U(B, g, 0).$$

This is because in the limit when the correlation is perfect, from equation (4.3) we have $q(m) = 0$ for all $m < N$, while $q(N) < 1$. Let $\tilde{p}$ be the value of p that solves

$$p^{N-1}U(B, g, 1) + (1 - p^{N-1})U(B, g, 0) = U(B, b, 0).$$

Then, for all $0 < p < \tilde{p}$, the limit of the left-hand-side of (4.4) as k goes to infinity is strictly smaller than the right-hand-side. Hence, for each p there exists $k(p)$ such that for all $k > k(p)$ there is an inflationary equilibrium with the probability of inflation strictly between 0 and p. Since this construction of $\tilde{p}$ and $k(p)$ holds for all p, by taking p arbitrarily close to 0, we can establish truth-telling (i.e., $p = 0$) as a limit point of a sequence of inflationary equilibria for k going to infinity. *Q.E.D.*

While strong correlation enhances credibility and improves welfare in decentralized rating, the opposite is true in centralized rating. To see this, note that the conditions made on the convergence of the sequence of the distributions π^k imply that in the limit of $k \to \infty$, there is no non-full support equilibrium by Proposition 1. Thus, centralized rating cannot improve upon individual rating when correlation is almost perfect.[19] Correlation of the underlying qualities reduces the manipulation room both under decentralized rating and under centralized rating. Under decentralized rating the constraint imposed by correlation makes it harder for a rater to fool the market with a good rating, and forces the individual raters to tone down the exaggeration. This then results in a greater ex ante payoff relative to individual rating. In contrast, strong correlation makes correlated randomization under centralized rating less effective.

For analysis involving non-extreme values of correlation, the notion of the "degree of dependence" is ambiguous, and a more specific description of the multivariate probability process is required. We illustrate how the degree of dependence affects the welfare properties of centralized rating versus decentralized rating using the probability distribution given in Joe (1997):

$$\pi_n = \binom{N}{n} \frac{\Pi_{i=0}^{n-1}(\pi + i\gamma)\Pi_{i=0}^{N-n-1}(1 - \pi + i\gamma)}{\Pi_{i=0}^{N-1}(1 + i\gamma)},$$

where $\gamma \geq -(N-1)^{-1} \min\{\pi, 1-\pi\}$. This probability mass function is fairly general as it encompasses four classes of probability distributions: (i) beta-binomial; (ii) binomial; (iii) hypergeometric; and (iv) Polya-Eggenberger distribution. It satisfies MTP_2 when $\gamma > 0$

[19] In Proposition 5 we have considered only the limit case of perfect positive correlation. For the model of $N = 2$, $\pi_0 = \pi_2 = \rho/2$ and $\pi_1 = 1 - \rho$ for some $\rho \in (0, 1)$, and $q^* > 1/2$, perfect negative correlation is well-defined. Under decentralized rating, there is unique inflationary equilibrium with negative correlation ($\rho < 1/2$). As ρ converges to 0, the equilibrium converges to truth-telling, with $p = 0$. Under centralized rating, when ρ is sufficiently small (precisely, when $\rho < 2(1 - q^*)/(2q^* - 1)$), the unique non-full support equilibrium has $l = 1$, $p(1; 1) = 1$, and $p(1; 0), p(2; 0) > 0$. As ρ decreases, $p(1; 0)$ increases and $p(2; 0)$ decreases. In the limit as ρ converges to 0, both $p(1; 0)$ and $p(2; 0)$ are strictly positive, so the limit equilibrium strategy is not truth-telling. However, the ex ante payoff of the agency in the limit equilibrium is the same as in truth-telling.

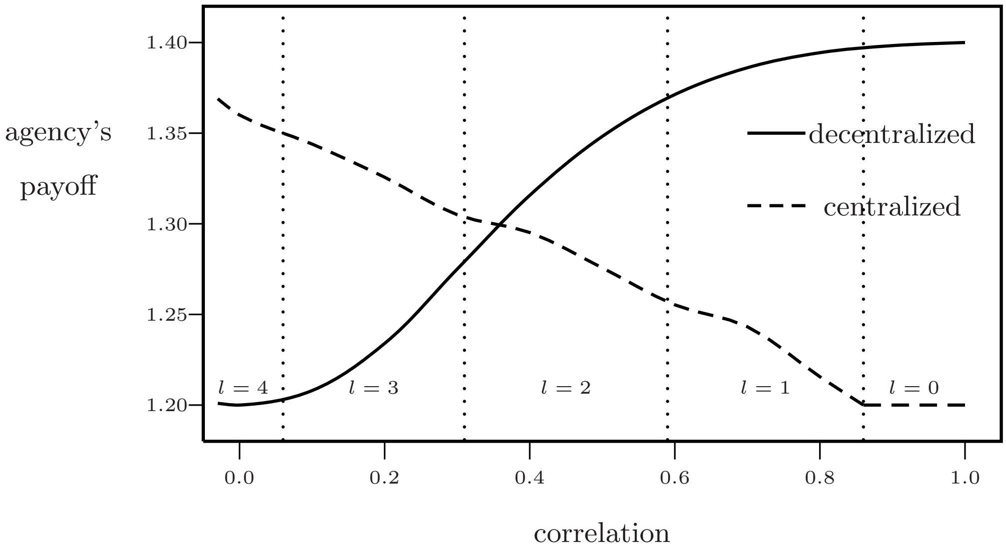

Figure 2

and satisfies MRR_2 when $\gamma < 0$. For this distribution, Joe (1997) shows that $\text{E}[Y_i] = \pi$ and $\text{cor}(Y_i, Y_j) = (1 + \gamma^{-1})^{-1}$. Therefore, the higher the absolute value of γ, the greater is the degree of (positive or negative) dependence among the qualities of different clients. We use this distribution, with $N = 10$ and $\pi = 0.5$, to calculate the equilibria under centralized rating and the inflationary equilibria under decentralized rating for different degrees of dependence.[20] Figure 2 shows the ex ante welfare for the rating agency under the two scenarios for different values of γ, for payoff functions $U(G, g, q) = 1 + q$, $U(B, b, q) = 0.8(1 + q)$, $U(G, b, q) = 0.6(1 + q)$ and $U(B, g, q) = 0.5(1 + q)$. For centralized rating, equilibrium threshold decreases weakly with γ. When the correlation is about zero, there is a non-full supprt equilibrium with threshold $l = 4$. When $\gamma > 0.61$, the only equilibrium is the full support equilibrium ($l = 0$) with payoff equal to that under individual rating. For decentralized rating, equilibrium probability of inflation is at a maximum when $\gamma = 0$, and decreases monotonically as γ becomes either more positive or more negative. In the figure, centralized rating dominates decentralized rating for all $\gamma < 0.56$.

[20] When $N > 2$, the maximum degree of negative correlation among a group of exchangeable binary random variables is bounded away from -1. In our probability distribution with $N = 10$ and $\pi = 0.5$, the maximum degree of negative correlation is approximately -0.059. We cannot compare centralized rating with decentralized rating beyond this degree of negative correlation.

6. Concluding Remarks

Providers of information often care about the way their information is used. The desire to create favorable beliefs about its clients may cause the rating agency to inflate its assessment of the quality of its clients. The exuberant stock recommendations made during the internet boom, and the failure of auditors to raise alerts in a number of recent corporate scandals have heightened the public's concern about the potential conflict of interests inherent in situations where raters are advocates for the rated. Moore et al. (2005) study this kind of problems and their possible solutions from a variety of perspectives. Gentzkow and Shapiro (2006) study how competition and the concern for reputation may constrain biased reporting by the mass media. Chan, Li and Suen (forthcoming) use a signaling model to understand why grades in academia tend to be exaggerated. None of these papers, however, examines how the credibility of ratings can be improved by coordinating or decentralizing the rating decisions, which is the main focus of our paper.

In the literature on reputational cheap talk, a bad sender type may provide useful information to the receiver to establish the credibility as a good sender type so as to extract future surplus (Sobel 1985; Benabou and Laroque 1992; Morris, 2001; Morgan and Stockton, 2003). This effect arises in a cheap talk game where the sender has private information on both the relevant state-of-world and his personal bias. As a costly signaling model of credibility, our model of individual rating has a single source of private information. The equilibrium credibility of a good rating is quantifiable in our model and corresponds one-to-one with the welfare of the rating agency. These features make our model of credibility a natural benchmark for comparisons with centralized and decentralized rating schemes.

This paper is related to the small literature on multi-dimensional signaling (Quinzii and Rochet, 1985; Engers, 1987). This literature focuses on the conditions under which separation of types occurs. Technically, the models in the existing literature are concerned with multi-dimensional private information for the sender and one-dimensional signals. Our signaling model of centralized rating assumes exchangeability of the components of the state vector, so that the private information is the number of good clients, which is one-dimensional. However, the signal space is multi-dimensional, as a strategy specifies a number of good ratings for each number of good clients. As a result, the single crossing condition in the benchmark case of individual rating is not completely effective in either centralized rating or decentralized rating. This feature complicates the analysis but enriches the comparison analysis for the different schemes of rating. Chakraborty and Harbaugh (forthcoming) show that in a cheap talk game where a send and a receiver interact on several unrelated issues, the sender can credibly communicate to the receiver the ranking of the private signals even if the conflicts between them are too great to permit credible communication of the signal on any single issue.[21] Their result has the interpretation that bundling independent reports may help information transmission, which is related to our result for centralized rating. However, their result follows from the observation that the sender has no incentive to deceive the receiver about the ranking of the signals, while our analysis is based on coordination of the reports in a costly signaling model.

In the signaling literature, there are existing models that involve multiple senders (Bagwell and Ramsey, 1991; Hertzendorf and Overgaard, 2001). In these models, the senders know each other's types and interact with each other directly through their signals. In contrast, the raters in our model of decentralized rating have private information about their own private types and have no direct interaction except that their signals are jointly used by the receivers to make inference about the types of the senders. Our model is therefore a model of competing signals, rather than a model of competing senders.

Appendix

Proof of Lemma 1. (i) The difference of differences $[W(m';n')-W(m;n')]-[W(m';n)-W(m;n)]$ is equal to $(n'-n)$ times

$$[U(G,g,q(m'))-U(G,b,\hat{q}(m))]-[U(B,g,q(m'))-U(B,b,\hat{q}(m))],$$

which is positive by equation (2.2).

[21] The idea that linking decisions can be payoff-improving also appears in the literature on bundling in monopoly pricing (Adam and Yellen, 1976; McAfee, McMillan, Whinston, 1979) and incentive design (Maskin and Tirole, 1990; Jackson and Sonnenschein, forthcoming).

(ii) The difference between $W(m';n') - W(m;n')$ and $W(m';n) - W(m;n)$ is $(n'-n)$ times

$$[U(G,g,q(m')) - U(G,g,q(m))] - [U(B,g,q(m')) - U(B,g,q(m))],$$

which is positive by Assumption 2 since $q(m') > q(m)$.

(iii) The difference between $W(m';n') - W(m;n')$ and $W(m';n) - W(m;n)$ is $(n-n')$ times

$$[U(B,b,\hat{q}(m')) - U(B,b,\hat{q}(m))] - [U(G,b,\hat{q}(m')) - U(G,b,\hat{q}(m))],$$

which is positive by Assumption 2 since $q(m') > q(m)$. *Q.E.D.*

Proof of Lemma 2. Let $\overline{m}_1 = N$, and iteratively define $\underline{m}_k$ as the smallest integer such that $\{\underline{m}_k, \ldots, \overline{m}_k\} \subseteq T$ and $\overline{m}_{k+1}$ as the largest integer smaller than $\underline{m}_k$ such that $\overline{m}_{k+1} \in T$. We have the following claims regarding equilibrium and out-of-equilibrium beliefs.

(1) In any inflationary equilibrium, if $q(m) < q^*$ for some $m \in T$ and $m-1 \in T$, then $q(m-1) < q(m) < q^*$. Otherwise, $W(m-1;n) > W(m;n)$ for all $n \leq m-1$, and either $m \notin T$ or $q(m) = 1$, a contradiction in either case.

(2) If $q(m) < q(m')$ for all $m, m' \in \{\underline{m}_k, \ldots, \overline{m}_k\}$ and $m < m'$, then $p(n;n) < 1$ for all $n \in \{\underline{m}_k, \ldots, \overline{m}_k - 1\}$. Otherwise, $W(n;n) \geq W(n+1;n)$ implies $W(n;n') > W(n+1;n')$ for all $n' < n$ by part (ii) of Lemma 1, implying that either $n+1 \notin T$ or $q(n+1) = 1$, a contradiction in either case.

(3) For any $x \notin T$, if for each k such that $\underline{m}_k > x$ we have $q(m) < q(m') < q^*$ for all $m, m' \in \{\underline{m}_k, \ldots, \overline{m}_k\}$ and $m < m'$, then the out-of-equilibrium belief $\hat{q}(x) = 0$. Suppose instead $\hat{q}(x) > 0$. We will show that $W(x;n) \geq W(t_n;n)$ for any $n > x$ and $t_n \in T_n$ implies that $W(x;n-1) > W(t_{n-1};n-1)$ for any $t_{n-1} \in T_{n-1}$. An iteration of this result then leads to $\hat{q}(x) = 0$ by the refinement, a contradiction that establishes the claim. For any $n > x$, there are two cases. In the first case, either there is no k with $n = \overline{m}_k + 1$, which by claim (2) above implies that there exists a $t_{n-1} \in T_{n-1}$ such that $t_{n-1} \geq n$, or $n = \overline{m}_k + 1$ for some k but $p(n-1;n-1) < 1$, which implies again that there exists a $t_{n-1} \in T_{n-1}$ such that $t_{n-1} \geq n$. Then, since $W(x;n) \geq W(t_n;n)$, we have $W(x;n) \geq W(t_{n-1};n)$ by optimality, which implies $W(x;n-1) > W(t_{n-1};n-1)$ by part (i) of Lemma 1. In the

second case, $n = \overline{m}_k + 1$ for some k and $p(n-1; n-1) = 1$. Since $\hat{q}(x) > 0$ and $\hat{q}(n-1) = 0$, by part (iii) of Lemma 1, $W(x; n) \geq W(n-1; n)$ implies that $W(x; n-1) > W(n-1; n-1)$.
(4) If for some $\overline{m}_k > 0$ we have $q(m) < q^*$ for all $m > \overline{m}_k$, then $q(\overline{m}_k) < q^*$. To see this, suppose that $q(\overline{m}_k) \geq q^*$. By construction $\overline{m}_k + 1 \notin T$ and by claim (3) $\hat{q}(\overline{m}_k + 1) = 0$. Next, for any $n \leq \overline{m}_k$ and any $t_n \in T_n$, $W(\overline{m}_k + 1; n) \geq W(t_n; n)$ implies $W(\overline{m}_k + 1; n) \geq W(\overline{m}_k; n)$. Since $\hat{q}(\overline{m}_k + 1) = 0$, and $q(\overline{m}_k) \geq q^*$ by assumption, $W(\overline{m}_k + 1; n) \geq W(\overline{m}_k; n)$ implies $q(\overline{m}_k + 1) \geq q^*$. It follows that $W(\overline{m}_k + 1; \overline{m}_k + 1) > W(t_{\overline{m}_k+1}; \overline{m}_k + 1)$ for any $t_{\overline{m}_k+1} \in T_{\overline{m}_k+1}$. The refinement then implies $q(\overline{m}_k + 1) = 1$, a contradiction.

Using the above four claims, we now establish that in any equilibrium $q(m) \geq q^*$ for all $m \in T$. Suppose instead $q(m) < q^*$ for some $m \in T$. Then, $q(N) < q^*$; otherwise, $W(N; n) > W(m; n)$ for all $n \leq m$, contradicting the assumption that $m \in T$. Claims (1) and (4) above then imply that $q(m) < q^*$ for all $m \in T$ and $m > 0$. If $l > 0$, we have $W(0; 0) > W(m; 0)$ for all $m \in T$ regardless of $\hat{q}(0)$, a contradiction. If $l = 0$ and $1 \in T$, we then have $p(0; 0) = 1$ and $q(1) = 1$, again a contradiction. Finally, if $l = 0$ and $1 \notin T$, since $\hat{q}(1) = 0$ by claim (3) above and $q(t_1) < q^*$ for any $t_1 \in T_1$, we have that $W(1; 1) > W(t_1; 1)$ whenever $W(1; 0) \geq W(0; 0)$, which then implies $q(1) = 1$ by the refinement, again a contradiction.

(i) For the first part of the lemma, note that if $q(m) > q^*$ for some $m > 0$ and $m \in T$, then $W(m; 0) > W(0; 0)$. This contradicts the assumption that $l = 0$. Thus, $q(m) = q^*$ for all $m > 0$ such that $m \in T$.

(ii) For the second part, note that if $q^* \leq q(m') < q(m)$ or if $q^* < q(m') = q(m)$ for some $m, m' \in T$, and $m > m'$, we have $W(m; n) > W(m'; n)$ for all $n \leq m'$, contradicting the assumption that $m' \in T$. Thus, it remains to prove that if $q(m) = q^*$ for some $m \in T$, then $q(m') = q^*$ for all $m \in T$ and $0 < m' < m$. To establish this last claim, suppose $q(m) = q^*$ for some $m \in T$. There are two cases. First suppose $m - 1 \in T$. If $q(m-1) > q^*$, then $W(m-1; n) > W(m; n)$ for all $n \leq m - 1$. This implies $q(m) = 1$, a contradiction. Thus $q(m-1) = q^*$. Next suppose $m - 1 \notin T$. Let $\overline{m}$ be the largest signal in T that is smaller than m. Since $q(m') = q^*$ for all $m' \in T$ and $m' \geq m$, we have $W(t_{\overline{m}+1}; \overline{m} + 1) = W(m'; \overline{m} + 1)$ for any $t_{\overline{m}+1} \in T_{\overline{m}+1}$. For all $n > \overline{m} + 1$, since $W(\overline{m} + 1; n) \geq W(t_n; n)$ for $t_n \in T_n$ implies $W(\overline{m} + 1; \overline{m} + 1) > W(t_n; \overline{m} + 1) =$

$W(t_{\overline{m}+1}, \overline{m}+1)$, it follows from the refinement that $\hat{q}(\overline{m}+1) = 0$. Given this, if $q(\overline{m}) > q^*$, then $W(\overline{m}+1; n) \geq W(t_n; n)$ for any $n \leq \overline{m}$ and any $t_n \in T_n$ implies $q(\overline{m}+1) > q^*$. It then follows that $W(\overline{m}+1; \overline{m}+1) > W(t_{\overline{m}+1}; \overline{m}+1)$, and therefore $q(\overline{m}+1) = 1$ by the refinement, a contradiction. Thus, $q(\overline{m}) = q^*$. *Q.E.D.*

Proof of Lemma 3. (i), (ii) Suppose $p(m'; m) > 0$ for some $m, m' \in T$ and $m' > m \geq l$. By optimality we have $W(m'; m) - W(m; m) \geq 0$. Since $q(m) > q(m')$ by Lemma 2, part (ii) of Lemma 1 implies $W(m'; n) - W(m; n) > 0$ and hence $p(m; n) = 0$ for all $n < m$. Part (i) of the lemma follows by setting $m = l$ and noting that $p(l; l) = 0$ implies $l \notin T$, a contradiction. Part (ii) follows by noting that for any $m \in T$ and $m > l$, $p(m; m) < 1$ implies that $q(m) = 1$, contradicting Lemma 2.

(iii) By optimality $W(\min T_m; m) \geq W(n; m)$ for all $n \geq \min T_m$. Since from Lemma 2 we have $q(\min T_m) > q(n)$, part (ii) of Lemma 1 implies that $W(\min T_m; m') > W(n; m')$ for all m' such that $m < m' \leq l \leq \min T_m$. Hence $p(n; m') = 0$, and $\max T_{m'} \leq \min T_m$.

(iv) Let $x > 0$ be the largest signal such that $x \notin T$. Note that in any inflationary equilibrium $x < N$. We first show by contradiction that $\hat{q}(x) = 0$. This claim follows from the refinement if $W(x; n) \geq W(t_n; n)$ for any $n > x$ and $t_n \in T_n$ implies that $W(x; x) > W(t_x; x)$ for any $t_x \in T_x$. To establish the latter claim, note that for any $n > x + 1$, by optimality $W(x; n) \geq W(t_n; n)$ implies that $W(x; n) \geq W(x+1; n)$. Since $\hat{q}(x) > 0$ and $\hat{q}(x+1) = 0$, part (iii) of Lemma 1 implies that $W(x; x+1) > W(x+1; x+1)$. Since $x + 1 \in T_{x+1}$ by (i) and (ii) above, we have $W(x; x+1) > W(t_{x+1}; x+1)$ for all $t_{x+1} \in T_{x+1}$, which by optimality implies $W(x; x+1) > W(t_x; x+1)$. Since $t_x \geq x+1$, by part (i) of Lemma 1, we have $W(x; x) > W(t_x; x)$.

Next, we claim that $q(x) = 1$. To see this, note that for each $n < x$ and any $t_n \in T_n$, by optimality $W(x; n) \geq W(t_n; n)$ implies $W(x; n) \geq W(x+1; n)$. Since $\hat{q}(x) = \hat{q}(x+1) = 0$, if $W(x; n) \geq W(x+1; n)$ then $q(x) > q(x+1)$. Since $t_x \geq x+1$, from Lemma 2 we have $q(x) > q(t_x)$. It then follows from part (ii) of Lemma 1 that $W(x; x) > W(t_x; x)$. By the refinement, $q(x) = 1$. Thus there is no $m < x$ such that $m \in T$.

(v) First, consider $\hat{q}(0)$. By (ii) and (iii) above, $p(N; 0) > 0$; otherwise, $q(N) = 1$, which is a contradiction. For any $n > 0$ and $t_n \in T_n$, if $W(0; n) \geq W(t_n; n)$ then by optimality $W(0;n) \geq W(N;n)$. By part (i) of Lemma 1, we have $W(0;0) > W(N;0)$. It follows from the refinement that $\hat{q}(0) = 0$.

Next, we show that $\hat{q}(m) = 0$ for any $m = \{1, \dots, l-1\}$. Suppose instead $\hat{q}(m) > 0$. We will show that $W(m;n) \geq W(t_n;n)$ for any $n > m$ and $t_n \in T_n$ implies $W(m;m) > W(t_m;m)$ for any $t_m \in T_m$, which leads to a contradiction by the refinement. First, for any $n > l$, by optimality $W(m;n) \geq W(t_n;n)$ implies $W(m;n) \geq W(l;n)$. Since $\hat{q}(m) > 0$ and $\hat{q}(l) = 0$, by part (iii) of Lemma 1 $W(m;l) > W(l;l)$. Second, for any n such that $m < n \leq l$, we have $t_m \geq n$. If $W(m;n) \geq W(t_n;n)$, optimality implies that $W(m;n) \geq W(t_m;n)$. It then follows from part (i) of Lemma 1 that $W(m;m) > W(t_m;m)$. Combining these two cases, we have $\hat{q}(m) = 0$, as desired.

Finally, consider $q(m)$ for any $m = \{1, \dots, l-1\}$. Suppose that $W(m;n) \geq W(t_n;n)$ for some $n < m$ and $t_n \in T_n$. By optimality $W(m;n) \geq W(t_m;n)$, which implies $q(m) > q(t_m)$ as $\hat{q}(m) = 0$. It then follows from part (ii) of Lemma 1 that $W(m;m) > W(t_m;m)$. By the refinement, $q(m) = 1$. *Q.E.D.*

Proof of Proposition 1. First, we derive necessary and sufficient equilibrium conditions for non-full support equilibria. Fix an inflationary strategy $\{p(m;n)\}$ that satisfies part (i) through part (iv) of Lemma 3 for some threshold $l = 1, \dots, N$, and let the out-of-equilibrium beliefs be given by part (v) of Lemma 3.

(1) Consider the incentives to deviate to an out-of-equilibrium signal $m < l$. The most profitable deviation among the out-of-equilibrium signals is $m = l - 1$. If $l \in T_{l-1}$, then a necessary equilibrium condition is

$$W(l;l-1) \geq W(l-1;l-1). \tag{A.1}$$

If $l \notin T_{l-1}$, then since $q(l-1) = q(l) = 1$ condition (A.1) is satisfied. In either case, by Lemma 1 this condition is also sufficient to imply that $p(l-1;n) = 0$ for any $n \neq l-1$ and thus rule out all out-of-equilibrium deviations. More precisely, condition (A.1) implies that $W(l;l) > W(l-1;l)$ by part (i) of Lemma 1. Since $q(l-1) \geq q(l)$, condition (A.1) implies that $W(l;n) \geq W(l-1;n)$ for all $n < l-1$ by part (ii) of Lemma 1. Finally, since $W(l;l) > W(l-1;l)$ and since $W(l;n) - W(l-1;n) = W(l;l) - W(l-1;l)$ for all $n > l$ as $\hat{q}(l-1) = \hat{q}(l) = 0$, we have $W(l;n) > W(l-1;n)$ for all $n > l$.

(2) For deviations among the signals that are chosen with positive probability in equilibrium, the following conditions are necessary: for each $n = 0, \ldots, l$ such that T_n is not a singleton,

$$W(t_n; n) = W(t'_n; n) \tag{A.2}$$

for all $t_n, t'_n \in T_n$; for each $n = 1, \ldots, l$ such that $\max T_n + 1 = \min T_{n-1}$,

$$W(\max T_n; n) \geq W(\max T_n + 1; n); \tag{A.3}$$

and for each $n = 0, \ldots, l-1$ such that $\min T_n - 1 = \max T_{n+1}$,

$$W(\min T_n; n) \geq W(\min T_n - 1; n). \tag{A.4}$$

(3) Conditions (A.2), (A.3) and (A.4) imply that $p(m; n) = 0$ for any $n \leq l$ and $m \geq l$, so that there is no profitable deviation to an equilibrium signal for any $n \leq l$. To see this, note first that since condition (A.1) implies that $q(l) > q^*$, we have $q(m) > q(m+1) > q^*$ for each $m = l, \ldots, N-1$. This can be established iteratively, starting from $m = l$. For any m, there is $n \leq l$ such that either $m, m+1 \in T_n$, or $m = \max T_n$ and $m + 1 = \min T_{n-1}$. Under the iterative assumption of $q(m) > q^*$, we can show that $q(m+1) > q^*$ and $q(m+1) < q(m)$, using condition (A.2) in the first case, and conditions (A.3) and (A.4) in the second case. Given that $q(m) > q(m+1) > q^*$ for each $m = l, \ldots, N-1$, we can iteratively establish that $W(m; n) \geq W(m+1; n)$ for any $n \leq l$ and $m \geq \max T_n$. First, since $q(\max T_n) > q(\max T_n + 1)$, using part (ii) of Lemma 1, we obtain condition (A.3) for n when $\max T_n = \min T_{n-1}$ from condition (A.4) for $n-1$. Next, for any $m \geq \max T_n + 1$, there exists $n' < n$ such that $W(m; n') \geq W(m+1; n')$ from either condition (A.2) for n' or from condition (A.3) for n'. Since $q(m) > q(m+1)$, part (ii) of Lemma 1 implies that $W(m; n) > W(m+1; n)$. A similar iterative argument establishes that $W(m; n) \geq W(m-1; n)$ for any $m \leq \min T_n$.

(4) Conditions (A.2), (A.3) and (A.4) imply that $p(m; n) = 0$ for any $n > m \geq l$, so that there is no profitable deviation to an equilibrium signal for any $n > l$. To establish this claim, first we show that $W(n; 0) - W(m; 0) \geq 0$ implies $W(n; n) - W(m; n) \geq 0$. This

is trivially true for any m if $n = m$. By (3) above, $q(m) > q(n) > q^*$ for all $n > m \geq l$. Now, the condition $W(n;0) - W(m;0) \geq 0$ is equivalent to:

$$\frac{U(B,g,q(m)) - U(B,g,q(n))}{U(B,g,q(n)) - U(B,b,0)} \leq \frac{n-m}{m}. \tag{A.5}$$

Similarly, the condition $W(n;n) - W(m;n) \geq 0$ is equivalent to:

$$\frac{U(G,g,q(m)) - U(G,g,q(n))}{U(G,g,q(n)) - U(G,b,0)} \leq \frac{n-m}{m}. \tag{A.6}$$

When Assumption 4 holds, $[U(G,g,q) - U(G,b,0)]/[U(B,g,q) - U(B,b,0)]$ is decreasing in $q \in (q^*, 1)$. So,

$$\frac{U(G,g,q(m)) - U(G,b,0)}{U(B,g,q(m)) - U(B,b,0)} \leq \frac{U(G,g,q(n)) - U(G,b,0)}{U(B,g,q(n)) - U(B,b,0)}.$$

This condition implies

$$\frac{U(B,g,q(m)) - U(B,g,q(n))}{U(B,g,q(n)) - U(B,b,0)} \leq \frac{U(G,g,q(m)) - U(G,g,q(n))}{U(G,g,q(n)) - U(G,b,0)}.$$

Hence, (A.5) implies (A.6). It then follows that if $p(m;n) > 0$ for some $n > m \geq l$, then $W(m;0) \geq W(n;0)$. By part (ii) of Lemma 1, $W(m;k) > W(n;k)$ for any $k \leq m$, which contradicts conditions (A.2) to (A.4).

Next, we construct an N-step, iterative algorithm, such that for each $l > 0$, the l-th step of the algorithm covers all possible non-full support equilibria.

Step 0. Consider the following full support equilibrium with $l = 0$. Assign $p(n;n) = 1$ for all $n > 0$; it remains to assign $p(n;0)$. Let the only component of the path variable P^0 be $p(0;0)$. Start the algorithm at $p(0;0) = 1$, and continuously decrease it to 0. For each $p(0;0) \in [0,1]$, let $p(n;0)$ be such that $W(n;0) = W(0;0)$. Define σ as $\sum_{n=0}^{N} p(n;0)$, which is a function of P^0.

Step l, with $l < N$. Consider a non-full support equilibrium with l. We have $p(n;n) = 1$ for all $n > l$, and $p(m;n) = 0$ for $m < l$ and $n \leq l$, and we need to assign $p(m;n)$ for all $m \geq l$ and $n \leq l$. Let the first component of the path variable P^l be $p(l;l)$. Fix any $p(l;l) \in [0,1]$ and decrease it continuously, starting from 1. For each $n = l+1, \ldots, N-1$, define iteratively $p(n;l)$ as the minimum of $1 - \sum_{k<n} p(k;l)$ and the value

such that $W(n;l) = W(l;l)$, where $q(n)$ and $q(l)$ are calculated under the assumption that all probability numbers yet to be assigned are zero. Let $p(N;l) = 1 - \sum_{k<N} p(k;l)$. Note that by construction $p(n;l) \geq 0$ for all n. Next, we iteratively assign values to $p(m;n)$ for $n < l$ and $m \geq l$. Suppose that for each $j < l$, the probability numbers $p(m;n)$ have been assigned for all $m \geq l$ and $n > j$, and denote the current path variable as P^l. Let $\underline{n}(j+1) = \min\{n : p(n;j+1)\} > 0$ and $\overline{n}(j+1) = \max\{n : p(n;j+1)\} > 0$. Assign $p(m;j) = 0$ for all $m < \overline{n}(j+1)$, and consider assigning the remaining probability numbers $p(m;j)$ for $m \geq \overline{n}(j+1)$. There are five cases. In each case, at every point in the algorithm, the beliefs are computed under the assumption that all probability numbers yet to be assigned are zero.

(1) $N > \overline{n}(j+1) > \underline{n}(j+1)$ and $W(\overline{n}(j+1);j+1) > W(\underline{n}(j+1);j+1)$. In this case, let $p(\overline{n}(j+1);j)$ be the minimum of 1 and the value such that $W(\overline{n}(j+1);j+1) = W(\underline{n}(j+1);j+1)$. (If $j=0$, let $p(\overline{n}(1);0)$ be such that $W(\overline{n}(1);1) = W(\underline{n}(1);1)$.) For each n strictly between $\overline{n}(j+1)$ and N, define iteratively $p(n;j)$ as the minimum of $1 - \sum_{k<n} p(k;j)$ and the value such that $W(n;j) = W(\overline{n}(j+1);j)$, and let $p(N;j) = 1 - \sum_{k<N} p(k;j)$. (If $j=0$, let $p(n;0)$ be the value such that $W(n;0) = W(\overline{n}(1);0)$ for all $n > \overline{n}(1)$.) Retain the same path variable P^l.

(2) $N > \overline{n}(j+1) > \underline{n}(j+1)$ and $W(\overline{n}(j+1);j+1) = W(\underline{n}(j+1);j+1)$. In this case, let $p(\overline{n}(j+1);j) = 0$. If $\overline{n}(j+1) = N-1$, let $p(N;j) = 1$. Otherwise, let $p(\overline{n}(j+1)+1;j)$ decrease from the minimum of 1 and the value of $p(\overline{n}(j+1)+1;j)$ that solves $W(\overline{n}(j+1);j) = W(\overline{n}(j+1)+1;j)$, to the minimum of 1 and the value that solves $W(\overline{n}(j+1);j+1) = W(\overline{n}(j+1)+1;j+1)$. (If $j=0$, let $p(\overline{n}(1)+1;0)$ decrease from the value that solves $W(\overline{n}(1);0) = W(\overline{n}(1)+1;0)$ to the value that solves $W(\overline{n}(1);1) = W(\overline{n}(1)+1;1)$.) At this point, the algorithm continues with $p(\overline{n}(j+1)+1;j)$ being the last component of the path variable P^l. We then assign each probability $p(n;j)$, for n strictly between $\overline{n}(j+1)+1$ and N, with the minimum of $1 - \sum_{k<n} p(k;j)$ and the value such that $W(n;j) = W(\overline{n}(j+1)+1;j)$. Let $p(N;j) = 1 - \sum_{k<N} p(k;j)$. (If $j=0$, let $p(n;0)$ be the value such that $W(n;0) = W(\overline{n}(1)+1;0)$ for all $n > \overline{n}(1)+1$.)

(3) $N > \overline{n}(j+1) = \underline{n}(j+1) = l$. Let $p(\overline{n}(j+1);j)$ decrease from the minimum of 1 and the value that solves $W(l;l-1) = W(l-1;l-1)$ to 0. (If $j=0$, let $p(\overline{n}(1);0)$ decrease

from the value that solves $W(l; l-1) = W(l-1; l-1)$ to 0.) At this point, the algorithm continues with $p(\overline{n}(j+1); j)$ being the last component of the path variable P^l. We then assign each probability $p(n; j)$, for n strictly between $\overline{n}(j+1)$ and N, with the minimum of $1 - \sum_{k<n} p(k; j)$ and the value such that $W(n; j) = W(\overline{n}(j+1); j)$. Let $p(N; j) = 1 - \sum_{k<N} p(k; j)$. (If $j = 0$, let $p(n; 0)$ be the value such that $W(n; 0) = W(\overline{n}(1); 0)$ for all $n > \overline{n}(1)$.)

(4) $N > \overline{n}(j+1) = \underline{n}(j+1) > l$. Define $\bar{t} = \min\{n : p(\overline{n}(j+1) - 1; n) > 0\}$ and $\underline{t} = \max\{n : p(\overline{n}(j+1); n) > 0\}$. Let $p(\overline{n}(j+1); j)$ decrease from the minimum of 1 and the value that solves $W(\overline{n}(j+1); \underline{t}) = W(\overline{n}(j+1) - 1; \underline{t})$ to the minimum of 1 and the value that solves $W(\overline{n}(j+1); \bar{t}) = W(\overline{n}(j+1) - 1; \bar{t})$. (If $j = 0$, let $p(\overline{n}(1); 0)$ decrease from the value that solves $W(\overline{n}(1); \underline{t}) = W(\overline{n}(1) - 1; \underline{t})$ to the value that solves $W(\overline{n}(1); \bar{t}) = W(\overline{n}(1) - 1; \bar{t})$.) At this point, the algorithm continues with $p(\overline{n}(j+1); j)$ being the last component of the path variable P^l. We then assign each probability $p(n; j)$, for n strictly between $\overline{n}(j+1)$ and N, with the minimum of $1 - \sum_{k<n} p(k; j)$ and the value such that $W(n; j) = W(\overline{n}(j+1); j)$. Let $p(N; j) = 1 - \sum_{k<N} p(k; j)$. (If $j = 0$, let $p(n; 0)$ be the value such that $W(n; 0) = W(\overline{n}(1); 0)$ for all $n > \overline{n}(1)$.)

(5) $\overline{n}(j+1) = N$. In this case, let $p(N; j) = 1$. (If $j = 0$ and $\overline{n}(1) = N$, $p(N; 0)$ is assigned as in case iv) above.) Retain the same path variable P^l.

Iterating the above procedure determines all probability numbers $p(m; n)$ for $m \geq l$ and $n \leq l$. The algorithm then assigns a value to σ, which is $\sum_{n=0}^{N} p(n; 0)$, as a function of the path variable P^l. Note that the path variable is uni-directional along the algorithm. We claim that as the path variable P^l decreases the value of σ strictly decreases continuously. This can be shown by induction. Clearly, in step 0 the function σ decreases continuously as the path variable P^0 decreases. At the end point of step 0, we have $p(0; 0) = 0$, $p(1; 1) = 1$, and $p(1; 0)$ is such that $W(0; 0) = W(1; 0)$. At the starting point of step 1, we have $p(1; 1) = 1$ and case 3) above applies, which implies that $p(1; 0)$ is such that $W(0; 0) = W(1; 0)$. Thus, the function σ is continuous when we move from step 0 to step 1. Within step 1, for a given value of $p(1; 1)$, either case (1) above applies and the path variable P^1 is $p(1; 1)$, or cases (2) to (4) apply and the path variable is $p(1; 1)$ together with $p(n; 0)$ for some $n \geq 1$. We can thus partition the entire path in step 1 into intervals, such that for each interval the path variable is either $p(1;1)$, or $p(1;1)$ and $p(n;0)$ for some $n \geq 1$ and $p(n;0)$ within some range $[\underline{p}(n;0), \overline{p}(n;0)]$ determined by the algorithm. The function σ is clearly continuous and strictly decreasing in the path variable within each interval. Moreover, for each interval where the path variable is $\tilde{p}(1;1)$ together with $p(n;0) \in [\underline{p}(n;0), \overline{p}(n;0)]$, we can verify that $\lim_{p(1;1)\uparrow\tilde{p}(1;1)} \sigma(P^1) = \lim_{p(n;0)\downarrow\underline{p}(n;0)} \sigma(P^1)$ and $\lim_{p(1;1)\downarrow\tilde{p}(1;1)} \sigma(P^1) = \lim_{p(n;0)\uparrow\overline{p}(n;0)} \sigma(P^1)$, so that σ is continuous at boundary points. Repeating this argument for each step l establishes continuity and monotonicity of the algorithm.

Finally, in step 0, when $P^0 = 1$, we have $\sigma > 1$. If $\sigma > 1$ at the end of step $N-1$, then there is a non-full support equilibrium with threshold $l = N$. Otherwise, there is a unique value $\hat{P}^l$ such that $\sigma(\hat{P}^l) = 1$, and the probability numbers generated by the algorithm at $\hat{P}^l$ constitute an equilibrium with threshold l. Further, if $l > 0$ at $\hat{P}^l$, the equilibrium has $q(l) > q^*$; otherwise, the equilibrium has $q(m) = q^*$ for all m. Finally, since every non-full support equilibrium corresponds to a distinct value of the path variable in the algorithm where the value of σ equals 1, the equilibrium with $l > 0$ and $q(l) > q^*$ is unique when it exists. The necessary and sufficient condition for the existence of an equilibrium with $l > 0$ and $q(l) > q^*$ is $\sigma > 1$ at $P^0 = 0$, which is equivalent to $q^* < 1 - \pi_0$. *Q.E.D.*

PROOF OF LEMMA 4. For each $m = 1, \ldots, N$, define $R^G(m) = \sum_{k=0}^{m} r^G(k)$ and $R^B(m) = \sum_{k=0}^{m} r^G(k)$. Then, letting

$$\mathcal{B}_n(m) = \begin{cases} \sum_{k=0}^{m-n} \beta(N-n, k, p) & \text{if } n \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

we have $R^G(m) = \sum_{n=1}^{N} \pi_n^G \mathcal{B}_n(m)$. Similarly, we can write $R^B(m) = \sum_{n=1}^{N} \pi_{n-1}^B \mathcal{B}_n(m)$. A direct probability argument involving independent Bernoulli trials establishes that

$$\mathcal{B}_n(m) = \mathcal{B}_{n+1}(m) + \beta(N-n-1, m-n, p)(1-p)$$

for $n < m$, and so $\mathcal{B}_n(m) > \mathcal{B}_{n+1}(m)$. For $n \geq m$, it is trivially true that $\mathcal{B}_n(m) \geq \mathcal{B}_{n+1}(m)$. Hence $\mathcal{B}_n(m)$ is decreasing in n for all m. Fact 1 implies that $\{\pi_n^G\}$ first order stochastically dominates $\{\pi_{n-1}^B\}$ under MTP_2, and the reverse is true under MRR_2.

It follows that $R^G(m) \le R^B(m)$ under MTP_2, and the reverse inequality holds under MRR_2. *Q.E.D.*

Proof of Lemma 5. Using $\pi_n = f_n N!/((N-n)!n!)$, we can rewrite $q(m)$ as

$$q(m) = \frac{1}{m} \frac{\sum_{n=1}^{m} (f_n/p^n)(1/((m-n)!(n-1)!))}{\sum_{n=0}^{m} (f_n/p^n)(1/((m-n)!n!)}.$$

Using the above formula, we have that $q(m+1) - q(m)$ has the same sign as

$$m \left(\sum_{n=1}^{m+1} \frac{f_n/p^n}{(m+1-n)!(n-1)!} \right) \left(\sum_{n=0}^{m} \frac{f_n/p^n}{(m-n)!n!} \right) - (m+1) \left(\sum_{n=1}^{m} \frac{f_n/p^n}{(m-n)!(n-1)!} \right) \left(\sum_{n=0}^{m+1} \frac{f_n/p^n}{(m+1-n)!n!} \right).$$

The above expression is a polynomial of $1/p$, of order $(2m+1)$. The constant in the polynomial is 0. For each $k = 1, \ldots, m$, the coefficient of p^{-k} is given by

$$\sum_{n=0}^{k-1} \Big(m \frac{f_n}{(m-n)!n!} \frac{f_{k-n}}{(m+1-k+n)!(k-n-1)!} - (m+1) \frac{f_n}{(m+1-n)!n!} \frac{f_{k-n}}{(m-k+n)!(k-n-1)!} \Big).$$

We will show that the above coefficient is positive for each k under MTP_2 and negative under MRR_2. To see this, note that the above sum is of the form $\sum_{n=0}^{k_*} c_n f_n f_{k-n}$, where $k_* = k/2$ if k is even and $(k-1)/2$ if k is odd. Using the sum over n, direct calculation yields

$$c_0 = \frac{k-1}{m!(m-k+1)!(k-1)!}.$$

For each $n = 1, \ldots, k_* - 1$, note that there are four terms of $f_n f_{k-n}$ in the above sum, so adding them up we have

$$c_n = \frac{(m+1)((k-n)(k-n-1) + n(n-1)) - 2mn(k-n)}{n!(m+1-n)!(m+1-k+n)!(k-n)!}.$$

Moreover, c_{k_*} is given by an identical formula with $n = k_*$ if k is odd, and is given by the same formula divided by 2 if k is even. By direct manipulations, we can show that each c_n for $n = 1, \ldots, k_* - 1$, and c_{k_*} when k is odd, can be decomposed into two terms

$$c_n = \frac{k-2n-1}{(m-n)!n!(m-k+n+1)!(k-n-1)!} - \frac{k-2(n-1)-1}{(m-n+1)!(n-1)!(m-k+n)!(k-n)!}.$$

For k even, we have

$$c_{k_*} = -\frac{1}{(m-k/2+1)!(k/2-1)!(m-k/2)!(k/2)!}.$$

Note that the second part of c_1 is equal to c_0; for each $n = 1, \ldots, k_* - 1$, the first part of c_n is equal to the second part of c_{n+1}; the first part of c_{k_*} is 0. It follows that we can write the sum over n in the form of

$$\sum_{n=0}^{k_*-1} \alpha_n (f_n f_{k-n} - f_{n+1} f_{k-n-1}),$$

for some $\alpha_n > 0$, $n = 0, \ldots, k_* - 1$. By Fact 2, the coefficient of each p^{-k} is positive under MTP_2 and negative under MRR_2. The case of $k \geq m+1$ can be established in a similar fashion. *Q.E.D.*

Proof of Lemma 6. For each $n = 1, \ldots, N$, we have

$$\pi_n^B = \frac{\pi_n}{1-\pi} \frac{\binom{N-1}{N-n-1}}{\binom{N}{N-n}} = \frac{\pi_n}{1-\pi} \frac{N-n}{N}.$$

It follows from equation (4.2) that for each $m = 1, \ldots, N$,

$$r^B(m) = \sum_{n=0}^{m-1} \frac{N-n}{N(1-\pi)} \pi_n \frac{m-n}{(N-n)p} \beta(N-n, m-n, p) = \sum_{n=0}^{m} \pi_n \beta(N-n, m-n, p) \frac{m-n}{N(1-\pi)p}.$$

For each $m = 1, \ldots, N$, define

$$\mathcal{N}(m) = \sum_{n=1}^{m} \pi_n \beta(N-n, m-n, p) n;$$
$$\mathcal{D}(m) = \sum_{n=1}^{m} \pi_n \beta(N-n, m-n, p) m.$$

Since $\sum_{m=1}^{N} r^B(m) = 1$, we have

$$\sum_{m=1}^{N} \Big(\mathcal{D}(m) - \mathcal{N}(m) \Big) = N(1-\pi)p.$$

Note that since $r^G(m) = \mathcal{N}(m)/(\pi N)$, from $\sum_{m=1}^N r^G(m) = 1$ we have

$$\sum_{m=1}^{N} \mathcal{N}(m) = N\pi,$$

and hence

$$\sum_{m=1}^{N} \mathcal{D}(m) = N(\pi + (1-\pi)p).$$

Finally, note that from equation (4.3) we have

$$q(m) = \frac{\mathcal{N}(m)}{\mathcal{D}(m)}.$$

Since $r^B(m) = (\mathcal{N}(m) - \mathcal{D}(m))/(N(1-\pi)p)$, we have

$$\sum_{m=1}^{N} r^B(m)q(m) = \frac{1}{N(1-\pi)p} \sum_{m=1}^{N} \mathcal{D}(m)q(m)(1-q(m)).$$

Since $q(1-q)$ is concave in q, Jensen's inequality implies the above is less than or equal to

$$\frac{\sum_{m=1}^N \mathcal{D}(m)}{N(1-\pi)p} \left(\frac{\sum_{m=1}^N q(m)\mathcal{D}(m)}{\sum_{m=1}^N \mathcal{D}(m)} \right) \left(1 - \frac{\sum_{m=1}^N q(m)\mathcal{D}(m)}{\sum_{m=1}^N \mathcal{D}(m)} \right).$$

The lemma follows immediately. *Q.E.D.*

References

W.J. Adam and J.L. Yellen, 1976, "Commodity Bundling and the Burden of Monopoly," *Quarterly Journal of Economics* 90, 475–498.

K. Bagwell and G. Ramsey, 1991, "Oligopoly Limit Pricing," *Rand Journal of Economics* 22, pp. 155–172.

J.S. Banks and J. Sobel, 1987, "Equilibrium Selection in Signaling Games," *Econometrica* 55, pp. 647–662.

R. Benabou and G. Laroque, 1992, "Using Privileged Information to Manipulate Markets: Insiders, Gurus, and Credibility," *Quarterly Journal of Economics* 107, pp. 921–958.

A. Chakraborty and R. Harbaugh, “Comparative Cheap Talk,” forthcoming *Journal of Economic Theory*.

W. Chan, H. Li and W. Suen, “A Signaling Theory of Grade Inflation,” forthcoming *International Economic Review*.

V. Crawford and J. Sobel, 1982, “Strategic Information Transmission,” *Econometrica* 50, pp. 1431–1451.

M. Engers, 1987, “Signalling with Many Signals,” *Econometrica* 55, pp. 663–674.

M. Gentzkow and J. Shapiro, 2006, “Media Bias and Reputation,” *Journal of Political Economy* 114, pp. 280–316.

M. Hertzendorf and P.B. Overgaard, 2001, “Price Competition and Advertising Signals: Signaling by Competing Senders,” *Journal of Economics and Management Strategy* 10, pp. 621–662.

M.O. Jackson and H.F. Sonnenschein, “Overcoming Incentive Constraints by Linking Decisions,” forthcoming, *Econometrica*.

H. Joe, 1997, *Multivariate Models and Dependence Concepts*. London: Chapman & Hall.

S. Karlin and Y. Rinott, 1980, “Classes of Orderings of Measures and Related Correlation Inequalities: I. Multivariate Totally Positive Distributions,” *Journal of Multivariate Analysis* 10, pp. 467–498.

E.L. Lehmann, 1966, “Some Concepts of Dependence,” *Annals of Mathematical Statistics* 37, pp. 1137–1153.

E. Maskin and J. Tirole, 1990, “The Principal-agent Relationship with an Informed Principal: The Case of Private Values,” Econometrica 58, pp. 379–409.

R.P. McAfee, J. McMillan and M. Whinston, 1979, “Multiproduct Monopoly, Commodity Bundling and the Correlation of Values,” *Quarterly Journal of Economics* 100, pp. 371–383.

P. Milgrom and R.J. Weber, 1982, “A Theory of Auctions and Competitive Bidding,” *Econometrica* 50, pp. 1089–1122.

D.A. Moore, D.M. Cain, G. Loewenstein and M. Bazerman, 2005, eds., *Conflicts of Interest: Problems and Solutions from Law, Medicine and Organizational Settings*. London: Cambridge University Press.

J. Morgan and P. Stockton, 2003, “An Analysis of Stock Recommendations,” *RAND Journal of Economics* 34, pp. 183–203.

S. Morris, 2001, “Political Correctness,” *Journal of Political Economy* 109, pp. 231-265.

M. Quinzii and J.-C. Rochet, 1985, “Multidimensional Signalling,” *Journal of Mathematical Economics* 14, pp. 261–284.

J. Sobel, 1985, “A Theory of Credibility,” *Review of Economic Studies* 52, pp. 557-573.