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Monotone strategyproofness ☆

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ABSTRACT

We propose a way to compare the extent of preference misrepresentation between two strategies. We define a preference revelation mechanism to be *monotone strategyproof* if declaring a "more truthful" preference ordering dominates (with respect to the true preferences) declaring a "less truthful" preference ordering. Our main result states that a mechanism is strategyproof if, and only if, it is monotone strategyproof. This result holds for any deterministic social choice function on any domain; for probabilistic social choice functions it holds under a mild assumption on the domain.

1. Introduction

Truthful revelation is a primary goal in mechanism design. Ideally, it is a dominant strategy to truthfully reveal one's preferences, and a mechanism that induces such a dominant strategy for all agents and all preference profiles is said to be **strategyproof**. Non-trivial strategyproof mechanisms do not always exist if other desired properties are also imposed (Gibbard, 1973; Satterthwaite, 1975), but a number of environments have been identified for which non-trivial strategyproof mechanisms exist, e.g. voting, two-sided matching, house allocation, or auctions.[1]

Strategyproof mechanisms induce a radical division between strategies, for they distinguish the truthful strategy from all other strategies. All non-truthful strategies are deemed undesirable regardless of their other characteristics; a lie is a lie, whether big or small. This gave the prior literature little reason to scrutinize misrepresentations in strategyproof

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[1] See for instance Moulin (1980) for voting with single-peaked preferences, Dubins and Freedman (1981) and Roth (1982) for two-sided matching. See also Barberà (2011) for a recent survey.

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mechanisms, for instance by measuring how much they deviate from the truth. We argue that this is an important omission and we focus in this paper specifically on non-truthful strategies in strategyproof mechanisms.

We believe there is a need for a general tool to analyze misrepresentations. There is indeed now growing evidence that strategyproof mechanisms perform poorly in the laboratory (see Chen (2008) for a survey).[2] Actually, experimental data from games with a dominant strategy also exhibit seemingly irrational behavior.[3] Overall, most experimental analysis of strategyproof mechanisms cannot go further than acknowledging the percentage of subjects not being truthful, and analyzing how this percentage varies when changing some environment parameters or the mechanism itself. However, the existing studies have not been able to rank non-truthful strategies on how close they are to the true preferences, save for some specific cases.[4] This is a serious limitation because what makes strategyproof mechanisms appealing is, among other things, their ability to generate quality data about individuals' preferences. Such information is crucial if one wishes to run counterfactuals and test potential new policies. Policy makers (and econometricians) may prefer a mechanism with a large percentage of individuals not being truthful but "close" to the truth over a mechanism with a smaller percentage of misrepresentations but consisting of large deviations from the truth.

From a theoretical perspective we argue that studying misrepresentations can help understanding further the anatomy of strategyproof mechanisms. By its definition, strategyproofness imposes the existence of a dominant strategy in the mechanism. But does it also impose any structure on misrepresentations? To address this question we classify misrepresentations so as to be able to rank strategies on how much they misrepresent the true preferences. Our contention is that such a classification must be linked to the cost of misrepresenting preferences. Drawing on the intuition for strategyproofness, small misrepresentations should have a lower impact on agents' welfare than large ones, or, put differently, small deviations should dominate large ones.[5] We call a mechanism satisfying this property **monotone strategyproof**. One might conjecture that imposing monotonicity between payoffs and distance from the truth would be more restrictive than the usual incentive compatibility, i.e., that some strategyproof mechanisms may not be monotone strategyproof. Our main contribution here is to show that monotone strategyproofness is actually equivalent to strategyproofness. This seemingly counterintuitive result turns out to be straightforward to show and holds for a very general class of environments.

Our result is derived within a typical environment where each individual has a strict preference relation over a finite set of alternatives and participates in a strategyproof mechanism.[6] We first devise a measure to compare the degree of preference misrepresentation. Given two preference orderings P_i and P_i', we define the **Kemeny set** of P_i and P_i' as the pairs of alternatives that are not ordered in the same way under these two preferences.[7] We compare the degree of misrepresentation by comparing Kemeny sets: Given a true preference ordering P_i, an ordering P_i' is defined to be more truthful than P_i'' when the Kemeny set of P_i' and P_i is a subset of that of P_i'' and P_i. That is, P_i' is more truthful than P_i'' when P_i'' has relatively more elements whose order disagrees with P_i. In this context, a mechanism is said to be monotone strategyproof if a more truthful strategy always dominates a less truthful one.[8] It is straightforward to see that monotone strategyproofness implies strategyproofness. Our main result (Theorem 2) states that the reverse also holds under a mild assumption on the domain of the mechanism. For deterministic social choice functions this equivalence actually holds for *any* environment (Theorem 1).

We compare strategies by comparing their Kemeny sets. A natural question is whether a non-truthful strategy P_i' that dominates another non-truthful strategy P_i'' is necessary closer to the true preferences in the way we define it. In other words, is Kemeny set inclusion equivalent to the dominance relation? It turns out that this equivalence is true for deterministic mechanisms, but not for the general case. For non-deterministic mechanisms we show how one preference ordering may dominate another without Kemeny set inclusion. This observation illustrates the complication added by non-deterministic mechanisms.

Two closely related papers are Carroll (2012) and Sato (2013). Like us, they also compare "large" and "small" misrepresentations, but they address a different question than we do. Both Carroll and Sato characterize conditions under which "local" strategyproofness implies "global" strategyproofness, that is, conditions under which restricting misrepresentations that only switch the ranking of two consecutive alternatives in one's preferences is enough to characterize strategyproofness. So their concern is more about the transitivity of strategyproofness.[9] Another related paper is Cho (2014). While considering closely related issues to ours, the analysis in Cho (2014) is constrained by a more restrictive environment. Cho studies probabilistic assignment mechanisms (Carroll, Sato and us consider any social choice mechanism). Cho's main contribution

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[2] See for instance Cason et al. (2008) for the pivotal and the Groves–Clarke mechanisms, Chen and Sönmez (2006) or Calsamiglia et al. (2010) in a matching context.

[3] See Palacios-Huerta and Volij (2009) for the centipede game, Kagel and Levin (1986) for auction games or Andreoni (1995) for public good games.

[4] Chen and Sönmez (2006) and Calsamiglia et al. (2010) for instance analyze which type of alternative is likelier to be displaced in the preference orderings.

[5] See Jackson (1992) for a similar argument in the case of in an exchange economy.

[6] As we discuss at the end of this paper (in the Discussion section), our strategyproof equivalence result does not necessarily hold for weak preferences. It is surprising, as it holds (for a very general class of environments) under stochastic mechanisms.

[7] The cardinality of this set is the well-known *Kemeny distance* (Kemeny, 1959).

[8] The equivalent definition for stochastic mechanisms simply replaces dominance with stochastic dominance.

[9] Another, somewhat less related paper, is Pathak and Sönmez (2013), who also focus on misrepresentation of preferences. However, Pathak and Sönmez are interested in comparing mechanisms—and therefore consider mechanisms that are not strategyproof—while we are interested in comparing misrepresentations under strategyproof mechanisms.

consists of proposing several ways to compare probabilistic assignments, and he shows that Sato's result continues to hold with these new notions of assignment comparison. As a by-product, Cho finds equivalence between monotone strategyproofness (that he calls "lie-monotonicity") and strategyproofness for stochastic mechanisms under specific domain conditions.[10] This result is covered in our Corollary 1. We obtain results for a more general domain, and also consider deterministic social choice functions and mechanisms with cardinal types.

We outline the environment we consider in Section 2. Monotone strategyproofness is defined and shown to be equivalent to strategyproofness in Section 3. In Section 4 we discuss the relevance of using Kemeny sets to compare strategies and show how similar result can be obtained when agents have cardinal utility functions over outcomes. We conclude in Section 5.

2. Preliminaries

Let N be a set of agents and X a finite set of alternatives. We shall focus in this paper on the incentives from an individual agent's perspective, henceforth called agent i.[11] A preference P_i for agent i over X is a linear order on X. Given a preference relation P_i we denote by R_i the weak ordering associated with P_i, i.e., xR_ix' implies xP_ix' or $x = x'$.[12]

A **preference profile** is a list P of preferences for each agent $i \in N$, $P = \times_{i\in N} P_i$. We follow the usual convention to denote by P_{-i} the profile $(P_1, \dots, P_{i-1}, P_{i+1}, \dots, P_n)$. The set of all possible preferences, called the **universal domain**, is denoted $\mathcal{P}$. A **domain**, denoted by $\mathcal{D}$, is a non-empty subset of $\mathcal{P}$. Note that $\mathcal{D}$ does not need to be a product of individual domains. Given a domain $\mathcal{D}$, we denote by $\mathcal{D}_i$ the set of preferences that are admissible for individual i and $\mathcal{D}_{-i}$ the profiles P_{-i} that are admissible for individuals in $N\backslash\{i\}$.

A sequence of preference orderings $(P^1, \dots, P^\ell)$ satisfies the **non-restoration property** if whenever for some $x, x' \in X$ and some $h < \ell$ we have xP^hx' and $x'P^{h+1}x$ then it implies that $x'P^{h'}x$ for each $h' > h + 1$.

A **lottery** is a vector of probabilities $\pi \in \mathbb{R}^{|X|}$. We denote by $\Delta(X)$ the set of all lotteries over X. A **social choice function** (or a **mechanism**) on a domain $\mathcal{D}$ is a mapping $\varphi : \mathcal{D} \to \Delta(X)$. Given a profile P, we denote by $\varphi_x(P)$ the probability of alternative x under the lottery $\varphi(P)$. The social choice function is **deterministic** if for each $P \in \mathcal{D}^N$, $\varphi(P)$ is a degenerate lottery. In this case (abusing notation) we shall denote by $\varphi(P_i, P_{-i})$ the alternative x such that $\varphi_x(P_i, P_{-i}) = 1$.

Given preference orderings P_i, P_i', P_i'', we say that P_i' **(stochastically) dominates** P_i'' **with respect to** P_i, denoted $P_i' \gg^{P_i} P_i''$, when

$$\text{for each } P_{-i}, \text{for each } x \in X, \qquad \sum_{x' : x'R_ix} \varphi_{x'}(P_i', P_{-i}) \geq \sum_{x' : x'R_ix} \varphi_{x'}(P_i'', P_{-i})\,. \tag{1}$$

For a deterministic social choice function, condition (1) can be rewritten as

$$\text{for each } P_{-i}, \qquad \varphi(P_i', P_{-i})\, R_i\, \varphi(P_i'', P_{-i})\,. \tag{1'}$$

Definition 1. A social function φ is **strategyproof** on a domain $\mathcal{D}$ if for each agent $i \in N$, and for each $P_i, P_i' \in \mathcal{D}$, P_i dominates P_i' with respect to P_i.

Observe that the sets of individuals, the (true) preference profile P, and a social choice function φ on a domain $\mathcal{D}$ induce a strategic form game $\Gamma^\varphi = \langle N, \mathcal{D}, P\rangle$, where N is the set of players, $\mathcal{D}$ is the set of (pure) strategy profiles, the outcome of a strategy profile P is given by $\varphi(P)$, and each player $i \in N$ evaluates the outcome $\varphi(P)$ using his true preferences P_i. In this context, a social choice function φ is strategyproof if in the game form Γ^φ the truthful strategy P_i is a (weakly) dominant strategy for each player i.

3. Monotone strategyproofness

One natural way to compare two preference orderings is by counting the number of pairs of alternatives whose relative rank differs between the two orderings. This method is known as the *Kemeny distance* (Kemeny, 1959). We propose instead to compare preference orderings with what we call the **Kemeny sets** of the preference orderings.

Definition 2. Given two preference orderings P_i, P_i', the **Kemeny set of** P_i **and** P_i' is the set of all pairs $(x, x') \in X \times X$ that are not ordered identically in P_i and P_i',

$$K(P_i, P_i') = \{(x, x') \in X \times X \;:\; \text{either } x'P_ix \,\&\, xP_i'x', \text{ or } xP_ix' \,\&\, x'P_i'x\}. \tag{2}$$

[10] Cho's definition of lie-monotonicity is more restrictive than our notion of monotone strategyproofness, as he only compares preference orderings that differ only in the relative ranking of two consecutive alternatives. Under his domain restriction this turns out to be equivalent to monotone strategyproofness.

[11] Thus, the set of individuals need not be finite nor countable.

[12] In the main sections of the paper, we derive out result for strict preferences. We discuss the limitations of our results under weak preferences in Section 4.1.

We are now ready to introduce our main concept:

Definition 3. A social choice function is **monotone strategyproof** on a domain $\mathcal{D}$ if for each $i \in N$, each $P_i \in \mathcal{D}$ and each pair $P'_i, P''_i \in \mathcal{D}$ such that $K(P'_i, P_i) \subset K(P''_i, P_i)$, $P'_i \gg^{P_i} P''_i$.

Note that if a social choice function is monotone strategyproof it is obviously strategyproof. Indeed, $K(P_i, P_i) = \emptyset$ implies that P_i dominates any other preference ordering P'_i. The next theorem states that the converse also holds for deterministic social choice functions on any domain.

Theorem 1. *Let φ be a deterministic social choice function on a domain $\mathcal{D}$. Then φ is strategyproof if, and only if, it is monotone strategyproof.*

Proof. Let P_i, P'_i and P''_i be such that $K(P'_i, P_i) \subset K(P''_i, P_i)$ with $P_i, P'_i, P''_i \in \mathcal{D}$. Let P_{-i} be any profile, and let $x' = \varphi(P'_i, P_{-i})$ and $x'' = \varphi(P''_i, P_{-i})$ and assume that $x' \neq x''$.[13]

Observe that if $x' P''_i x''$, then φ cannot be strategyproof. This is because if P''_i were the true preferences, then individual i could benefit by reporting P'_i instead of P''_i. Similarly, it cannot be that $x'' P'_i x'$. So it must be that $x' P'_i x''$ and $x'' P''_i x'$. Since $K(P'_i, P_i) \subset K(P''_i, P_i)$, we have $x' P_i x''$. That is, $\varphi(P'_i, P_{-i}) P_i \varphi(P''_i, P_{-i})$. □

Observe that Theorem 1 holds for any domain, but only for deterministic social choice functions. If we want to consider non-deterministic social choice functions a result similar to Theorem 1 can be obtained under certain conditions on the domain. Before presenting those conditions some definitions are in order.

For any two preferences P_i and P'_i we first construct the set of connected components of the graph $G(P_i, P'_i) = (X, K(P_i, P'_i))$, where X is the set of vertices and $K(P_i, P'_i)$ is the set of edges. That is, in the graph $G(P_i, P'_i)$ there is an edge between x and x' if $(x, x') \in K(P_i, P'_i)$, i.e., if the relative order of x and x' differ between P_i and P'_i. Two alternatives x and x' are **connected** in $G(P_i, P'_i)$ if there exists a sequence $(x_1, \ldots, x_k)$ with $x = x_1$, $x' = x_k$ such that $(x_h, x_{h+1}) \in K(P_i, P'_i)$ for each $h < k$. A **connected component** is a set of alternatives $C \subseteq X$ such that any two alternatives in C are connected in $G(P_i, P'_i)$ and no alternative in C is connected with an alternative in $X \backslash C$.

For instance, if $K(P_i, P'_i) = \{(x_1, x_2), (x_2, x_4), (x_3, x_5)\}$, then the graph $G(P_i, P'_i)$ has three edges: between x_1 and x_2, between x_2 and x_4, and between x_3 and x_5. The connected components in $G(P_i, P'_i)$ are $\{x_1, x_2, x_4\}$ and $\{x_3, x_5\}$.

For a subset of alternatives $C \subset X$, let P_i **restricted to** C, denoted $P_i|_C$, be a preference ordering defined on C such that for any $x, y \in C$, $x P_i|_C\, y$ if, and only if $x P_i y$. We say that a preference ordering P'_i is a **complete reversal** of P_i when for any x and y, $x P_i y$ if, and only if $y P'_i x$.

A domain $\mathcal{D}$ is **weakly connected** if for any two distinct preferences P_i and P'_i there exists a sequence $(P^1, \ldots, P^k)$ that satisfies the non-restoration property where $P^1 = P_i$, $P^k = P'_i$ and for each $h < k$, and the graph $G(P^h_i, P^{h+1}_i)$ has exactly one connected component C, and furthermore $|C| \leq 3$, or $P^{h+1}|_C$ is a complete reversal of $P^h|_C$.[14]

Theorem 2. *Let φ be a social choice function on a weakly connected domain $\mathcal{D}$. Then φ is strategyproof if, and only if, it is monotone strategyproof.*

The proof of Theorem 2 will invoke the Lemmas 1 and 2 that we present below.

Lemma 1. *Let P_i, P'_i and P''_i be preference orderings from a domain $\mathcal{D}$, and let $C \subset X$ be a unique connected component of the graph $G(P'_i, P''_i)$. Suppose that*

$$\sum_{y R_i x \ \&\ y \in C} \varphi_y(P'_i, P_{-i}) \geq \sum_{y R_i x \ \&\ y \in C} \varphi_y(P''_i, P_{-i}), \quad \text{for all } x \in X \text{ and } P_{-i} \in \mathcal{D}_{-i}. \tag{3}$$

Then $P'_i \gg^{P_i} P''_i$.

Proof. Since C is the unique connected component, $P'_i|_{X \backslash C} = P''_i|_{X \backslash C}$. Let A be the set of alternatives that are above C in P'_i, $A = \{x : x P'_i y \text{ for all } y \in C\}$. The set C being the unique connected component implies that $P'_i|_A = P''_i|_A$, and thus we also have $A = \{x : x P''_i y \text{ for all } y \in C\}$. Similarly, we can define the set B of alternatives below C, $B = X \backslash (A \cup C) = \{x : y P'_i x \text{ for all } y \in C\} = \{x : y P''_i x \text{ for all } y \in C\}$. Thus, $P'_i|_B = P''_i|_B$.

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[13] If $\varphi(P'_i, P_{-i}) = \varphi(P''_i, P_{-i})$ for any profile P_{-i}, then P'_i and P''_i are equivalent strategies and thus P'_i trivially dominates P''_i.

[14] We could allow in the definition that two consecutive preference orderings in the sequence, say, P^h_i and P^{h+1}_i are identical, in which case the graph $G(P^h_i, P^{h+1}_i)$ would have no connected component. To avoid taking care of those trivial cases we require that along the sequence there is always at least one connected component.

By strategyproofness, $P'_i \gg^{P'_i} P''_i$ and $P''_i \gg^{P''_i} P'_i$ imply that, for any $P_{-i} \in \mathcal{D}_{-i}$ and any $x \in A$, $\sum_{yR'_ix} \varphi_y(P'_i, P_{-i}) \geq \sum_{yR'_ix} \varphi_y(P''_i, P_{-i})$ and $\sum_{yR''_ix} \varphi_y(P''_i, P_{-i}) \geq \sum_{yR''_ix} \varphi_y(P'_i, P_{-i})$, respectively. Since $P'_i|_A = P''_i|_A$, for each of the above inequalities both sides must be identical. It follows then that $\varphi_x(P'_i, P_{-i}) = \varphi_x(P''_i, P_{-i})$, for each $x \in A$ and all $P_{-i} \in \mathcal{D}_{-i}$.

Let x_1 be the highest alternative in $P'_i|_B$ (so x_1 is also the highest alternative ranked in $P''_i|_B$) and let x'_0 and x''_0 be the lowest alternatives ranked in $P'_i|_C$ and $P''_i|_C$, respectively. Finally, let P_{-i} be any profile in $\mathcal{D}_{-i}$. By definition, $P'_i \gg^{P'_i} P''_i$ implies

$$\sum_{yR'_ix'_0} \varphi_y(P'_i, P_{-i}) \geq \sum_{yR'_ix'_0} \varphi_y(P''_i, P_{-i}), \tag{4}$$

and $P''_i \gg^{P''_i} P'_i$ implies

$$\sum_{yR''_ix''_0} \varphi_y(P''_i, P_{-i}) \geq \sum_{yR''_ix''_0} \varphi_y(P'_i, P_{-i})\,. \tag{5}$$

Observe that $\{x : xR'_ix'_0\} = \{x : xP'_ix_1\} = A \cup C = \{x : xR''_ix''_0\} = \{x : xP'_ix_1\}$. So Eqs. (4) and (5) imply $\sum_{xP'_ix_1} \varphi_x(P'_i, P_{-i}) = \sum_{xP''_ix_1} \varphi_x(P''_i, P_{-i})$. Again using $P'_i \gg^{P'_i} P''_i$ and $P''_i \gg^{P''_i} P'_i$ we obtain $\sum_{xR'_ix_1} \varphi_x(P'_i, P_{-i}) = \sum_{xR'_ix_1} \varphi_x(P''_i, P_{-i})$. Therefore, $\varphi_{x_1}(P'_i, P_{-i}) = \varphi_{x_1}(P''_i, P_{-i})$. Continuing with the alternatives ranked below x_1 (which are ordered identically in P'_i and P''_i) we then obtain, for each $x \in B$, $\varphi_x(P'_i, P_{-i}) = \varphi_x(P''_i, P_{-i})$.

We now show that $P'_i \gg^{P_i} P''_i$, that is,

$$\sum_{y:yR_ix} \varphi_y(P'_i, P_{-i}) \geq \sum_{y:yR_ix} \varphi_y(P''_i, P_{-i}) \text{ for all } x \in X \text{ and } P_{-i} \in \mathcal{D}_{-i}. \tag{6}$$

Since $\varphi_y(P'_i, P_{-i}) = \varphi_y(P''_i, P_{-i})$ for each $y \notin C$ and any $P_{-i} \in \mathcal{D}_{-i}$,

$$\sum_{y:yR_ix\,\&\,y\notin C} \varphi_y(P'_i, P_{-i}) = \sum_{y:yR_ix\,\&\,y\notin C} \varphi_y(P''_i, P_{-i})\,, \text{ for all } x \in X \text{ and } P_{-i} \in \mathcal{D}_{-i}. \tag{7}$$

Summing Eqs. (3) and (7) yields (6), the desired result. □

Lemma 2. *Let P_i, P'_i and P''_i be preference orderings from a domain $\mathcal{D}$, and let C be a unique connected component of the graph $G(P'_i, P''_i)$ and suppose $P'_i|_C = P_i|_C$. Then $P'_i \gg^{P_i} P''_i$.*

Proof. Let C be the (unique) connected component of $G(P'_i, P''_i)$. Note that by Lemma 1 it suffices to show

$$\sum_{yR_ix\,\&\,y\in C} \varphi_y(P'_i, P_{-i}) \geq \sum_{yR_ix\,\&\,y\in C} \varphi_y(P''_i, P_{-i}) \text{ for all } x \in X \text{ and } P_{-i} \in \mathcal{D}_{-i}. \tag{8}$$

Since $P'_i|_C = P_i|_C$, for any $x \in X$ there exists a unique $z(x) \in C$ such that

$$\{y \,:\, yR_ix \,\&\, y \in C\} = \{y \,:\, yR'_iz(x) \,\&\, y \in C\}\,. \tag{9}$$

Since φ is strategyproof, using an argument similar as in the proof of Lemma 1 we have

$$P'_i \gg^{P'_i} P''_i \Leftrightarrow \sum_{yR'_iz(x)\,\&\,y\in C} \varphi_y(P'_i, P_{-i}) \geq \sum_{yR'_iz(x)\,\&\,y\in C} \varphi_y(P''_i, P_{-i}) \quad \text{for all } x \in X \text{ and } P_{-i} \in \mathcal{D}_{-i} \tag{10}$$

From Eq. (9), we have, for any $x \in X$ and any $P_{-i} \in \mathcal{D}_{-i}$,

$$\sum_{yR'_iz(x)\,\&\,y\in C} \varphi_y(P'_i, P_{-i}) = \sum_{yR_ix\,\&\,y\in C} \varphi_y(P'_i, P_{-i}) \tag{11}$$

and

$$\sum_{yR'_iz(x)\,\&\,y\in C} \varphi_y(P''_i, P_{-i}) = \sum_{yR_ix\,\&\,y\in C} \varphi_y(P''_i, P_{-i}) \tag{12}$$

Plugging Eqs. (11) and (12) into Eq. (10) yields Eq. (8), the desired result. □

Proof of Theorem 2. Let P_i, P'_i and P''_i be such that $K(P'_i, P_i) \subset K(P''_i, P_i)$. Since the domain is weakly connected, there exists a sequence $P^1_i, \ldots, P^\ell_i$ that satisfies the non-restoration property where $P'_i = P^1_i$, $P''_i = P^\ell_i$, and for each $h < \ell$, the graph $G(P_i^h, P_i^{h+1})$ has only one connected component C, such that either $|C| \leq 3$ or $P^{h+1}|_C$ is a complete reversal of $P^h|_C$. Notice that the non-restoration property implies that $K(P_i^h, P_i) \subset K(P_i^{h+1}, P_i)$. Since the stochastic dominance relation is transitive it is sufficient to show that for any $h < \ell$ we have $P_i^h \gg^{P_i} P_i^{h+1}$.

Let C be the (unique) connected component of $G(P_i^h, P_i^{h+1})$. If $|C| \neq 3$, then by weak connectedness $P_i^{h+1}|_C$ is a complete reversal of $P_i^h|_C$.[15] Since $K(P_i^h, P_i) \subset K(P_i^{h+1}, P_i)$ we thus have $P_i^h|_C = P_i|_C$. Then Lemma 2 implies $P_i^h \gg^{P_i} P_i^{h+1}$.

Now consider $|C| = 3$. So $2 \leq |K(P_i^h, P_i^{h+1})| \leq 3$.[16] If $|K(P_i^h, P_i^{h+1})| = 3$, then $P_i^{h+1}|_C$ is a complete reversal of $P_i^h|_C$, and thus we are back to the previous paragraph. So, $|K(P_i^h, P_i^{h+1})| = 2$. From $K(P_i^h, P_i) \subset K(P_i^{h+1}, P_i)$, we have either $|K(P_i^h|_C, P_i|_C)| = 0$ or $|K(P_i^h|_C, P_i|_C)| = 1$. In the former case, $P_i^h|_C = P_i|_C$ and thus by Lemma 2 we have $P_i^h \gg^{P_i} P_i^{h+1}$. If $|K(P_i^h|_C, P_i|_C)| = 1$, by Kemeny set inclusion $|K(P_i^{h+1}|_C, P_i|_C)| = 3$, i.e., $P_i^{h+1}|_C$ is a complete reversal of $P_i|_C$.

Using again the argument in the proof of Lemma 1, we have for any $P_{-i} \in \mathcal{D}_{-i}$ and $y \notin C$, $\varphi_y(P_i^h, P_{-i}) = \varphi_y(P_i^{h+1}, P_{-i})$. So $P_i^{h+1} \gg^{P_i^{h+1}} P_i^h$ is tantamount to

$$\sum_{y R_i^{h+1} x \ \& \ y \in C} \varphi_y(P_i^{h+1}, P_{-i}) \geq \sum_{y R_i^{h+1} x \ \& \ y \in C} \varphi_y(P_i^h, P_{-i}), \quad \text{for all } x \in X \text{ and } P_{-i} \in \mathcal{D}_{-i}. \tag{13}$$

Let $C = \{x_1, x_2, x_3\}$, and assume without loss of generality that $x_1 P_i x_2 P_i x_3$. So, because $P_i^{h+1}|_C$ is a complete reversal of $P_i|_C$, it must be that $x_3 P_i^{h+1} x_2 P_i^{h+1} x_1$.

So Eq. (13) implies, for any $P_{-i} \in \mathcal{D}_{-i}$

$$\varphi_{x_3}(P_i^{h+1}, P_{-i}) \geq \varphi_{x_3}(P_i^h, P_{-i}) \tag{14}$$

$$\varphi_{x_3}(P_i^{h+1}, P_{-i}) + \varphi_{x_2}(P_i^{h+1}, P_{-i}) \geq \varphi_{x_3}(P_i^h, P_{-i}) + \varphi_{x_2}(P_i^h, P_{-i}) \tag{15}$$

$$\varphi_{x_3}(P_i^{h+1}, P_{-i}) + \varphi_{x_2}(P_i^{h+1}, P_{-i}) + \varphi_{x_1}(P_i^{h+1}, P_{-i}) = \varphi_{x_3}(P_i^h, P_{-i}) + \varphi_{x_2}(P_i^h, P_{-i}) + \varphi_{x_1}(P_i^h, P_{-i}) \tag{16}$$

where the last equality follows from $\varphi_y(P_i^h, P_{-i}) = \varphi_y(P_i^{h+1}, P_{-i})$ for all $y \notin C$. Eqs. (14) and (16) imply

$$\varphi_{x_1}(P_i^h, P_{-i}) + \varphi_{x_2}(P_i^h, P_{-i}) \geq \varphi_{x_1}(P_i^{h+1}, P_{-i}) + \varphi_{x_2}(P_i^{h+1}, P_{-i}) \tag{17}$$

Similarly, Eqs. (16) and (15) imply

$$\varphi_{x_1}(P_i^h, P_{-i}) \geq \varphi_{x_1}(P_i^{h+1}, P_{-i}) \tag{18}$$

Again, since $\varphi_y(P_i^h, P_{-i}) = \varphi_y(P_i^{h+1}, P_{-i})$ for all $y \notin C$, Eqs. (16), (17) and (18) imply that for any $x \in X$, $\sum_{y R_i x} \varphi_y(P_i^h, P_{-i}) \geq \sum_{y R_i x} \varphi_y(P_i^{h+1}, P_{-i})$, i.e., $P_i^h \gg^{P_i} P_i^{h+1}$. This completes the proof of Theorem 2. □

A domain $\mathcal{D}$ is **strongly connected** if for any two distinct preferences P_i and P_i' there exists a sequence $(P^1, \ldots, P^k)$ that satisfies the non-restoration property where $P^1 = P_i$, $P^k = P_i'$ and for each $h < k$, and the graph $G(P_i^h, P_i^{h+1})$ has exactly one connected component C such that $|C| = 2$.[17] Sato (2013) showed that the single-peaked domain is strongly connected and Carroll (2012) showed that the (maximal) single-crossing domain is also strongly connected.[18] Note that the universal domain is obviously strongly connected. Clearly, any strongly connected domain is also weakly connected. Therefore, following corollary holds.

Corollary 1. *Let φ be a social choice function on a strongly connected domain $\mathcal{D}$. Then φ is strategyproof if, and only if, it is monotone strategyproof.*

The next example shows that when the domain is not weakly connected then strategyproofness and monotone strategyproofness are no longer equivalent.

Example 1. Let $X = \{x_1, x_2, x_3, x_4\}$, and let $\mathcal{D}$ be the domain composed of the three preference orderings depicted in Table 1. Note that $K(P_i', P_i) = \{(x_3, x_4)\}$ and $K(P_i'', P_i) = \{(x_3, x_4), (x_1, x_2), (x_1, x_3), (x_1, x_4)\}$, i.e., $K(P_i', P_i) \subset K(P_i'', P_i)$. However, the domain $\mathcal{D}$ is not weakly connected.

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[15] Note that if $|C| = 2$, the only way in which C could be a connected component is to represent a complete reversal, i.e., $P_i^{h+1}|_C$ is a complete reversal of $P_i^h|_C$.

[16] If $|K(P_i^h, P_i^{h+1})| = 1$ then $|C| \neq 3$.

[17] Sato (2013) calls a strongly connected domain a *connected domain that satisfies the non-restoration property*. The *strongly path-connected domain* defined by Chatterji et al. (2013) rests on a similar notion but the notion of connectedness is imposed on alternatives and not preferences.

[18] Both single-peakedness and single-crossingness assume the existence of an ordering of alternatives, and admissible preferences are obtained using this ordering. It is of course possible to have a (small) domain that satisfies the properties required by single-peakedness or single-crossingness but does not contain enough preferences such that any pair of preferences in the domain are connected.

Table 1
A domain not strongly connected.

P_i	P'_i	P''_i
x_1	x_1	x_2
x_2	x_2	x_4
x_3	x_4	x_3
x_4	x_3	x_1

Table 2
Probabilities of each alternative under P_i, P'_i and P''_i.

	P_i	P'_i	P''_i
x_1	.51	.51	.1
x_2	.3	.3	.7
x_3	.18	.01	.1
x_4	.01	.18	.1

Table 3
A domain with weak preferences.

R_i	R'_i	R''_i
x_1	x_1	x_2, x_3
x_2	x_2, x_3	x_1
x_3		

Let φ be a mechanism such that, for any P_{-i}, the probability to obtain alternative $x \in X$ for each of the preferences in $\mathcal{D}$ is given by Table 2. It can be verified that φ is strategyproof, yet it is not monotone strategyproof as $P'_i \gg^{P_i} P''_i$ does not hold. Indeed, $.82 = \sum_{x\,:\,xR_ix_3} \varphi_x(P'_i, P_{-i})) < \sum_{x\,:\,xR_ix_3} \varphi_x(P''_i, P_{-i})) = .9$. □

4. Discussion

4.1. Weak preferences

Until now we have only considered the case of strict preference domains. In this section we question whether our results extend to the domains with weak preferences. A weak preference relation R_i for agent i over X is a complete, reflexive and transitive binary relation on X. Given a preference relation R_i we denote by P_i and I_i the corresponding strict and indifference preference relation, respectively. That is, xP_ix' if xR_ix' and *not* $x'R_ix$, and xI_ix' if both xR_ix' and $x'R_ix$ hold. We denote by $\mathcal{R}$ the domain of all possible preference profiles over X.

The natural extension of the Kemeny set inclusion for weak preference relations—when comparing two preference orderings with respect to a third one—is the notion of *intermediate preferences* introduced by Grandmont (1978).

Definition 4. R'_i is between R_i and R''_i (noted as $R'_i \in (R_i, R''_i)$) if for all $x, x' \in X$,

(a) xR_ix' and xR''_ix' imply xR'_ix'.
(b) xP_ix' and xP''_ix' imply xP'_ix'.
(c) $(xI_ix'$ and $xP''_ix')$ or $(xP_ix'$ and $xI''_ix')$ imply xR'_ix'.

Observe that for a triple (P_i, P'_i, P''_i) of strict preferences $K(P'_i, P_i) \subset K(P''_i, P_i)$ implies that condition *(b)* of Definition 4 holds. For weak preferences, a natural definition of monotone strategyproofness would be that for any triple of preference relations (R_i, R'_i, R''_i) such that $R'_i \in (R_i, R''_i)$, it holds that R'_i dominates R''_i with respect to R_i. One could then conjecture that monotone strategyproofness would be equivalent to strategyproofness in this setting. However, as the following example shows, there exist situations where this property does not hold.

Example 2. Let $X = \{x_1, x_2, x_3\}$, and let $\mathcal{D}$ be the domain composed of the three preference orderings depicted in Table 3 where by the notation in the table, under R'_i, agent i is indifferent between x_2 and x_3, but strictly prefers x_1 to both x_2 and x_3.

It is easy to verify that $R'_i \in (R_i, R''_i)$, that is, R'_i is between R_i and R''_i. Let φ be a strategyproof mechanism such that for some R_{-i} we have $\varphi(R'_i, R_{-i}) = x_3$ and $\varphi(R''_i, R_{-i}) = x_2$. So $\varphi(R''_i, R_{-i})R_i\varphi(R'_i, R_{-i})$, so we cannot have R'_i dominating R''_i with respect to R_i. □

In other words, the equivalence between strategyproofness and monotone strategyproofness is not assured when considering domains with weak preferences. Note that it is not just the presence of indifferences—stochastic mechanisms also allow for indifferences between lotteries, and yet, Theorem 2 establishes the strategyproofness equivalence result for stochastic mechanisms (for a wide class of environments). This is because the indifference resulting from lotteries and indifferences embedded in weak preferences are different in their nature, and they differently impact strategyproofness conditions.[19]

Strategyproofness imposes more restrictions on feasible outcomes in the case of a stochastic mechanism on a domain with strict preferences than in the case of a mechanism on a domain with weak preferences. This follows from the fact that for a given set of alternatives X, strategyproofness of a stochastic mechanism on a domain with strict preferences yields $|X|$ distinct inequalities that the outcome needs to satisfy (cf. conditions in Eq (1)). When the domain includes weak preferences, some of the inequalities are redundant, and therefore there are fewer than $|X|$ distinct inequalities that the outcome needs to satisfy to for a mechanism to be strategyproof.[20] This redundancy allows then for a larger set of feasible outcomes under weak preferences, which may entail in a violation of the strategyproofness/monotone strategyproofness equivalence result.

4.2. Comparing preferences

Theorems 1 and 2 show that Kemeny set inclusion captures dominance relations between different strategies in a strategyproof mechanism. One natural question to address is whether the converse also holds, i.e., when a preference ordering P_i' dominates another ordering P_i'', is it necessarily the case that $K(P_i', P_i)$ is a subset of $K(P_i'', P_i)$? In other words one may ask whether the partial order over preferences induced by the Kemeny set relation is the weakest possible order such that the equivalence between monotone strategyproofness and strategyproofness holds.

To investigate this question, first note that we may be limited in the set of alternatives we can compare. To see this, suppose that for some alternatives x and y, and preference orderings P_i' and P_i'', there is no P_{-i} such that $\varphi(P_i', P_{-i}) = x$ and $\varphi(P_i'', P_{-i}) = y$. If this happens, it is impossible to know how P_i' and P_i'' compare those two alternatives, and thus we cannot say anything about Kemeny set inclusion. So when comparing two preference orderings P_i' and P_i'', we can only consider pairs of alternatives x, y such that, for some P_{-i}, $\varphi(P_i', P_{-i}) = x$ and $\varphi(P_i'', P_{-i}) = y$.

Given a deterministic mechanism φ, the **joint range** of two preference orderings P_i and P_i' is the set of pairs of alternatives (x, y) for which there exists a profile P_{-i} such that, for some preference profile P_{-i}, one ordering yield x and the other y. Formally, the joint range of P_i and P_i', denoted $J_\varphi(P_i, P_i')$, is

$$J_\varphi(P_i, P_i') = \Big\{(x, y) \;:\; \exists P_{-i} \text{ such that either } \varphi(P_i, P_{-i}) = x \;\&\; \varphi(P_i', P_{-i}) = y \\ \text{or } \varphi(P_i, P_{-i}) = y \;\&\; \varphi(P_i', P_{-i}) = x\Big\}. \tag{19}$$

Definition 5. Given three preference orderings P_i, P_i' and P_i'', the **Kemeny set of P_i' with respect to P_i on joint range with** P_i'' is the set of all pairs $(x, x') \in X \times X$ that are not ordered identically in P_i and P_i' and that belong to the joint range of P_i' and P_i'', i.e.,

$$\widehat{K}(P_i', P_i, P_i'') \equiv K(P_i', P_i) \cap J_\varphi(P_i', P_i''). \tag{20}$$

Proposition 1. *Let φ be a deterministic strategyproof social choice function, and let (P_i, P_i', P_i'') be any triple of preferences. Then P_i' dominates P_i'' with respect to P_i if, and only if, $\widehat{K}(P_i', P_i, P_i'') \subseteq \widehat{K}(P_i'', P_i, P_i')$.*

Proof. The *if* direction is a direct corollary of Theorem 1. Consider the *only if* direction, and let P_i, P_i' and P_i'' be such that $P_i' \gg^{P_i} P_i''$. We need to show that $\widehat{K}(P_i', P_i, P_i'') \subseteq \widehat{K}(P_i'', P_i, P_i')$. So we only need to check the Kemeny set inclusion for the pairs that are in the joint range of P_i' and P_i''. Accordingly, let (x, y) be any pair of alternatives in $J_\varphi(P_i', P_i'')$. We then have to show that

$$(x, y) \in K(P_i', P_i) \quad \Rightarrow \quad (x, y) \in K(P_i'', P_i). \tag{21}$$

Observe that if $(x, y) \notin K(P_i', P_i)$, then Eq. (21) is trivially satisfied. Suppose then that $(x, y) \in K(P_i', P_i)$. Without loss of generality, suppose that $xP_i'y$. If $(x, y) \notin K(P_i'', P_i)$, then we have $yP_i''x$. Since $(x, y) \in J_\varphi(P_i', P_i'')$, there exists P_{-i} such that $\varphi(P_i', P_{-i}) \neq \varphi(P_i'', P_{-i})$ and $\{\varphi(P_i', P_{-i}), \varphi(P_i'', P_{-i})\} = \{x, y\}$. If $\varphi(P_i', P_{-i}) = y$, then $P_i' \gg^{P_i'} P_i''$ implies that

[19] Specifically, under stochastic mechanism on strict preferences individuals always have strict preferences over the corners of the simplex. Thus, while the use of a stochastic mechanism introduces possible indifferences between strategies, it does not induce indifferences between alternatives of the kind we have with weak preferences.

[20] For example, if yI_iz, then $\{x' : x'R_iy\} \equiv \{x' : x'R_iz\}$, and therefore the Eq (1) inequality for y constitutes the same expression as the inequality for Z. Thus, one of them is redundant.

$\varphi(P''_i, P_{-i}) \neq x$. So it must be that $\varphi(P'_i, P_{-i}) = x$ and $\varphi(P''_i, P_{-i}) = y$. But then $(x, y) \in K(P'_i, P_i)$ and xP'_iy implies yP_ix, i.e., $\varphi(P''_i, P_i)P_i\varphi(P'_i, P_i)$. This contradicts $P'_i \gg^{P_i} P''_i$. So we have $(x, y) \in K(P''_i, P_i)$. □

Kemeny set inclusion (or betweenness) is more restrictive than Kemeny set inclusion on the joint range. The next example illustrates this point with the median voter with single-peaked preferences, where P'_i dominating P''_i with respect to P_i does not imply that $K(P_i, P'_i) \subset K(P_i, P''_i)$.

Example 3. Let φ be the standard median voter social choice function, and let the domain be the single-peaked preference domain. It is well known that for the single-peaked domain the median voter rule is strategyproof (Moulin, 1980). Let L be the order of the alternatives under which the domain is single-peaked. For simplicity, assume here that there is an odd number of individuals.

Let i be an individual and P_i his true preferences where x is the most preferred alternative (the *peak*) according to P_i. Consider now two preference orderings (that are single-peaked under the order L), P'_i and P''_i, where x' and x'' are their respective peaks. Suppose that $xLx'Lx''$. Then P'_i dominates P''_i with respect to P_i.[21]

Suppose that there exists a pair of alternatives, say x_1 and x_2, such that $(x_1, x_2) \in K(P'_i, P_i)$ yet $(x_1, x_2) \notin K(P''_i, P_i)$. We claim that $(x_1, x_2) \notin J_\varphi(P'_i, P''_i)$. To see this, without loss of generality suppose that $\varphi(P'_i, P_{-i}) = x_1$ and $\varphi(P''_i, P_{-i}) = x_2$. Consider first the case when x_1Lx'. If x_1Lx', then by choosing P''_i instead of P'_i individual i cannot change the outcome, so $x_1 = x_2$, a contradiction. So, $x'Lx_1$. Using symmetric argument we obtain x_2Lx''. So we have $xLx'Lx_1Lx_2Lx''$, which contradicts $(x_1, x_2) \in K(P'_i, P_i)$. If x_2Lx_1, a similar argument leads to a contradiction, too. It is important to note that $(x_1, x_2) \in K(P'_i, P_i)$ and $(x_1, x_2) \notin K(P''_i, P_i)$ does not contradict single-peakedness. So, even though P'_i dominates P''_i we can still have $K(P'_i, P_i) \not\subseteq K(P''_i, P_i)$. However, it cannot be that $\widehat{K}(P'_i, P_i, P''_i) \not\subseteq \widehat{K}(P''_i, P_i, P'_i)$. □

4.3. Cardinal environments

The concept of monotone strategyproofness can be easily adapted to cardinal environments, i.e., when an individual is characterized by a utility vector over the set of alternatives and individuals have expected utility preferences over lotteries. A few more definitions are needed before going further.

A **type space** is a non-empty subset of $\times_{i\in N}\mathbb{R}^{|X|}$, and an **individual's type** is a vector in $\mathbb{R}^{|X|}$. Given a type space T, a **mechanism** is a mapping $\varphi : T \to \Delta(X)$. We denote by u_i a generic type of individual i, and $u = (u_i)_{i\in N}$ is a **type profile**. Given a (true) type profile u and a reported type profile u', the expected utility of individual i is given by the inner product $u_i \cdot \varphi(u)$. A mechanism is **incentive compatible** on a type space T if, for each $i \in N$, for each $u \in T$ and each $u' \in T$ such that $u_{-i} = u'_{-i}$, we have $u_i \cdot (\varphi(u_i, u_{-i}) - \varphi(u'_i, u_{-i})) \geq 0$.

Definition 6. A mechanism is **monotone incentive compatible** on a type space T if, for each $i \in N$, for each $u \in T$ and each $u', u'' \in T$ such that $u_{-i} = u'_{-i} = u''_{-i}$ and $u'_i = \alpha \cdot u_i + (1-\alpha) \cdot u''_i$ for some $\alpha \in [0, 1]$, we have $u_i \cdot (\varphi(u'_i, u_{-i}) - \varphi(u''_i, u_{-i})) \geq 0$.

We can now introduce the counterpart of Theorem 2 for cardinal mechanisms.[22]

Proposition 2. *A mechanism is incentive compatible if, and only if, it is monotone incentive compatible.*

Proof. That a monotone incentive compatible mechanism is also incentive compatible is obvious.[23] Consider then an incentive compatible mechanism φ on a type space T. Let u_i be any admissible type for individual i, and u'_i, u''_i such that $u'_i = (1-\alpha) \cdot u_i + \alpha \cdot u''_i$ for some $\alpha \in [0, 1]$. Note that if $\alpha = 0$ or $\alpha = 1$ then $u'_i = u_i$ or u''_i and thus we trivially have $u_i \cdot (\varphi(u'_i, u_{-i}) - \varphi(u''_i, u_{-i})) \geq 0$. So assume $\alpha \in (0, 1)$. Since φ is incentive compatible,

$$u'_i \cdot (\varphi(u'_i, u_{-i}) - \varphi(u''_i, u_{-i})) \geq 0$$
$$u''_i \cdot (\varphi(u''_i, u_{-i}) - \varphi(u'_i, u_{-i})) \geq 0$$

Multiplying the second constraint by α and adding up the two inequalities and rearranging yields

$$(u'_i - \alpha u''_i) \cdot (\varphi(u'_i, u_{-i}) - \varphi(u''_i, u_{-i})) \geq 0$$

Note that $u'_i - \alpha u''_i = (1-\alpha) \cdot u_i$. Since $\alpha \in (0, 1)$ we obtain $u_i \cdot (\varphi(u'_i, u_{-i}) - \varphi(u''_i, u_{-i})) \geq 0$. □

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[21] Consider any profile P_{-i}. Observe that if $\varphi(P_i, P_{-i}) = \varphi(P''_i, P_{-i})$ then $\varphi(P'_i, P_{-i}) = \varphi(P_i, P_{-i})$, and thus $\varphi(P'_i, P_{-i})R_i\varphi(P''_i, P_{-i})$. So, assume $\varphi(P_i, P_{-i}) \neq \varphi(P''_i, P_{-i})$. If $\varphi(P'_i, P_{-i}) = \varphi(P_i, P_{-i})$ or $\varphi(P'_i, P_{-i}) = \varphi(P''_i, P_{-i})$ then again $\varphi(P'_i, P_{-i})R_i\varphi(P''_i, P_{-i})$. So, suppose that $\varphi(P_i, P_{-i}) \neq \varphi(P'_i, P_{-i}) \neq \varphi(P''_i, P_{-i})$. This implies that $\varphi(P_i, P_{-i}) = x$, $\varphi(P'_i, P_{-i}) = x'$ and $\varphi(P''_i, P_{-i}) = x''$. So, $\varphi(P'_i, P_{-i})P_i\varphi(P''_i, P_{-i})$.

[22] The proof of Proposition 2 is built on the proof of Proposition 1 in Carroll (2012).

[23] Take any u_i and u''_i and set $u'_i = 0 \cdot u_i + (1-0) \cdot u''_i$.

A straightforward application of Proposition 2 is for incentive compatible auction mechanisms with private values. Consider the case when individuals' types are real numbers (their value of the good to be auctioned). Our result then simply says that if an individual's private value for the auctioned good is, say, x, then bidding $x' < x$ dominates bidding $x'' < x'$.

5. Conclusions

We showed that for strategyproof mechanisms one can meaningfully compare the extent of preference misrepresentation by comparing pairs of alternatives. We defined the concept of monotone strategyproofness, which captures the link between incentives and the extent of a misrepresentation: a larger extent of misrepresentation makes the individual (weakly) worse off. Remarkably, requiring monotone strategyproofness does not reduce the set of strategyproof social choice functions. This result shows that imposing strategyproofness (or incentive compatibility) does not only consist of imposing the existence of *one* dominating strategy (the one corresponding to the true type/preferences) but also imposes the existence of a large collection of dominance relations between strategies, thereby providing further evidence that strategyproofness is a very demanding property. The works of Nehring and Puppe (2007) or Barberà et al. (2010) share some similarities with ours in the sense that we all address the question of which additional property, or feature, is implied by strategyproofness. Barberà et al. (2010) characterize domains of preferences under which any strategyproof social choice function is also group strategyproof, and Nehring and Puppe (2007) show that when the domain is a subdomain of generalized single-peaked preferences then any strategyproof social choice function takes the form of voting by issues.[24]

Our results also shed light on the complications that arise when social choice functions are non-deterministic, or are defined on domains with indifferences. Strategyproofness in the non-deterministic case imposes that the truthful strategy stochastically dominates any other strategy. It is well known that the mere existence of a stochastically dominating strategy can be very challenging in a general setting, so it is not a surprise that one needs to impose some constraints on the domain to obtain the equivalence between strategyproofness and monotone strategyproofness for stochastic mechanisms. As for the case of domains with weak preferences we encounter stronger hurdles. Our discussion in Section 4.1 indeed suggests that obtaining a similar result for the case of weak preferences seems beyond reach.

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[24] The similarities between their works and ours stop here. The domains identified by Nehring and Puppe (2007) or Barberà et al. (2010) differ significantly from that of weakly connected preferences. Contrary to these two papers, our domain condition never consists of comparing or relating preferences *across* individuals.