HOMOTOPY PERTURBATION ALGORITHM USING LAPLACE TRANSFORM FOR GAS DYNAMICS EQUATION

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Abstract

In this paper, we apply a combined form of the Laplace transform method with the homotopy perturbation method to obtain the solution of nonlinear gas dynamics equation. This method is called the homotopy perturbation transform method (HPTM). This technique finds the solution without any discretization or restrictive assumptions and avoids the round-off errors. The fact that this scheme solves nonlinear problems without using Adomian's polynomials can be considered as a clear advantage of this algorithm over the decomposition method. The results reveal that the homotopy perturbation transform method (HPTM) is very efficient, simple and can be applied to other nonlinear problems.

Mathematics Subject Classification 2000: 39B12, 35F25

Key Words and Phrases: Laplace transform method, Homotopy perturbation method, Gas dynamics equation, He's Polynomials

1. INTRODUCTION

Nonlinear phenomena play a crucial role in applied mathematics and physics. We know that most of engineering problems are non-linear, and it is difficult to solve them analytically. The importance of obtaining the exact or approximate solutions of nonlinear partial differential equations in physics and mathematics is still a significant problem that needs new methods to discover exact or approximate solutions. Various powerful mathematical methods have been proposed for obtaining exact and approximate analytic solutions. Some of the classic analytic methods are Lyapunov's artificial small parameter method [1], perturbation techniques [2-4], \( \delta \)-expansion method [5] and Hirota bilinear method [6, 7]. In recent years, many research workers have paid attention to study the solutions of nonlinear partial differential equations by using various methods. Among these are the Adomian decomposition method (ADM) [8], He’s semi-inverse method [9], the tanh method, the homotopy perturbation method (HPM), the differential transform method and the variational iteration method (VIM) [10-17]. He [25-38] developed the homotopy perturbation method (HPM) by merging the standard homotopy and perturbation for solving various physical problems. It is worth mentioning that the HPM is applied without any discretization, restrictive assumption or transformation and is free from round off errors. The Laplace transform is totally incapable of handling nonlinear equations because of the difficulties that are caused by the nonlinear terms. Various ways have been proposed recently to deal with these nonlinearities such as the Adomian decomposition method [39] and the Laplace decomposition algorithm [40-44]. Furthermore, the homotopy perturbation method is also combined with the well-known Laplace transformation method [45] and the variational iteration method [47] to produce a highly effective technique for handling many nonlinear problems. In a recent paper
Khan and Wu [23] proposed the homotopy perturbation transform method (HPTM) for solving the nonlinear equations. It is worth mentioning that the HPTM is an elegant combination of the Laplace transformation, the homotopy perturbation method and He’s polynomials and is mainly due to Ghorbani [20, 21]. The homotopy perturbation transform method (HPTM) provides the solution in a rapid convergent series which may lead to the solution in a closed form. The advantage of this method is its capability of combining two powerful methods for obtaining exact solutions for nonlinear equations. Recently several techniques including the Adomian decomposition method and the homotopy perturbation method (HPM) have been used to handle nonlinear homogeneous gas dynamics equations [18, 19]. Inspired and motivated by the ongoing research in this area, we use the homotopy perturbation transform method (HPTM) in solving the following nonlinear homogeneous gas dynamics equation

\[
\frac{\partial u}{\partial t} + \frac{1}{2}(u^2)_x - u(1-u) = 0, \quad 0 \leq x \leq 1, \quad t > 0,
\]

with the initial condition

\[
u(x,0) = g(x).
\]

2. HOMOTOPY PERTURBATION TRANSFORM METHOD (HPTM)

To illustrate the basic idea of this method, we consider a general nonlinear partial differential equation with the initial conditions of the form:

\[
D u(x,t) + R u(x,t) + N u(x,t) = g(x,t),
\]

\[
u(x,0) = h(x), \quad u_t(x,0) = f(x),
\]

where \(D\) is the second order linear differential operator \(D = \frac{\partial^2}{\partial t^2}\), \(R\) is the linear differential operator of less order than \(D\), \(N\) represents the general nonlinear differential operator and \(g(x, t)\) is the source term. Taking the Laplace transform (denoted in this paper by \(L\)) on both sides of eq. (3), we get

\[
L[D u(x,t)] + L[R u(x,t)] + L[N u(x,t)] = L[g(x,t)].
\]

Using the differentiation property of the Laplace transform, we have

\[
L[u(x,t)] = \frac{h(x)}{s} + \frac{f(x)}{s^2} - \frac{1}{s^2} L[R u(x,t)] + \frac{1}{s^2} L[g(x,t)] - \frac{1}{s^2} L[N u(x,t)].
\]

Operating with the Laplace inverse on both sides of eq. (5) gives

\[
u(x,t) = G(x,t) - L^{-1}\left[\frac{1}{s^2} L[R u(x,t) + N u(x,t)]\right].
\]

where \(G(x,t)\) represents the term arising from the source term and the prescribed initial conditions. Now we apply the homotopy perturbation method

\[
u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t)
\]

and the nonlinear term can be decomposed as

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\[ N u(x,t) = \sum_{n=0}^{\infty} p^n H_n(u) \]  

(8)

for some He's polynomials \( H_n(u) \) (see [45, 46]) that are given by

\[ H_n(u_0,u_1,\ldots,u_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[ N \left( \sum_{j=0}^{\infty} p^j u_j \right) \right]_{p=0}, \quad n = 0, 1, 2, 3\ldots \]  

(9)

Substituting eq. (8) and eq. (7) in eq. (6), we get

\[
\sum_{n=0}^{\infty} p^n u_n(x,t) = G(x,t) - p \left( L^{-1} \left[ \frac{1}{s^2} L \left[ R \sum_{n=0}^{\infty} p^n u_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(u) \right] \right) \right)
\]

(10)

which is the coupling of the Laplace transform and the homotopy perturbation method using He's polynomials. Comparing the coefficient of like powers of \( p \), the following approximations are obtained.

\( p^0 : u_0(x,t) = G(x,t) \)

\[
\begin{aligned}
p^1 : u_1(x,t) &= -L^{-1} \left[ \frac{1}{s^2} L \left[ R u_0(x,t) + H_0(u) \right] \right], \\
p^2 : u_2(x,t) &= -L^{-1} \left[ \frac{1}{s^2} L \left[ R u_1(x,t) + H_1(u) \right] \right], \\
p^3 : u_3(x,t) &= -L^{-1} \left[ \frac{1}{s^2} L \left[ R u_2(x,t) + H_2(u) \right] \right], \\
\vdots
\end{aligned}
\]

(11)

3. EXPERIMENTAL EVALUATION

In this section we consider the following nonlinear homogeneous gas dynamics equation

\[
\frac{\partial u}{\partial t} + \frac{1}{2}(u^2)_x - u(u - 1) = 0, \quad 0 \leq x \leq 1, \quad t > 0,
\]

(12)

with the initial condition

\[ u(x,0) = g(x) = e^{-x}. \]

(13)

Taking the Laplace transform on both sides of eq. (12) subject to the initial condition (13), we have
\[ L[u(x,t)] = \frac{e^{-x}}{s} - \frac{1}{2s} L\left[\left(\frac{u^2}{x}\right)_x\right] + \frac{1}{s} L[u(1-u)]. \] (14)

The inverse of Laplace transform implies that
\[ u(x,t) = e^{-x} - L^{-1}\left[\frac{1}{s} L\left[\frac{1}{2} \left(\frac{u^2}{x}\right)_x - u(1-u)\right]\right]. \] (15)

Now, applying the homotopy perturbation method, we get
\[ \sum_{n=0}^{\infty} p^n u_n(x,t) = e^{-x} - p \left[ L^{-1}\left(\frac{1}{2} \left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)^2\right)_x \right] \]
\[ - \sum_{n=0}^{\infty} p^n u_n(x,t) \left(1 - \sum_{n=0}^{\infty} p^n u_n(x,t)\right) \] (16)

Comparing the coefficients of like powers of \( p \), we have
\[ p^0 : u_0(x,t) = e^{-x}, \]
\[ p^1 : u_1(x,t) = -L^{-1}\left[\frac{1}{s} L\left[\frac{1}{2} \left(u_0^2\right)_x - u_0(1-u_0)\right]\right] = e^{-x} t, \] (17)
\[ p^2 : u_2(x,t) = -L^{-1}\left[\frac{1}{s} L\left[\frac{1}{2} \left(u_0 u_1\right)_x - u_1 + u_0 u_1\right]\right] = e^{-x} \frac{t^2}{2}. \]

Proceeding in a similar manner, we have
\[ p^3 : u_3(x,t) = e^{-x} \frac{t^3}{6}, \] (18)
\[ \vdots \]
Therefore the solution \( u(x,t) \) is given by
\[ u(x,t) = e^{-x} \left(1 + t + \frac{1}{2} t^2 + \frac{1}{6} t^3 + \cdots\right) = e^{\ell - x}. \] (19)

It is obvious that a higher number of iterations make \( u_n(x,t) \) converge to the exact solution \( e^{\ell - x} \).

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4. CONCLUSIONS

In this paper, the homotopy perturbation transform method (HPTM) was successfully applied to study the homogeneous case of nonlinear gas dynamics with initial condition. The results show that the homotopy perturbation transform method (HPTM) is powerful and efficient technique in finding exact and approximate solutions for nonlinear differential equations. It is worth mentioning that HPTM is capable of reducing the volume of the computational work as compared to the classical methods while still maintaining the high accuracy of the numerical result; the size reduction amounts to an improvement of the performance of the approach. The fact that the HPTM solves nonlinear problems without using Adomian’s polynomials is a clear advantage of this technique over the decomposition method. In conclusion, the HPTM may be considered as a nice refinement in existing numerical techniques and might find the wide applications.

ACKNOWLEDGMENTS

The authors are grateful to the referees for their invaluable suggestions and comments for the improvement of the paper.

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Received July 2011